1) (13.1 #13)

The figure below shows a contour diagram for the monthly payment P as a function of the interest rate r% and the amount L of the loan.

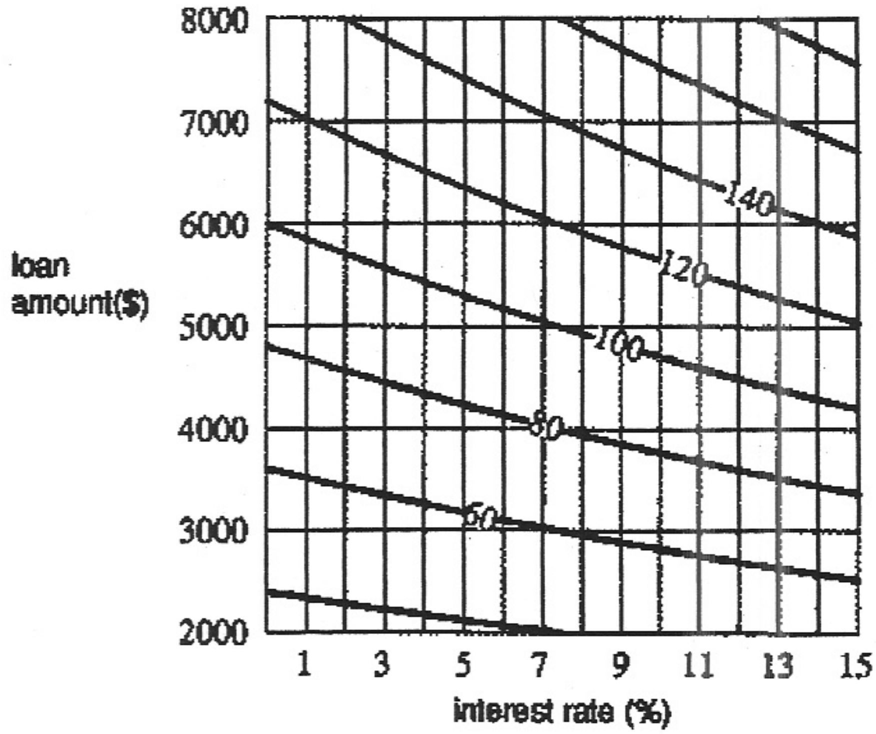
Estimate the given partial derivatives at the given points. In each case, give the units, and explain the everyday meaning of your answer.

$\frac{\partial P}{\partial r}$ and $\frac{\partial P}{\partial L}$ at:

a) r = 8 , L = 4000

b) r = 8 , L = 6000

c) r = 13 , L = 7000



2) (13.2 example 2)

Find both partial derivatives:

a) $f(x,y) = e^{x+3y}\sin(xy)$

b) $f(x,y) = y^2 e^{3x}$

c) $z = (3xy + 2x)^5$

3) (13.4 #16, #13, #18 and example 7)

Find $\circ$ z , the gradient of z for

a) $z = \frac{xe^y}{x+y}$

b) $f(x,y) = x + e^y$ at (1,1)

c) $f(x,y) = \frac{3}{2}x^5 - \frac{4}{7}y^6$

d) $z = (x + y)\, e^y$

4) (15.2 #12)

Evaluate $\int_1^4 \int_{\sqrt{y}}^{y} x^2 y^3 \, dx \, dy$

5) a) (16.1 #15)

Write a parameterization for a circle of radius 2 centered at the region traced out clockwise starting at the point (-2,0) when t = 0.

b) (Example 1, 16.2)

Give a parameterization for the circle of radius 1/2 centered at (-1,2).

6) Set up $\int_R f \, dA$ as an iterated integral in polar coordinates with R as below.

a) (15.5 #3)

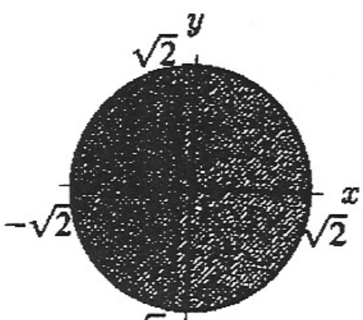


Figure 15.33

b) (15.5 #4)

4.

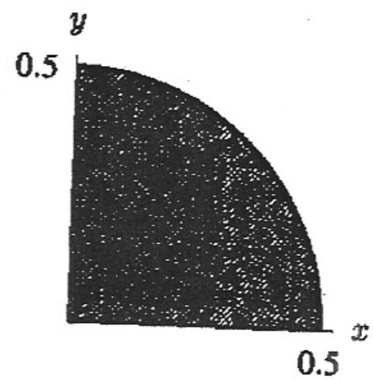


Figure 15.34

7) a) (12.4 example 3)

Find $(2\vec{i} + \vec{j} - 2\vec{k}) \times (3\vec{i} + \vec{k})$

b) (12.4 #6)

Find $(2\vec{i} - 3\vec{j} + \vec{k}) \times (\vec{i} + 2\vec{j} - \vec{k})$

8) (17.1 #1 d)
Sketch a graph of a vector field representing the force on a particle at different points in space as a result of another particle at the origin where the force is an attractive force whose magnitude increases as distance decreases.

9) Is the line integral $\int_C \vec{F} d\vec{r}$ positive, negative or zero for the given field $\vec{F}$ and curve C below?

a) (#3 18.1)

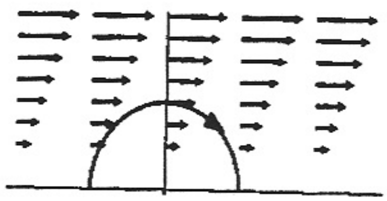


Figure 18.15

b) (#4 18.1)

4.

