Notes, Gene Ressler, 7 June 2005.

Disjoint circle merge (dcm). Let S be a set of n unit radius (area π) circles in the Cartesian plane with centers $\vec{p}_1, \ldots, \vec{p}_n$. Then $\Psi(S)$ is a single circle having area $n\pi$ with its center at $\vec{p} = \frac{1}{n}\sum_{i=1}^{n} \vec{p}_i$, the centroid of S. The "error" associated with S is $\varepsilon(S) = \sum_{i=1}^{n} |\vec{p}_i - \vec{p}|$.

Instance: A real number $E \geq 0$ and a set C of circles in the Cartesian plane, each with area π.

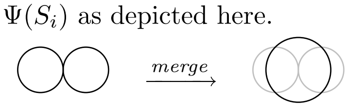
Question: Does there exist a partition of C into disjoint sets S_i such that the circles $\Psi(S_i)$ are also disjoint and $\max_i \varepsilon(S_i) \leq E$?

Call such a partition a *solution* of the instance.

Proof of NP completeness. It is easy to verify in polynomial time that the circles in a given solution are disjoint and that the error criterion is satisfied. Thus the problem is in NP.

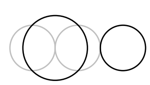
We complete the proof by reduction from planar 3-sat (p3sat). For a given instance F of p3sat, we construct a corresponding instance of dcm that has a solution if and only if F has a satisfying truth assigment.

We choose $E = 2$ and adopt the invariant that all non-disjoint circles in our construction are exactly tangent—their centers are 2 units apart. Thus all partition sets S_i in any solution must contain exactly two tangent circles. We say they *merge* to form $\Psi(S_i)$ as depicted here.



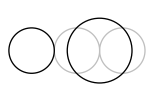
Any other kind of subset—with more than two circles or disjoint circles—entails error greater than 2 and so cannot exist in a solution.

Our construction relies on standard configurations of circles that we call *widgets*. A *flip-flop* widget is three tangent circles with colinear centers. It has two solutions. We designate the one shown below as the *true state*. Our convention as above is to draw the original circles from the set C in gray and the solution circles $\Psi(S_i)$ in black. When gray and black coincide, the case $|S_i| = 1$, we also use black.



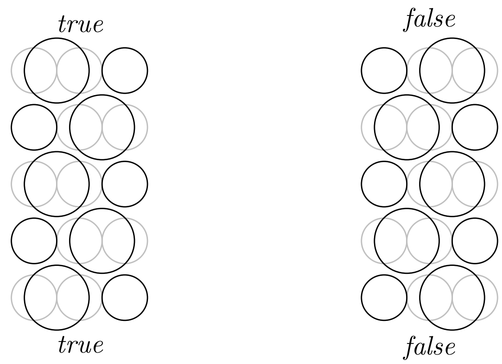
true

The other solution for the flip-flop widget is shown below—the *false state*.

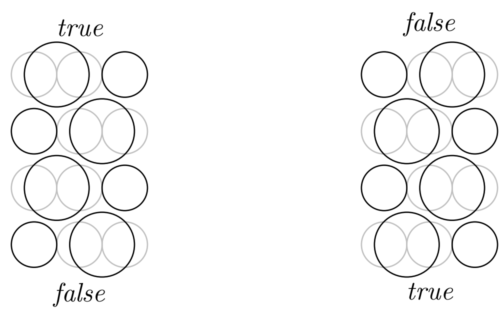


false

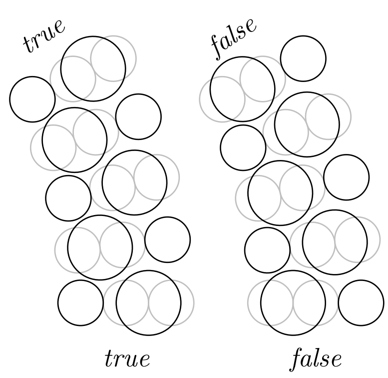
A *wire* widget is just a number of flip-flops arranged side-by-side with a separation of 5/2. This spacing causes the state of any one flip-flop to effectively determine all others in any solution. The section of wire below has a length of 5 flip-flops, but it could be extended to any number.

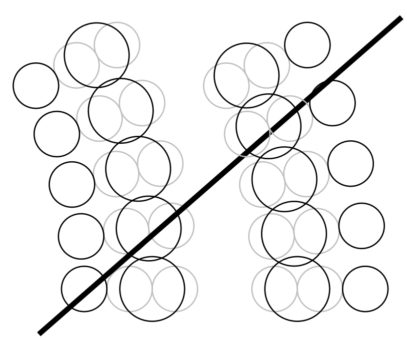


The flip-flops at the ends of any odd-length wire as shown above necessarily have the same truth value in any solution. We call these *plain wires*. Even-length wires have ends with opposite values as shown below. These are *inverting wires*.



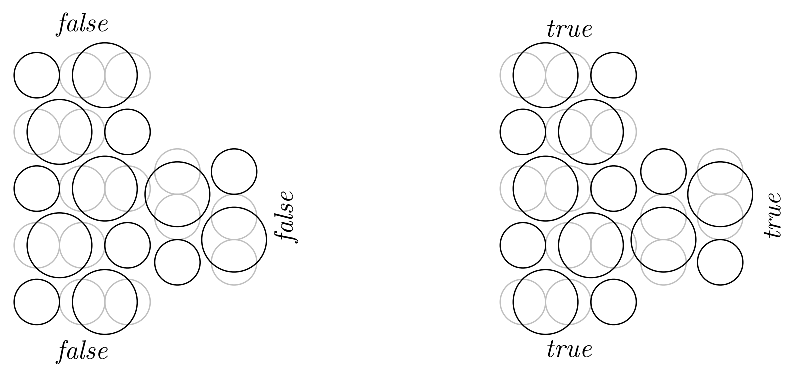
Wires can bend gently without changing their useful properties. Below is a bend with radius 20. Each pair of flip-flops subtends an arc of $\pi/26$.





