13.7 Tangent Planes and Normal Lines

Let $z = f(x,y)$ so that $F(x,y,z) = 0$

The normal line (line of impact) to the surface intersects the surface at a point $P(x_0, y_0)$ at a right angle, and is written as:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

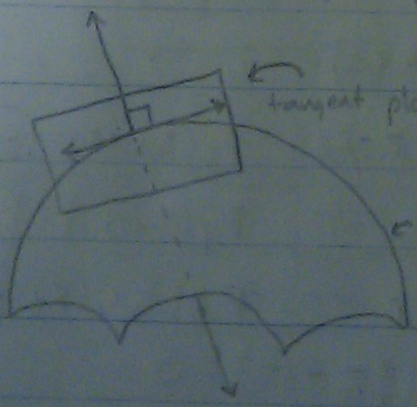
where $a = F_x(x_0, y_0, z_0)$ $\quad b = F_y(x_0, y_0, z_0)$

$c = F_z(x_0, y_0, z_0)$

- The tangent plane to the surface is the plain containing the line orthogonal to the normal line at $P(x_0, y_0, z_0)$ and is given by

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$$

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$



#7) $z = \sqrt{x^2+y^2}$ $(3,4,5)$ unit normal vector

$0 = \sqrt{x^2+y^2} - z$

$F(x,y,z) = (x^2+y^2)^{1/2} - z$

$F_x(x,y,z) = \frac{1}{2}(x^2+y^2)^{-1/2} 2x = \frac{x}{\sqrt{x^2+y^2}}$

$F_x(3,4,5) = \frac{(3)}{\sqrt{3^2+4^2}} = \frac{3}{5}$

$F_y(x,y,z) = \frac{1}{2}(x^2+y^2)^{-1/2} 2y = \frac{y}{\sqrt{x^2+y^2}}$

$F_y(3,4,5) = \frac{4}{\sqrt{25}} = \frac{4}{5}$

$F_z(x,y,z) = -1$

$F_z(3,4,5) = -1$

$\langle \frac{3}{5}, \frac{4}{5}, -1 \rangle$

$$\frac{\langle \frac{3}{5}, \frac{4}{5}, -1 \rangle}{\sqrt{(\frac{3}{5})^2+(\frac{4}{5})^2+(-1)^2}} = \frac{\langle \frac{3}{5}, \frac{4}{5}, -1 \rangle}{\sqrt{\frac{9}{25}+\frac{16}{25}+\frac{25}{25}}} = \frac{\langle \frac{3}{5}, \frac{4}{5}, -1 \rangle}{\sqrt{\frac{50}{25}}} = \left\langle \frac{3\sqrt{2}}{10}, \frac{2\sqrt{2}}{5}, \frac{-\sqrt{2}}{2} \right\rangle$$

#15) $z = 25 - x^2 - y^2$ $(3,1,15)$

$0 = 25 - x^2 - y^2 - z = F(x,y,z)$

$F_x(x,y,z) = 0 - 2x - 0 - 0 = -2x$ $F_x(3,1,15) = -6 = a$

$F_y(x,y,z) = 0 - 0 - 2y - 0 = -2y$ $F_y(3,1,15) = -2 = b$

$F_z(x,y,z) = 0 - 0 - 0 - 1 = -1$ $F_z(3,1,15) = -1 = c$

$-6(x-3) - 2(y-1) - 1(z-15) = 0$

$-6x + 18 - 2y + 1 - z + 15$

$6x + 2y + z = 35$

5-18 p 292 #7, 15, 23, 29, 35

$P(1, 1, \frac{\pi}{4})$

#33) $z = \arctan(\frac{y}{x})$ $\quad 0 = \arctan(\frac{y}{x}) - z$

$F(x, y, z) = \arctan(\frac{y}{x}) - z$

$F_x(x,y,z) = \frac{1}{1+(yx^{-1})^2}(-yx^{-2}) = \frac{-y}{x^2(1+y^2x^{-2})} = \frac{-y}{x^2+y^2}$

$F_x(1, 1, \frac{\pi}{4}) = \frac{-1}{1^2+1^2} = \boxed{\frac{-1}{2} = a}$

$F_y(x,y,z) = \frac{1}{1+(yx^{-1})^2}(x^{-1}) = \frac{1}{x(1+y^2x^{-2})} = \frac{1}{(x+\frac{y^2}{x^{-1}})} \xrightarrow{x\to} \frac{x}{x^2+y^2}$

$F_y(1, 1, \frac{\pi}{4}) = \frac{1}{1^2+1^2} = \boxed{\frac{1}{2} = b}$

$F_z(x,y,z) = \boxed{-1 = c}$

Tangent Plane

$a(x-x_0)+b(y-y_0)+c(z-z_0) = 0$

$-\frac{1}{2}(x-1)+\frac{1}{2}(y-1)-(z-\frac{\pi}{4}) = 0$

$-\frac{x}{2}+\frac{1}{2}+\frac{y}{2}-\frac{1}{2}-z+\frac{\pi}{4} = 0$

$-\frac{x}{2}+\frac{y}{2}-z = -\frac{\pi}{4}$

$\boxed{\frac{x}{2}-\frac{y}{2}+z = \frac{\pi}{4}}$ ← tangent plane (duh)

Normal Line (symmetric form)

$$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$$

$$\frac{x-1}{-\frac{1}{2}} = \frac{y-1}{\frac{1}{2}} = \frac{z-\frac{\pi}{4}}{-1}$$

multiply all by -2

$$\frac{x-1}{1} = \frac{y-1}{-1} = \frac{z-\frac{\pi}{4}}{-2}$$

Angle of Inclination (like angle between vectors)

The angle θ between the xy plane and another inclined plane with normal vectors are $\hat{k}$ and $\hat{n}$ respectively is given by:

$$\cos\theta = \frac{|\hat{n}\cdot\hat{k}|}{\|\hat{n}\|\,\|\hat{k}\|}$$

so: $\hat{k}$ normal to xy plane

$\hat{n}$ normal to the inc. plane

Example

#49) $3x^2+2y^2-z=15$ $\quad 3x^2+2y^2-z-15=0=F(x,y,z)$

$P(2,2,5)$ angle between the tangent plane of this and the xy plane

Gradient of the paraboloid:

$F_x(x,y,z)=6x \quad F(2,2,5)=12$

$F_y(x,y,z)=4y \quad F(2,2,5)=8$

$F_z(x,y,z)=-1 \quad F(2,2,5)=-1$

gradient at $(2,2,5)$: $\langle 12,8,-1\rangle$

(xy plane)

$$\cos\theta = \frac{|\langle 12,8,-1\rangle\cdot\langle 0,0,1\rangle|}{\sqrt{12^2+8^2+(-1)^2}\,\sqrt{\quad 1^2}}$$

$$\theta = \cos^{-1}\frac{\sqrt{209}}{209}$$

$$\theta \approx 1.501569600\,66$$