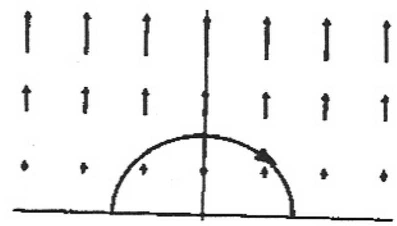
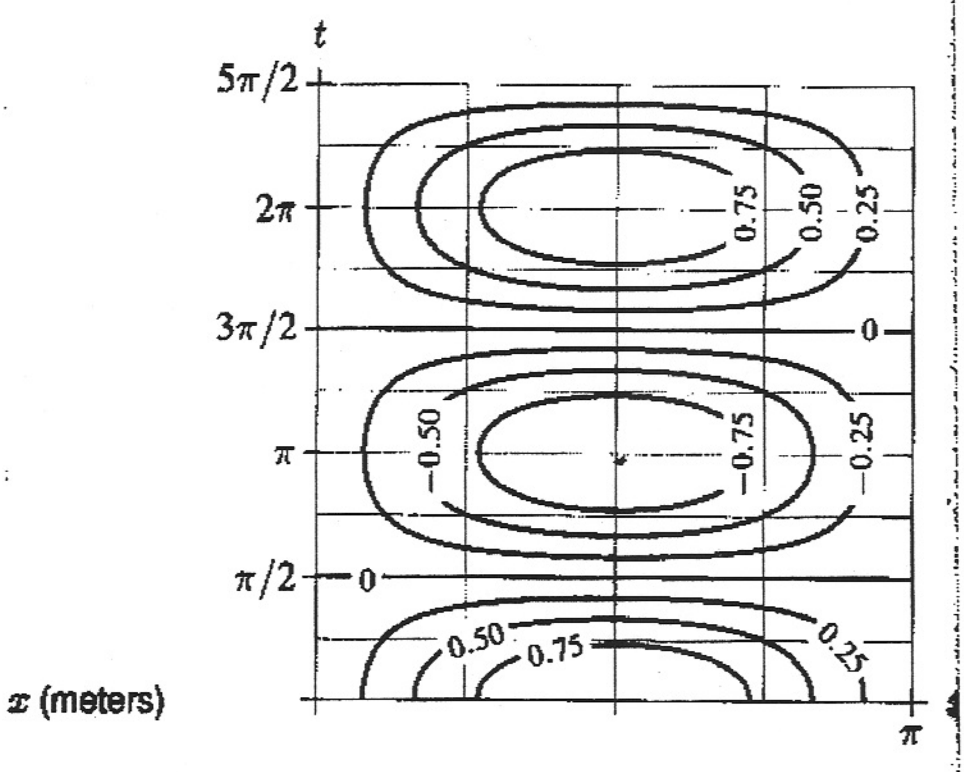


Figure 18.16

10) Chapter 14 Review #23
The contour diagram for f(x,t) = cos t sin x is given below.
Give the coordinates of any local extrema.



11) (14.2 #10)

Assume that two products are manufactured in quantities q_1 and q_2 and sold at prices of p_1 and p_2 respectively, and that the cost of producing them is given by

$$C = 2q_1^2 + 2q_2^2 + 10 .$$

a) Find the maximum profit that can be made, assuming the prices are fixed.

b) Find the rate of change of that maximum profit as p_1 increases.

12) (14.3 #26)

Suppose the quantity, q, of a product manufactured depends on the number of workers, W, and the amount of capital invested, K, and is represented by the Cobb-Douglas function

$$q = 6W^{\frac{3}{4}} K^{\frac{1}{4}}$$

In addition, labor costs are \$10 per worker and capital costs are \$20 per unit, and the budget is \$3000.

a) What are the optimum number of workers and the optimum number of units of capital?

b) Check that at the optimum values of W and K, the ratio of the marginal productivity of labor to the marginal productivity of capital is the same as the ratio of the cost of a unit of labor to the cost of a unit of capital.

c) Recompute the optimum values of W and K when the budget is increased by one dollar. Check that increasing the budget by \$1 allows the production of ℓ extra units of the good, where ℓ is the Lagrange multiplier.

13) (14.3 #18)

A company manufactures a product using inputs x, y, and z according to the production function

$$Q(x,y,z) = 20x^{\frac{1}{2}} y^{\frac{1}{4}} z^{\frac{2}{5}}$$

The input prices per unit are $20 for x, and $10 for y, and $5 for z. What inputs should the company use if it wishes to manufacture 1,200 products at minimum cost?

14) Find the local maxima, minima and saddle points of the given function. Justify your answers.

a) (14.Review #1)

$f(x,y) = \sin x + \sin y + \sin(x + y), \quad 0 < x < \prec, \quad 0 < y < \prec$.

b) (14 Review #2)

$f(x,y) = x^2 + y^3 - 3xy$

c) (14 Review #3)

$f(x,y) = xy + \ln x + y^2 - 10 \quad (x > 0)$

15) (14 Review #11)

Suppose that the quantity, Q, manufactured of a certain product depends on the number of units of labor, L, and of capital, K, according to the function

$$Q = 900\, L^{\frac{1}{2}} K^{\frac{2}{3}}$$

Suppose also that labor costs $100 per unit and that capital costs $200 per unit. What combination of labor and capital should be used to produce 36,000 units of the goods at minimum cost? What is that minimum cost?

16) (15.5 #14)

Consider the integral $\int_0^3 \int_{\frac{x}{3}}^1 f(x,y)\ dy\ dx$

a) Sketch the region R over which the integration is being performed.

b) Rewrite the integral with the order of integration reversed.

c) Rewrite the integral in polar coordinates.

17) (13 Review #35)

Suppose that the values of the function f(x,y) near the point x = 2, y = 3 are given in the table below.

y \ x	2.00	2.01
3.00	7.56	7.42
3.02	7.61	7.47

Estimate the following.

a) $\left.\frac{\partial f}{\partial x}\right|_{(2,3)}$ and $\left.\frac{\partial f}{\partial y}\right|_{(2,3)}$

b) The rate of change of f at (2,3) in the direction of the vector $\vec{i} + 3\vec{j}$.

c) The maximum possible rate of change of f as you move away from the point (2,3). In which direction should you move to obtain this rate of change?

d) Write an equation for the level curve through the point (2,3).

e) Find a vector tangent to the level curve of f through the point (2,3).

f) Find the differential of f at the point (2,3).
If dx = 0.03, dy = 0.04, find df. What does df represent?

