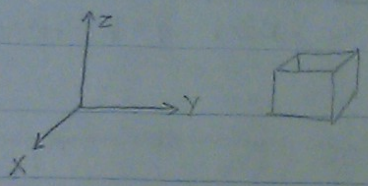
13.9 Optimization in 3 dimensions

#10) The material for constructing the base of an open box costs 1.5 times as much per unit area as the material for constructing the sides. For a fixed amount of money C, find the dimensions for a box, maximizing volume.



area of the base: xy

base cost: $(1.5)xy$

area of all sides: $2xz+2yz$

C_0 = total cost $= (1.5)xy + 2xz + 2yz$

(C_0: fixed cost)

volume $= xyz$

(we want volume as a function of only x and y)

(solve for z in terms of x and y and sub. into volume)

$$C_0 - 1.5xy = 2xz + 2yz$$

$$\frac{C_0 - 1.5xy}{(2x+2y)} = z \longrightarrow V = xy\left(\frac{C_0 - 1.5xy}{2x+2y}\right)$$

$$V(x,y) = \frac{C_0 xy - 1.5x^2y^2}{2x+2y}$$

(quotient rule) ↘

$$V_x(x,y) = \frac{(2x+2y)(C_0 y - 3xy^2) - (C_0 xy - 1.5x^2y^2)(2)}{(2x+2y)^2}$$

$$= \frac{(2C_0 xy - 6x^2y^2) + (2C_0 y^2 - 6xy^3) - 2C_0 xy + 3x^2y^2}{4(x+y)^2}$$

$$= \frac{-3x^2y^2 + 2C_0 y^2 - 6xy^3}{4(x+y)^2}$$

(continued...)

$$V_y(x,y) = \frac{(2x+2y)(C_0 x - 3x^2 y) - (C_0 xy - 1.5x^2y^2)(2)}{(2x+2y)^2}$$

(lo, d(hi), hi, d(lo))

$$= \frac{2C_0x^2 - 6x^3y + 2C_0xy - 6x^2y^2 - 2C_0xy + 3x^2y^2}{4(x+y)^2}$$

$$= \frac{-3x^2y^2 + 2C_0x^2 - 6x^3y}{4(x+y)^2}$$

If $V_x = V_y = 0$ and we see symmetry about $Y = X$, then we can let $Y = X$

(letting $Y=X$) $\rightarrow$ $V_x(x,x) = \frac{-3x^2x^2 + 2C_0x^2 - 6xx^3}{4(x+x)^2}$

$$= \frac{-3x^4 + 2C_0x^2 - 6x^4}{16x^2} = \frac{2C_0x^2 - 9x^4}{16x^2}$$

do not cancel x^2

$V_x(x,x) = 0$

$0 = -9x^4 + 2C_0x^2$

$0 = -x^2(9x^2 - 2C_0)$

$-x^2 = 0$, $x = 0$ DNE

$9x^2 - 2C_0 = 0$

$9x^2 = 2C_0$

$x^2 = \frac{2C_0}{9} \Rightarrow \boxed{x = \frac{\sqrt{2C_0}}{3}}$

$V_x(x,x)$ DNE

$16x^2 \neq 0$

$x \neq 0$

since $x = y$, then $\boxed{y = \frac{\sqrt{2C_0}}{3}}$

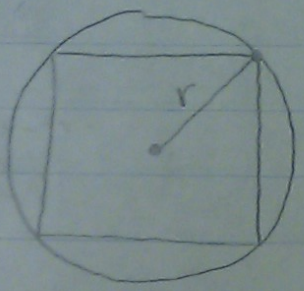
$$z = \frac{C_0 - 1.5xy}{2x+2y} = \frac{C_0 - \frac{3}{2}\left(\frac{\sqrt{2C_0}}{3}\right)\left(\frac{\sqrt{2C_0}}{3}\right)}{2\left(\frac{\sqrt{2C_0}}{3} + \frac{\sqrt{2C_0}}{3}\right)}$$

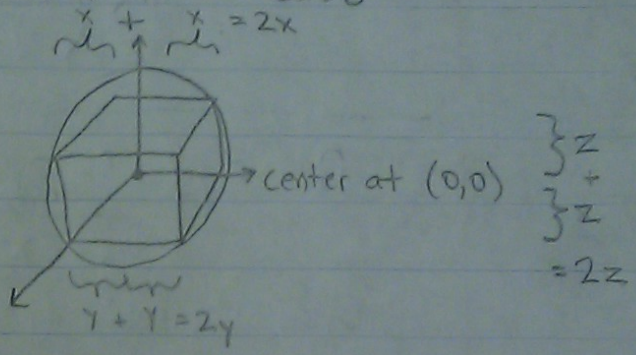
(continued...)

