13.8 Extrema of functions of two variables

Extreme Value Theorem

Let f be a continuous function of x and y in the region R in the xy plane

- there is at least one max. and min. value for f on the region R

Relative Extrema

Let f be defined at the point (x_0, y_0) on R

① f has a minimum value at (x_0, y_0) of $f(x_0, y_0)$; if $f(x_0, y_0) \leq f(x, y)$ for all (x, y) in R

② f has a maximum value at (x_0, y_0) of $f(x_0, y_0)$; if $f(x_0, y_0) \geq f(x, y)$ for all (x, y) in R

Critical Point

Let f be defined on an open region R containing (x_0, y_0). (x_0, y_0) is a critical point if:

① both $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$

OR

② $f_x(x_0, y_0)$ OR $f_y(x_0, y_0)$ does not exist

* All extrema come from critical points

#2) Find extrema by examining:

$g(x,y) = 9-(x-3)^2-(y+2)^2$

If we let $(x,y)=(3,-2)$ then $g(3,-2)=9$

$g(x,y) < 9$ for all $(x,y) \neq (3,-2)$

thus we have a maximum value of 9 at $(3,-2)$

#6) Same, but complete the square

$f(x,y) = -x^2-y^2+4x+8y-11$

$f(x,y) = (-x^2+4x)+(-y^2+8y)-11$

$f(x,y) = -(x^2-4x)-(y^2-8y)-11$

$f(x,y) = -(x^2-4x+4)-(y^2-8y+16)-11+(4+16)$

$f(x,y) = -(x-2)^2-(y-4)^2+9$

$f(x,y) = 9-(x-2)^2-(y-4)^2$

$f(x,y) = (2,4)$ then

$f(2,4) = 9$

$f(x,y) < 9$ for all $(x,y) \neq 4$

we have a maximum of 9 at $(2,4)$

#8) $f(x,y) = -x^2-5y^2-10x-30y-62$

$f_x(x,y) = -2x+10$ $\quad$ $f_y(x,y) = -10y-30$

$0 = -2x+10$ $\quad$ $0 = -10y-30$

$x = 5$ $\quad$ $y = -3$

Critical point: $(5,-3)$

$f(5,-3) = -5^2-5(-3)^2+10(5)-30(-3)-62$

$= -25-45+50+90-62$

$= 8$

(continue)

max. or min.?

$$F(x,y) = (-x^2+10x)+(-5y^2-3y)-62$$
$$= -(x^2-10x)-(5y^2+3y)-62$$
$$= -(x^2-10x+25)-5(y^2+6y+9)-62+25+45$$
$$= -(x-5)^2-5(y+3)^2+8$$
$$= 8-k-5(y,3)^2$$

everything else < 8

Saddle points – points of inflection in space

- sort of a 3 dimensional change in concavity

picture ⟶ think Pringles

Theorem – Second Partials Test

Let f have a continuous second derivative on R containing (a,b) where $f_x(a,b) = f_y(a,b) = 0$

Let $d = f_{xx}(a,b)\,f_{yy}(a,b) - [f_{xy}(a,b)]^2$

determinate ↗

$$\rightarrow d = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - f_{xy}f_{yx}$$

since $f_x(a,b) = f_y(a,b) = 0$ these are the same

① If $d > 0$ and $f_{xx}(a,b) > 0$ then f has a min at (a,b) ("concave up")

② If $d > 0$ and $f_{xx}(a,b) < 0$ then f has a max. at (a,b) ("concave down")

③ If $d < 0$ then (a,b) is a saddle point

④ If $d = 0$ then **Inconclusive**

#46) $f(x,y) = x^3 + y^3 - 6x^2 + 9y^2 + 12x + 27y + 19$

Critical Points:

$f_x(x,y) = 3x^2 - 12x + 12$

$0 = 3x^2 - 12x + 12$

$0 = x^2 - 4x + 4$

$0 = (x-2)^2$

$x = 2$ → $(2,-3)$ ← $y = -3$

$f_y(x,y) = 3y^2 + 18y + 27$

$0 = 3y^2 + 18y + 27$

$0 = y^2 + 6y + 9$

$0 = (y+3)^2$

$f_{xx}(x,y) = 6x - 12$

$f_{xx}(2,-3) = 6(2) - 12$

$= 0$

$f_{yy}(x,y) = 6y + 18$

$f_{yy}(2,-3) = -16 + 18$

$= 0$

$f_{xy}(x,y) = 0$

$d = 0(0) - 0^2 = 0$ Inconclusive

#22) $g(x,y) = 120x + 120y - xy - x^2 - y^2$

$g_x(x,y) = 120 + 0 - y - 2x - 0$

$0 = 120 - y - 2x$

$y = 120 - 2x$

$y = 120 - 2(40)$

$y = 40$

$g_y(x,y) = 0 + 120 - x - 2y$

$0 = 120 - x - 2y$

$0 = 120 - x - 2(120 - 2x)$

$0 = 120 - x - 240 + 4x$

$0 = -120 + 3x$

$120 = 3x$

$x = 40$

$(40,40)$

$f_{xx}(x,y) = -2$ $f_{yy}(x,y) = -2$

$f_{xy}(x,y) = 0 - 1 - 0 = -1$

$d = (-2)(-2) - (-1)^2$

$= 3$

$d > 0$, $f_{xx} < 0$ so max. @ $(40,40)$

$g(40,40) = 4800$