18) (16.2 #23)

Suppose $\vec{r} = \cos t\vec{i} + \sin t\vec{j} + 2t\vec{k}$ represents the position of a particle on a helix, where z is the height of the particle above the ground.

a) Is the particle ever moving downwards? When?

b) When does the particle reach a point 10 units above the ground?

c) What is the velocity of the particle when it is 10 units above the ground?

d) Suppose the particle leaves the helix and moves along the tangent line to the spiral at this point. What is the equation of the tangent line?

19) (16.2 #22)

Consider the motion of the particle given by the parametric equations

$$x = t^3 - 3t, \quad y = t^2 - 2t$$

where the y-axis is vertical and the x-axis is horizontal.

a) Does the particle ever come to a stop? If so, when and where?

b) Is the particle ever moving straight up or down?
If so, when and where?

c) Is the particle ever moving straight horizontally right or left?
If so, when and where?

20. 17.2 Example 2

The velocity of a flow at (x,y) is $F(x,y) = \vec{i} + x\vec{j}$

a. Sketch the velocity field and show the path of motion of an object in the flow that is at the point (-2,2) at time t = 0.

b. Find the system of differential equations associated with the field and show that

$$x = t - 2$$
$$y = 0.5t^2 - 2t + 2$$

satisfies the system.

21. 18.1 # 6

Say whether the vector field $\vec{F}(x,y) = x\vec{i} + y\vec{j}$ has positive, negative, or zero circulation around the curve shown in Figure 18.18 below. Explain.

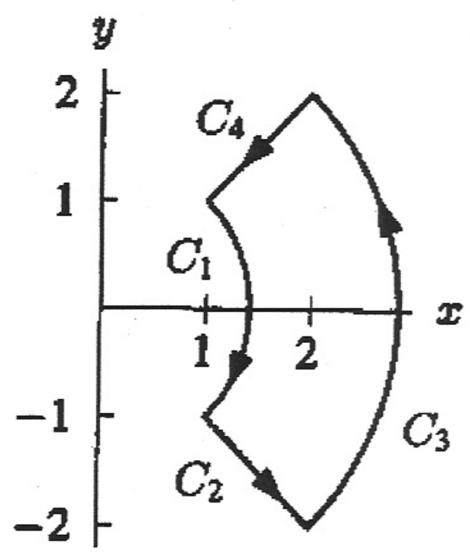


Figure 18.18: **The closed curve** $C = C_1 + C_2 + C_3 + C_4$

23. 18.2 #5

a. Compute the line integral of the vector field $\vec{F}(x,y) = e^x\vec{i} + e^y\vec{j}$ along the path C which is part of the ellipse $x^2 + 4y^2 = 4$ joining the point (0,1) to (2,0) in the clockwise direction by parameterizing the curve.

b. Find a potential function for $\vec{F}(x,y)$ and use the Fundamental Theorem of Calculus for Line Integrals to find the line integral from part a. (Your answers for parts a and b should be the same.)

24. 18.3 #4

Let $\vec{F}(x,y) = (x^2 - y^2)\vec{i} - 2xy\vec{j}$. Show that this is a gradient vector field by finding the potential function f.

25. 18.4 #11

a. Sketch $\vec{F}(x,y) = y\vec{i}$
and hence decide the sign of the circulation of $\vec{F}(x,y) = y\vec{i}$ around the unit circle centered at the origin and traversed counterclockwise.

b. Use Green's Theorem to compute the circulation in part a exactly.

26. Find an equation of the tangent plane to the function sin(2x+y) at the point (0,0,0)

27. a. Explain why the function $f(x,y) = x^2 + y^2$ has a global maximum and a global minimum on the region $x^4 + y^4 \leq 2$.

b. For the function $f(x,y) = x^2 + y^2$ find the global maximum and global minimum on the region $x^4 + y^4 \leq 2$.

28. a. Sketch and describe the region of integration for the triple integral

$$\int_0^4 \int_{\frac{3x}{2}}^{6} \int_0^{8-\frac{y}{3}-\frac{x}{2}} xdzdydx.$$

b. Find the average distance from the yz-plane for the points in the region you sketched in part a.

29. The temperature at any point in the plane is given by the function

$$T(x,y) = \frac{100}{x^2 + y^2}$$

a) Find the direction of the greatest increase in temperature at the point (3,2).
b) Find the rate of increase in temperature at the point (3,2) in the direction of greatest increase.
c) If a bug moves on the plane at the rate of 5 units per second starting at the point (3,2) and moving in the direction of greatest increase, how fast is the temperature increasing? Include units in your answer.
d) In what direction should the bug move so that there is no change in temperature?

30. Refer to the following table of fuel efficiency, f, in miles per gallon, for a subcompact car as a function of speed, s, in miles per hour (mph), and altitude, a, in feet above sea level.

Fuel Efficiency in mpg

		a feet			
		0	2000	4000	6000
s	35	35.3	35.0	34.6	33.8
mph	40	36.7	36.2	35.4	34.3
	45	37.2	36.8	35.7	34.5
	50	35.9	35.8	35.2	34.1
	55	34.1	33.9	33.5	33.2

a. If the car is driven at 45 miles per hour at 4000 feet above sea level, how many miles per gallon will it get?

a. ____________

b. Explain what f(s,4000) represents in terms of the fuel efficiency of the car.

c. Sketch a graph of fuel efficiency as a function of altitude on the grid below for each speed given in the table. Describe the effect of altitude on fuel efficiency.

Calculus III Final Exam Review Solutions

All review problems are from your text. Section numbers are given. Solutions provided are copied from the text or its solutions manual, compiled here for your convenience.

1. 13.1 #13

13. (a) Estimate $\partial P/\partial r$ and $\partial P/\partial L$ by using difference quotients and reading values of P from the graph:

$$\frac{\partial P}{\partial r}(8,4000) \approx \frac{P(16,4000)-P(8,4000)}{16-8} = \frac{100-80}{8} = 2.5,$$

and

$$\frac{\partial P}{\partial L} \approx \frac{P(8,5000)-P(8,4000)}{5000-4000} = \frac{100-80}{1000} = 0.02.$$

$P_r(8,4000) \approx 2.5$ means that at an interest rate of 8% and a loan amount of \$4000 the monthly payment increases by approximately \$2.50 for every one percent increase of the interest rate. $P_L(8,4000) \approx 0.02$ means the monthly payment increases by approximately \$0.02 for every \$1 increase in the loan amount at an 8% rate and a loan amount of \$4000.