$$= \frac{\left(C_0 - \frac{2C_0}{6}\right)}{\left(\frac{4\sqrt{2C_0}}{3}\right)} = \frac{2C_0}{3} \cdot \frac{3}{4\sqrt{2C_0}}$$

$$= \frac{C_0}{2\sqrt{2C_0}} \xrightarrow{\text{times } \frac{\sqrt{2C_0}}{\sqrt{2C_0}}} \frac{C_0\sqrt{2C_0}}{4C_0} = \boxed{\frac{\sqrt{2C_0}}{4} = z}$$

#12) Show that the rectangular box of maximum volume inscribed in a sphere of radius r is a cube





sphere: $x^2+y^2+z^2=r^2$

volume of box: $V=(2x)(2y)(2z)$

Solve the sphere equation for z in terms of x and y

$z^2 = r^2-x^2-y^2$ $\quad z = \pm\sqrt{r^2-x^2-y^2}$ $\quad z = \sqrt{r^2-x^2-y^2}$

this is taken care of in the 2 scalar in the box volume equation

Substitute:

$$V(x,y) = (2x)(2y)(2\sqrt{r^2-x^2-y^2}) = 8xy\sqrt{r^2-x^2-y^2}$$
$$= 8xy(r^2-x^2-y^2)^{1/2}$$

$$V_x(x,y) = 8y(r^2-x^2-y^2)^{1/2} + 8xy\left[\tfrac{1}{2}(r^2-x^2-y^2)^{-1/2}(-2x)\right]$$
$$= 8y(r^2-x^2-y^2)^{1/2} - \frac{8x^2y}{(r^2-x^2-y^2)^{1/2}} \xrightarrow{\text{common denominator}} = \frac{8y(r^2-x^2-y^2)-8x^2y}{(r^2-x^2-y^2)^{1/2}}$$

(continued...)

$$V_x(x,y) = \frac{8yr^2 - 8x^2y - 8y^3 - 8x^2y}{\sqrt{r^2-x^2-y^2}}$$

$$= 8y\left(\frac{r^2-x^2-y^2-x^2}{\sqrt{r^2-x^2-y^2}}\right) = 8y\left(\frac{r^2-2x^2-y^2}{\sqrt{r^2-x^2-y^2}}\right)$$

$$V_x(x,y) = (8x)(r^2-x^2-y^2)^{1/2} + 8xy\left[\frac{1}{2}(r^2-x^2-y^2)^{-1/2}(-2y)\right]$$

(same simplification as above...)

$$= 8x\left(\frac{r^2-2y^2-x^2}{\sqrt{r^2-x^2-y^2}}\right)$$

$$0 = 8y\left(\frac{r^2-2x^2-y^2}{\sqrt{r^2-x^2-y^2}}\right) \qquad 0 = 8x\left(\frac{r^2-2y^2-x^2}{\sqrt{r^2-x^2-y^2}}\right)$$

$8y=0$ and $(r^2-x^2-y^2)=0$ — $8x=0$

wrong: we'd have no box

no box — wrong

$r^2-2y^2-x^2=0$

$$r^2-2x^2-y^2 = r^2-2y^2-x^2$$

$$-2x^2-y^2 = -2y^2-x^2$$

$$y^2 = x^2$$

$$y = \pm\sqrt{x^2}$$

$$y = \pm|x|$$

(we need positive lengths so...)

$$\boxed{Y = X}$$

$$z = \sqrt{r^2-x^2-y^2}$$

$$z = \sqrt{r^2-x^2-(x)^2}$$

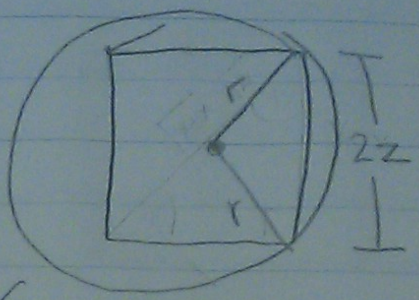
$$z = \sqrt{r^2-2x^2}$$

(continued...)

$r^2 = x^2 + y^2 + z^2$ $\quad x = y$ $\quad r^2 = 2x^2 + z^2$

$V = 8xy\sqrt{r^2 - x^2 - y^2}$ $\quad V = 8x^2\sqrt{r^2 - 2x^2}$

45°, 45°, 90° triangle



$x^2 + x^2 = d^2$

$2x^2 = d^2$

$\sqrt{2x^2} = d$

diameter $= \sqrt{2}x$

$2r^2 = 4z^2$