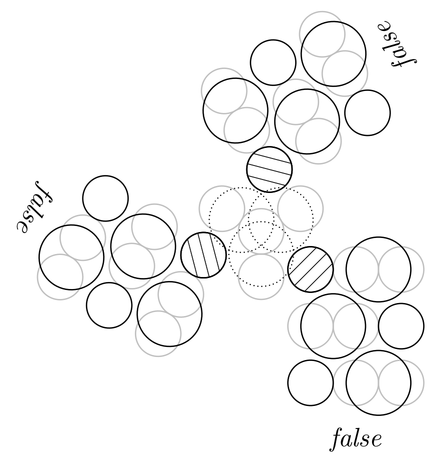
The slashed drawings on the right show that non-alternating merge patterns cannot exist in a solution even with the bend.

A *tee* widget is two wires carefully positioned to form a three-way branch.



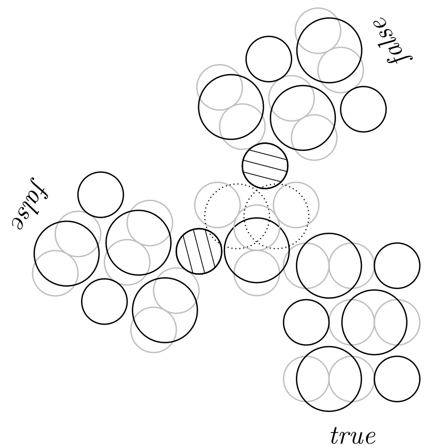
Again it is not hard to see that these two are the only admissible solutions for this set of circles. All three ends of the tee must have the same truth value in any solution.

A *clause* widget is shown here.



Solutions for the clause widget behave like an "or gate" due to the four inner circles, which are tangent. The other circles serve as input wires. The precise positioning of the hatched circles of the wires is essential. When all three inputs are false as shown, there is no solution because no pair among the inner four circles can be merged. Prospective merges are shown with dotted lines, but these cannot be part of a solution because each one overlaps a hatched circle.

If at least one of the inputs is true as in the following sketch, however, any single pair among the center four circles can be merged to form a solution as shown below.



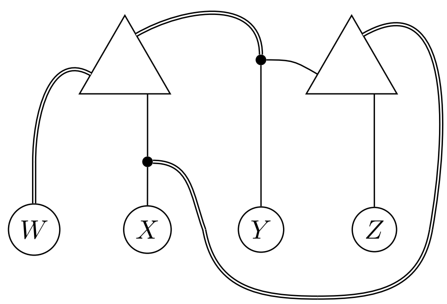
The reader can easily verify that if one or both of the remaining inputs changes to *true*, a solution to F is found with equal ease.

The remainder of our construction is a job of plumbing. Starting with an empty Cartesian plane, add one flip-flop widget for each variable in F and one clause widget for each clause. Connect each clause widget by wires to the variable widgets that correspond to its literals. If a literal is negated, use an inverting wire, otherwise a plain one. When the same variable must be connected to more than one clause, use a tee widget. Allow enough space between variables and clauses in the plane so that wires are bent with radii of no less than 20. Finally, ensure that no wires cross. This is always possible due to the planar property of P3SAT.

To make this discussion concrete, consider the expression

$$(\overline{W} \vee X \vee \overline{Y}) \wedge (\overline{X} \vee Y \vee Z).$$

This transforms, using the procedure above, to the following configuration of widgets, which is shown schematically.



The circular nodes are the flip-flop widgets for variables. Triangular nodes are clause widgets. Single lines are plain wires. Double lines are inverting wires. Dots are tees.

With this, the construction is complete. It is tedious but not hard to show that the time to compute a planar "wiring diagram" for F as shown above and also the number of circles required are bounded by polynomial functions of the length of a reasonable encoding of F.

It remains to show as claimed at the outset that the constructed instance of DCM has a solution with $E \leq 2$ if and only if there is a satisfying truth assignment for F. Since a satisfying truth assignment directly corresponds to a solution of merged pairs of circles according to the diagrams above, the "if" of this statement is clearly true.

The final step is to show that if there is no satisfying truth assignment, then there is no circle merge solution. For this purpose, note that every DCM solution directly implies a satisfying truth assignment for the corresponding instance of P3SAT. Each solution of a clause widget corresponds directly to truth values for its three literals, one of which must be true. These values must be consistent with other uses of the same variables due to the properties of tees and wires; i.e., each tee and wire has only two possible states, each corresponding to the value of a variable or its complement. It follows that if given a DCM solution, we can merely read the corresponding P3SAT solution by inspecting the variable flip-flop widget states.

To complete the proof, let us assume that indeed F is unsatisfiable, yet we have found a circle merge solution—pairs of tangent circles have been merged until there are no overlaps. This is an immediate contradiction of the assumption because we can read a satisifying truth assignment from the solution as described above. Therefore no solution exists. $\square$