(b) Using difference quotients and reading from the graph

$$\frac{\partial P}{\partial r}(8,6000) \approx \frac{P(14,6000)-P(8,6000)}{14-8} = \frac{140-120}{6} = 3.33,$$

and

$$\frac{\partial P}{\partial L}(8,6000) \approx \frac{P(8,7000)-P(8,6000)}{7000-6000} = \frac{140-120}{1000} = 0.02.$$

Again, we see that the monthly payment increases with increases in interest rate and loan amount. The interest rate is $r = 8\%$ as in part (a), but here the loan amount is $L = \$6000$. Since $P_L(8,4000) \approx P_L(8,6000)$, the increase in monthly payment per unit increase in loan amount remains the same as in part a). However, in this case, the effect of the interest rate is different: here the monthly payment increases by approximately \$3.33 for every one percent increase of interest rate at $r = 8\%$ and loan amount of \$6000.

(c)

$$\frac{\partial P}{\partial r}(13,7000) \approx \frac{P(19,7000)-P(13,7000)}{19-13} = \frac{180-160}{6} = 3.33,$$

and

$$\frac{\partial P}{\partial L}(13,7000) \approx \frac{P(13,8000)-P(13,7000)}{8000-7000} = \frac{180-160}{1000} = 0.02.$$

The figures show that the rates of change of the monthly payment with respect to the interest rate and loan amount are roughly the same for $(r,L) = (8,6000)$ and $(r,L) = (13,7000)$.

2. 13.2 Example 2

Example 2 Compute the partial derivatives with respect to x and with respect to y for the following functions.
(a) $f(x, y) = y^2 e^{3x}$ (b) $z = (3xy + 2x)^5$ (c) $g(x, y) = e^{x+3y} \sin(xy)$

Solution (a) This is the product of a function of x (namely e^{3x}) and a function of y (namely y^2). When we differentiate with respect to x, we think of the function of y as a constant, and vice versa. Thus,

$$f_x(x, y) = y^2 \frac{\partial}{\partial x}(e^{3x}) = 3y^2 e^{3x},$$

$$f_y(x, y) = e^{3x} \frac{\partial}{\partial y}(y^2) = 2ye^{3x}.$$

(b) Here we use the chain rule:

$$\frac{\partial z}{\partial x} = 5(3xy + 2x)^4 \frac{\partial}{\partial x}(3xy + 2x) = 5(3xy + 2x)^4(3y + 2).$$

$$\frac{\partial z}{\partial y} = 5(3xy + 2x)^4 \frac{\partial}{\partial y}(3xy + 2x) = 5(3xy + 2x)^4 3x = 15x(3xy + 2x)^4.$$

(c) Since each function in the product is a function of both x and y, we need to use the product rule for each partial derivative:

$$g_x(x, y) = \left(\frac{\partial}{\partial x}(e^{x+3y})\right) \sin(xy) + e^{x+3y} \frac{\partial}{\partial x}(\sin(xy)) = e^{x+3y} \sin(xy) + e^{x+3y} y \cos(xy).$$

$$g_y(x, y) = \left(\frac{\partial}{\partial y}(e^{x+3y})\right) \sin(xy) + e^{x+3y} \frac{\partial}{\partial y}(\sin(xy)) = 3e^{x+3y} \sin(xy) + e^{x+3y} x \cos(xy).$$

For functions of three or more variables, we find partial derivatives by the same method: Differentiate with respect to one variable, regarding the other variables as constants. For a function $f(x, y, z)$, the partial derivative $f_x(a, b, c)$ gives the rate of change of f with respect to x along the line $y = b$, $z = c$.

3. a. 13.4 #16

16. Since the partial derivatives are

$$z_x = \frac{e^y(x+y) - xe^y}{(x+y)^2} = \frac{ye^y}{(x+y)^2}$$
$$z_y = \frac{xe^y(x+y) - xe^y}{(x+y)^2} = \frac{e^y(x^2+xy-x)}{(x+y)^2}$$

we have

$$\nabla z = \frac{ye^y}{(x+y)^2}\vec{i} + \frac{e^y(x^2+xy-x)}{(x+y)^2}\vec{j}$$

b. 13.4 example 7

Example 7 Use the gradient to find the directional derivative of $f(x,y) = x + e^y$ at the point $(1,1)$ in the direction of the vectors $\vec{i} - \vec{j}, \vec{i} + 2\vec{j}, \vec{i} + 3\vec{j}$.

Solution In Example 5 we found

$$\text{grad}\, f(1,1) = \vec{i} + e\vec{j}.$$

A unit vector in the direction of $\vec{i} - \vec{j}$ is $\vec{s} = (\vec{i} - \vec{j})/\sqrt{2}$, so

$$f_{\vec{s}}(1,1) = \text{grad}\, f(1,1) \cdot \vec{s} = (\vec{i} + e\vec{j}) \cdot \left(\frac{\vec{i} - \vec{j}}{\sqrt{2}}\right) = \frac{1-e}{\sqrt{2}} \approx -1.215.$$

A unit vector in the direction of $\vec{i} + 2\vec{j}$ is $\vec{v} = (\vec{i} + 2\vec{j})/\sqrt{5}$, so

$$f_{\vec{v}}(1,1) = \text{grad}\, f(1,1) \cdot \vec{v} = (\vec{i} + e\vec{j}) \cdot \left(\frac{\vec{i} + 2\vec{j}}{\sqrt{5}}\right) = \frac{1+2e}{\sqrt{5}} \approx 2.879.$$

A unit vector in the direction of $\vec{i} + 3\vec{j}$ is $\vec{w} = (\vec{i} + 3\vec{j})/\sqrt{10}$, so

$$f_{\vec{w}}(1,1) = \text{grad}\, f(1,1) \cdot \vec{w} = (\vec{i} + e\vec{j}) \cdot \left(\frac{\vec{i} + 3\vec{j}}{\sqrt{10}}\right) = \frac{1+3e}{\sqrt{10}} \approx 2.895.$$

Now look back at the answers and compare with the value of $\|\text{grad}\, f\| = \sqrt{1+e^2} \approx 2.896$. One answer is not close to this value; the other two, $f_{\vec{v}} = 2.879$ and $f_{\vec{w}} = 2.895$, are close but slightly smaller than $\|\text{grad}\, f\|$. Since $\|\text{grad}\, f\|$ is the maximum rate of change of f at the point, we would expect for *any* unit vector $\vec{u}$:

$$f_{\vec{u}}(1,1) \le \|\text{grad}\, f\|.$$

with equality when $\vec{u}$ is in the direction of grad f. Since $e \approx 2.718$, the vectors $\vec{i} + 2\vec{j}$ and $\vec{i} + 3\vec{j}$ both point roughly, but not exactly, in the direction of the gradient vector grad $f(1,1) = \vec{i} + e\vec{j}$. Thus, the values of $f_{\vec{v}}$ and $f_{\vec{w}}$ are both close to the value of $\|\text{grad}\, f\|$. The direction of the vector $\vec{i} - \vec{j}$ is not close to the direction of grad f and the value of $f_{\vec{s}}$ is not close to the value of $\|\text{grad}\, f\|$.

c. 13.4 #18

18. Since the partial derivatives are

$$\frac{\partial f}{\partial x} = \frac{15}{2}x^4 - 0 = \frac{15}{2}x^4$$
$$\frac{\partial f}{\partial y} = 0 - \frac{24}{7}y^5 = -\frac{24}{7}y^5$$

we have

$$\operatorname{grad} f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} = \left(\frac{15}{2}x^4\right)\vec{i} - \left(\frac{24}{7}y^5\right)\vec{j}.$$

d. 13.4 #13

13. Since the partial derivatives are

$$z_x = e^y, \quad \text{and} \, z_y = xe^y + e^y + ye^y,$$

we have

$$\nabla z = e^y\vec{i} + e^y(1 + x + y)\vec{j}.$$

4. 15.2 #12

$$\int_1^4 \int_{\sqrt{y}}^{y} x^2 y^3 \, dx dy = \int_1^4 y^3 \frac{x^3}{3}\Big|_{\sqrt{y}}^{y} dy$$
$$= \frac{1}{3}\int_1^4 (y^6 - y^{9/2})\, dy$$
$$= \frac{1}{3}\left(\frac{y^7}{7} - \frac{y^{11/2}}{11/2}\right)\Big|_1^4$$
$$= \frac{1}{3}\left[\left(\frac{4^7}{7} - \frac{4^{11/2}\times 2}{11}\right) - \left(\frac{1}{7} - \frac{2}{11}\right)\right] \approx 656.082$$

See Figure 15.10.

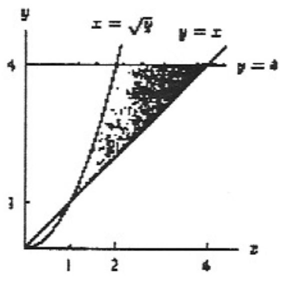


Figure 15.10

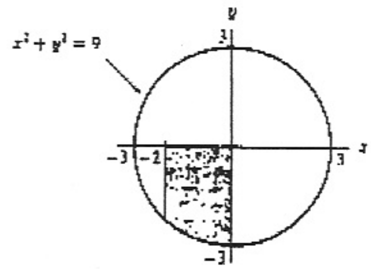


Figure 15.11

5 a. 16.1#15

15. The parameterization $x = 2\cos t$, $y = 2\sin t$, $0 \le t \le 2\pi$, is a circle of radius 2 traced out counterclockwise starting at the point $(2,0)$. To start at $(-2,0)$, put a negative in front of the first coordinate

$$x = -2\cos t \quad y = 2\sin t, \qquad 0 \le t \le 2\pi.$$

5b. Example 1 in 16.2

Example 1 Give a parameterization for the circle of radius $\frac{1}{2}$ centered at the point $(-1, 2)$.

Solution The circle of radius 1 centered at the origin is parameterized by the vector-valued function

$$\vec{r}_1(t) = \cos t\vec{i} + \sin t\vec{j}, \quad 0 \le t \le 2\pi.$$

The point $(-1, 2)$ has the position vector $\vec{r}_0 = -\vec{i} + 2\vec{j}$. The position vector, $\vec{r}(t)$, of a point on the circle of radius $\frac{1}{2}$ centered at $(-1, 2)$ is found by adding $\frac{1}{2}\vec{r}_1$ to $\vec{r}_0$. (See Figures 16.21 and 16.22.) Thus,

$$\vec{r}(t) = \vec{r}_0 + \tfrac{1}{2}\vec{r}_1(t) = -\vec{i} + 2\vec{j} + \tfrac{1}{2}(\cos t\vec{i} + \sin t\vec{j}) = (-1 + \tfrac{1}{2}\cos t)\vec{i} + (2 + \tfrac{1}{2}\sin t)\vec{j},$$

or, equivalently,

$$x = -1 + \tfrac{1}{2}\cos t, \quad y = 2 + \tfrac{1}{2}\sin t, \qquad 0 \le t \le 2\pi.$$

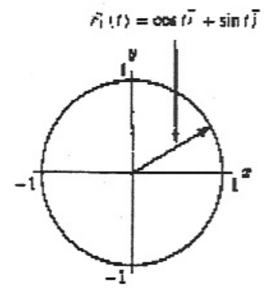


Figure 16.21: The circle $x^2 + y^2 = 1$ parameterized by $\vec{r}_1(t) = \cos t\vec{i} + \sin t\vec{j}$

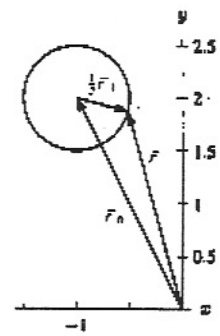


Figure 16.22: The circle of radius $\frac{1}{2}$ and center $(-1, 2)$ parameterized by $\vec{r}(t) = \vec{r}_0 + \frac{1}{2}\vec{r}_1(t)$

6. 15.5 a. #3 b. #4

3.

$$\int_0^{2\pi} \int_0^{\sqrt{2}} f\, r dr\, d\theta$$

4.

$$\int_0^{\pi/2} \int_0^{1/2} f\, r dr\, d\theta$$

7. a. 12.4 Example 3

Example 3 Find the cross product of $\vec{v} = 2\vec{i} + \vec{j} - 2\vec{k}$ and $\vec{w} = 3\vec{i} + \vec{k}$ and check that the cross product is perpendicular to both $\vec{v}$ and $\vec{w}$.

Solution Writing $\vec{v} \times \vec{w}$ as a determinant and expanding it into three two-by-two determinants, we have

$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & -2 \\ 3 & 0 & 1 \end{vmatrix} = \vec{i}\begin{vmatrix} 1 & -2 \\ 0 & 1 \end{vmatrix} - \vec{j}\begin{vmatrix} 2 & -2 \\ 3 & 1 \end{vmatrix} + \vec{k}\begin{vmatrix} 2 & 1 \\ 3 & 0 \end{vmatrix}$$

$$= \vec{i}\,(1(1) - 0(-2)) - \vec{j}\,(2(1) - 3(-2)) + \vec{k}\,(2(0) - 3(1))$$

$$= \vec{i} - 8\vec{j} - 3\vec{k}.$$

To check that $\vec{v} \times \vec{w}$ is perpendicular to $\vec{v}$, we compute the dot product:

$$\vec{v} \cdot (\vec{v} \times \vec{w}) = (2\vec{i} + \vec{j} - 2\vec{k}) \cdot (\vec{i} - 8\vec{j} - 3\vec{k}) = 2 - 8 + 6 = 0.$$

Similarly,

$$\vec{w} \cdot (\vec{v} \times \vec{w}) = (3\vec{i} + 0\vec{j} + \vec{k}) \cdot (\vec{i} - 8\vec{j} - 3\vec{k}) = 3 + 0 - 3 = 0.$$

Thus, $\vec{v} \times \vec{w}$ is perpendicular to both $\vec{v}$ and $\vec{w}$.

b. 12.4 #6

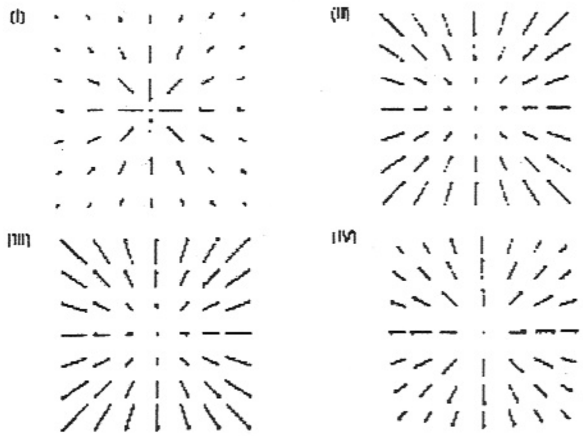
6. $\vec{a} = 2\vec{i} - 3\vec{j} + \vec{k}$, and $\vec{b} = \vec{i} + 2\vec{j} - \vec{k}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -3 & 1 \\ 1 & 2 & -1 \end{vmatrix} = \vec{i} + 3\vec{j} + 7\vec{k}$$

8. 17.1 #1 d

Each vector field in Figures (I)–(IV) represents the force on a particle at different points in space as a result of another particle at the origin. Match up the vector fields with the descriptions below.

(a) A repulsive force whose magnitude decreases as distance increases, such as between electric charges of the same sign.
(b) A repulsive force whose magnitude increases as distance increases.
(c) An attractive force whose magnitude decreases as distance increases, such as gravity.
(d) An attractive force whose magnitude increases as distance increases.



Notice that for a repulsive force, the vectors point outward, away from the particle at the origin, for an attractive force, the vectors point toward the particle. So we can match up the vector field with the description as follows:

(a) IV
(b) III
(c) I
(d) II

9. 18.1 a. #3 b. #4

3. Positive, because the vector field points in the same direction as the path.
4. Zero, because, by symmetry, the positive integral along the left half of the path cancels the negative integral along the right half.

10. Local Max in center of encircling contours that are increasing at $\left(\frac{\pi}{2}, 2\pi\right)$ and $\left(\frac{\pi}{2}, 0\right)$

Local Min in center of encircling contours that are decreasing at $\left(\frac{\pi}{2}, \pi\right)$

11. 14.2 #10

10. (a) The revenue $R = p_1q_1 + p_2q_2$. Profit $= P = R - C = p_1q_1 + p_2q_2 - 2q_1^2 - 2q_2^2 - 10$.

$$\frac{\partial P}{\partial q_1} = p_1 - 4q_1 = 0 \quad \text{gives } q_1 = \frac{p_1}{4}$$
$$\frac{\partial P}{\partial q_2} = p_2 - 4q_2 = 0 \quad \text{gives } q_2 = \frac{p_2}{4}$$

Since $\frac{\partial^2 P}{\partial q_1^2} = -4$, $\frac{\partial^2 P}{\partial q_2^2} = -4$ and $\frac{\partial^2 P}{\partial q_1 \partial q_2} = 0$, at $(p_1/4, p_2/4)$ we have that the discriminant, $D = (-4)(-4) > 0$ and $\frac{\partial^2 P}{\partial q_1^2} < 0$, thus P has a local maximum value at $(q_1, q_2) = (p_1/4, p_2/4)$. Since P is quadratic in q_1 and q_2, this is a global maximum. So $P = \frac{p_1^2}{4} + \frac{p_2^2}{4} - 2\frac{p_1^2}{16} - 2\frac{p_2^2}{16} - 10 = \frac{p_1^2}{8} + \frac{p_2^2}{8} - 10$ is the maximum profit.

(b) The rate of change of the maximum profit as p_1 increases is

$$\frac{\partial}{\partial p_1}(\max\ P) = \frac{2p_1}{8} = \frac{p_1}{4}.$$

12. 14.3 #26

26. (a) Let c be the cost of producing the product. Then $c = 10W + 20K = 3000$. At optimum production,

$$\nabla q = \lambda \nabla c.$$

$$\nabla q = \left(\tfrac{9}{2}W^{-\frac{1}{4}}K^{\frac{1}{4}}\right)\vec{i} + \left(\tfrac{3}{2}W^{\frac{3}{4}}K^{-\frac{3}{4}}\right)\vec{j}, \text{ and } \nabla c = 10\vec{i} + 20\vec{j}. \text{ Equating we get}$$

$$\tfrac{9}{2}W^{-\frac{1}{4}}K^{\frac{1}{4}} = \lambda 10, \quad \text{and} \quad \tfrac{3}{2}W^{\frac{3}{4}}K^{-\frac{3}{4}} = \lambda 20.$$

Dividing yields $K = \frac{1}{6}W$, so substituting into c gives

$$10W + 20\left(\frac{1}{6}W\right) = \frac{40}{3}W = 3000.$$

Thus $W = 225$ and $K = 37.5$. Substituting both answers to find λ gives

$$\lambda = \frac{\frac{9}{2}(225)^{-\frac{1}{4}}(37.5)^{\frac{1}{4}}}{10} = 0.2875.$$

We also find the optimum quantity produced, $q = 6(225)^{\frac{3}{4}}(37.5)^{\frac{1}{4}} = 862.57$.

(b) At the optimum values found above, marginal productivity of labor is given by

$$\left.\frac{\partial q}{\partial W}\right|_{(225,37.5)} = \left.\frac{9}{2}W^{-\frac{1}{4}}K^{\frac{1}{4}}\right|_{(225,37.5)} = 2.875,$$

and marginal productivity of capital is given by

$$\left.\frac{\partial q}{\partial K}\right|_{(225,37.5)} = \left.\frac{3}{2}W^{\frac{3}{4}}K^{-\frac{3}{4}}\right|_{(225,37.5)} = 5.750.$$

The ratio of marginal productivity of labor to that of capital is

$$\frac{\frac{\partial q}{\partial W}}{\frac{\partial q}{\partial K}} = \frac{1}{2} = \frac{10}{20} = \frac{\text{cost of a unit of L}}{\text{cost of a unit of K}}.$$

(c) When the budget is increased by one dollar, we substitute the relation $K_1 = \frac{1}{6}W_1$ into $10W_1 + 20K_1 = 3001$ which gives $10W_1 + 20(\frac{1}{6}W_1) = \frac{40}{3}W_1 = 3001$. Solving yields $W_1 = 225.075$ and $K_1 = 37.513$, so $q_1 = 862.86 = q + 0.29$. Thus production has increased by 0.29 $\approx \lambda$, the Lagrange Multiplier.

26. Find an equation of the tangent plane to the function sin(2x+y)
at the point (0,0,0)

$$z = f_x(0,0)x + f_y(0,0)y + f(0,0) = 2x + y$$

26. <u>z = 2x + y</u>

27. a. Explain why the function $f(x,y) = x^2 + y^2$
has a global maximum and a global minimum on the region $x^4 + y^4 \le 2$.

f is a continuous function and the region described is closed and bounded.

b. For the function $f(x,y) = x^2 + y^2$ find the global maximum and global minimum on the region $x^4 + y^4 \le 2$.

1. Find the CP's inside the region.
2. Find local extrema on the boundary using LaGrange Multipliers
3. The max value will be found at the largest f at each of the points from parts 1 and 2 and the min value at the smallest of these points.

1. $f_x = 2x, f_y = 2y \Rightarrow 2x = 0, 2y = 0 \Rightarrow CP = (0,0)$ which is in the region.

2.
$$\begin{cases} g_x = 4x^3, g_y = 4y^3 \\ \nabla f = \lambda \nabla g \\ 2x = \lambda(4x^3) \\ 2y = \lambda(4y^3) \\ x^4 + y^4 = 2 \\ \lambda = \dfrac{2x}{4x^3} = \dfrac{2y}{4y^3} \Rightarrow \dfrac{1}{2x^2} = \dfrac{1}{2y^2} \Rightarrow x = \pm y, x \ne 0, y \ne 0 \\ x^4 + y^4 = 2 \Rightarrow 2x^4 = 2 \Rightarrow x = \pm 1, y = \pm 1 \\ x = 0 \Rightarrow y = \pm\sqrt[4]{2}, y = 0 \Rightarrow x = \pm\sqrt[4]{2} \end{cases}$$

Possible boundary extrema are: $(0,\pm\sqrt[4]{2}), (\pm\sqrt[4]{2},0), (1,1), (1,-1), (-1,1), (-1,-1)$

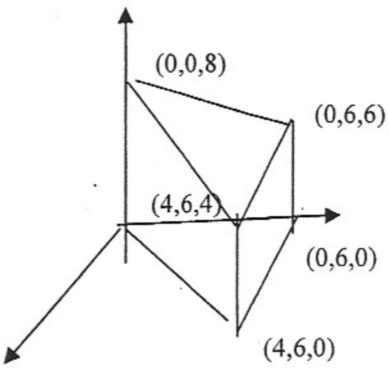
3.
$$\begin{cases} f(0,0) = 0 \\ f(0,\pm\sqrt[4]{2}) = f(\pm\sqrt[4]{2},0) = \sqrt{2} \\ f(\pm 1,\pm 1) = 2 \end{cases}$$

Max. <u>2</u>
Min <u>0</u>

28. a. Sketch and describe the region of integration for the triple integral

$$\int_0^4 \int_{\frac{3x}{2}}^6 \int_0^{8-\frac{y}{3}-\frac{x}{2}} x\,dz\,dy\,dx .$$

Top plane z = 8-y/3-x/2
Bottom plane z =0
over region in xy plane
line y = 3x/2 to y =6
from x = 0 to x = 4
at (0,0) z = 8 on top plane
at (0,6) z = 6
at (4,6) z = 4
back boundary of region is trapezoid in yz plane
Front boundary is trapezoid in plane y=3x/2
Right boundary is trapezoid in plane y = 6



b. Find the average distance from the yz-plane for the points in the region you sketched in part a. Circle your final answer.

x = distance from the yz plane so just evaluate the integral and divide by the volume of the region

$$V = \int_0^4 \int_{\frac{3x}{2}}^6 \int_0^{8-\frac{y}{3}-\frac{x}{2}} 1\,dz\,dy\,dx = \int_0^4 \int_{\frac{3x}{2}}^6 8 - y/3 - x/2\,dy\,dx = \int_0^4 8y - y^2/6 - xy/2\Big|_{3x/2}^{6} dx$$

$$= \int_0^4 (48-6-3x)-(12x-\frac{3x^2}{8}-\frac{3x^2}{4})dx = \int_0^4 (42-15x+\frac{9x^2}{8})dx = 42x-\frac{15x^2}{2}+\frac{9x^3}{24}\Big|_0^4 = 72$$

$$\int_0^4 \int_{\frac{3x}{2}}^6 \int_0^{8-\frac{y}{3}-\frac{x}{2}} x\,dz\,dy\,dx = \int_0^4 \int_{\frac{3x}{2}}^6 zx \Big|_0^{8-\frac{y}{3}-\frac{x}{2}} dy\,dx = \int_0^4 \int_{\frac{3x}{2}}^6 8x - \frac{xy}{3} - \frac{x^2}{2} dy\,dx$$

$$= \int_0^4 8xy - \frac{xy^2}{6} - \frac{x^2 y}{2}\Big|_{3x/2}^{6} dx = \int_0^4 (48x-6x-3x^3)-(12x^2-\frac{3x^3}{8}-\frac{3x^3}{4})dx$$

$$= \int_0^4 (42x-15x^2+\frac{9x^3}{8})dx = 21x^2-5x^3+\frac{9x^4}{32}\Big|_0^4 = 88 \qquad \frac{1}{V}\int_0^4 \int_{\frac{3x}{2}}^6 \int_0^{8-\frac{y}{3}-\frac{x}{2}} x\,dz\,dy\,dx = \frac{88}{72} = \frac{11}{9}$$

28. ______11/9_____

29. The temperature at any point in the plane is given by the function

$$T(x,y) = \frac{100}{x^2 + y^2}$$

a) Find the direction of the greatest increase in temperature at the point (3,2).
b) Find the rate of increase in temperature at the point (3,2) in the direction of greatest increase.
c) If a bug moves on the plane at the rate of 5 units per second starting at the point (3,2) and moving in the direction of greatest increase, how fast is the temperature increasing? Include units in your answer.
d) In what direction should the bug move so that there is no change in temperature?

a) The direction of greatest increase will be the direction of the gradient.

$$\nabla T = T_x i + T_y j = \frac{-200x}{(x^2+y^2)^2} i + \frac{-200y}{(x^2+y^2)^2} j$$

$$\nabla T(3,2) = \frac{-600}{169} i - \frac{400}{169} j$$

b) The magnitude of greatest increase is

$$\|\nabla T(3,2)\| = \frac{\sqrt{(-600)^2 + (-400)^2}}{169} = \frac{200\sqrt{13}}{169}$$

c) $$\frac{5units}{\sec} \cdot \frac{200\sqrt{13}}{169} \frac{\deg rees}{unit} = \frac{1000\sqrt{13}}{169} \deg rees/\sec.$$

e) There is no change in any direction perpendicular to the gradient, such as $\frac{-400}{169} i + \frac{600}{169} j$

30. Refer to the following table of fuel efficiency, f, in miles per gallon, for a subcompact car as a function of speed, s, in miles per hour (mph), and altitude, a, in feet above sea level.

Fuel Efficiency in mpg

Fuel Efficiency in mpg

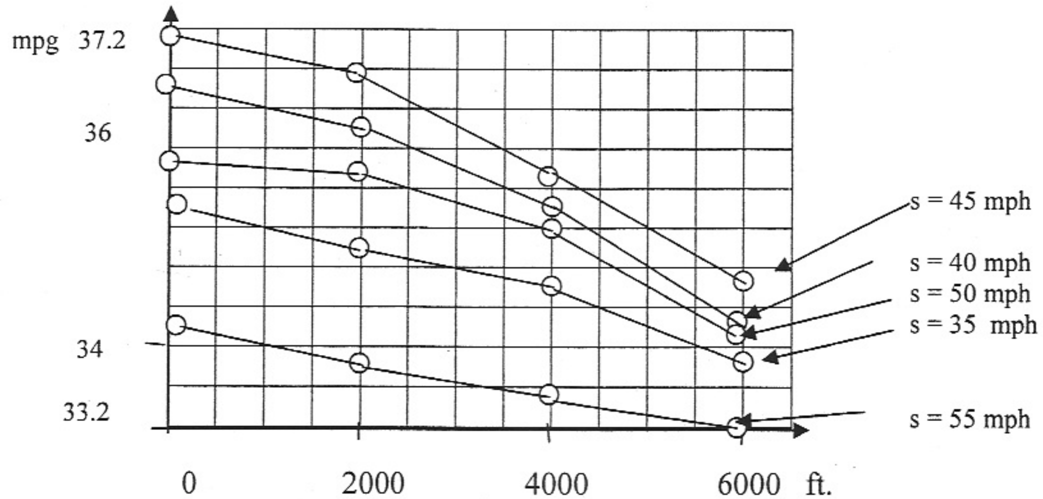
		a feet			
		0	2000	4000	6000
s mph	35	35.3	35.0	34.6	33.8
	40	36.7	36.2	35.4	34.3
	45	37.2	36.8	35.7	34.5
	50	35.9	35.8	35.2	34.1
	55	34.1	33.9	33.5	33.2

a. If the car is driven at 45 miles per hour at 4000 feet above sea level, how many miles per gallon will it get?

a. ____35.7 mpg____

b. Explain what f(s,4000) represents in terms of the fuel efficiency of the car.
It represents fuel efficiency as a function of speed at an altitude of 4000 feet.

c. Sketch a graph of fuel efficiency as a function of altitude on the grid below for each speed given in the table. Describe the effect of altitude on fuel efficiency.



As altitude increases, fuel efficiency decreases, with the biggest drop, 2.7 mpg, at 45 mph. Except at 55 mph, the decrease in fuel efficiency is more pronounced as altitude increases. At 6000 feet, differences in fuel efficiency due to speed are much less.