13.5 The chain rule

Theorem - One independent variable

Let $w = f(x,y)$ and $x = g(t)$ and $y = h(t)$ where f, g and h are all differentiable functions. Then,

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$

Examples

① $w = \ln\left(\frac{y}{x}\right)$ $\quad x = \cos t$ $\quad y = \sin t$

$\frac{\partial w}{\partial x} = \left(\frac{x}{y}\right)\left(y(-x^{-2})\right) = -\frac{x}{x^2} = \frac{-1}{x}$

$\frac{\partial w}{\partial y} = \frac{1}{y}$ $\quad$ (since $\ln\left(\frac{x}{y}\right) = \ln x - \ln y$)

using these identities...

$\frac{dx}{dt} = -\sin t$ $\quad \frac{dy}{dt} = \cos t$

$\frac{dw}{dt} = \left(\frac{-1}{x}\right)(-\sin t) + \left(\frac{1}{y}\right)(\cos t)$

$\frac{dw}{dt} = \left(\frac{-1}{\cos t}\right)(-\sin t) + \left(\frac{1}{\sin t}\right)(\cos t)$

we could have stopped here...

$= \frac{\sin t}{\cos t} + \frac{\cos t}{\sin t}$

$= \frac{\sin^2 t}{\sin t \cos t} + \frac{\cos^2 t}{\sin t \cos t}$

$= \frac{\sin^2 t + \cos^2 t}{\sin t \cos t}$

$= \frac{1}{\sin t \cos t}$

$= \boxed{\csc t \sec t}$

or...

By substitution...

$w = \ln\left(\frac{\sin t}{\cos t}\right)$

$= \ln(\tan t)$

$\frac{dw}{dt} = \frac{1}{\tan t}\sec^2 t$

$= \frac{\cos t}{\sin t}\sec^2 t$

$= \frac{1}{\sin t}\sec t$

$= \boxed{\csc t \sec t}$

same

② $w = \arctan(x+y)$ $\quad x = t-1$ $\quad y = 1+2t$

$$\frac{\partial w}{\partial x} = \frac{1}{1+(x+y)^2}(1) \qquad \frac{\partial w}{\partial y} = \frac{1}{1+(x+y)^2}(1)$$

$$\frac{dx}{dt} = 1 \qquad \frac{dy}{dt} = 2$$

$$\frac{dw}{dt} = \frac{1}{1+(x+y)^2}(1) + \frac{1}{1+(x+y)^2}(2)$$

$$= \frac{3}{1+(x+y)^2} \xrightarrow{\text{terms of } t} \frac{3}{1+((t-1)+(1+2t))^2} = \boxed{\frac{3}{1+9t^2}}$$

By substitution:

$$w = \arctan(x+y) = \arctan(t-1+1+2t) = \arctan(3t)$$

$$\frac{dw}{dt} = \frac{1}{1+(3t)^2}(3) = \boxed{\frac{3}{1+9t^2}}$$ Bamf.

Theorem – Two Independent Variables

Let $w = f(x,y)$ and $x = g(s,t)$ and $y = h(s,t)$ where f, g, h are differentiable then

$$\left.\begin{aligned} \frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} \\ \frac{\partial w}{\partial t} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} \end{aligned}\right\} \text{partials!}$$

Example

$w = 3x^2 + 2xy \qquad x = 3s+4t \quad y = t^2 - s$

$\frac{\partial w}{\partial x} = 6x+2y \qquad \frac{\partial w}{\partial y} = 2x$

$\frac{\partial x}{\partial s} = 3 \qquad \frac{\partial x}{\partial t} = 4$

$\frac{\partial y}{\partial s} = -1 \qquad \frac{\partial y}{\partial t} = 2t$

$\frac{\partial w}{\partial s} = (6x+2y)(3) + (2x)(-1)$

$= 18x + 6y - 2x = 16x + 6y$

$= 16(3s+4t) + 6(t^2 - s) = 48s + 64t + 6t^2 - 6s = \boxed{42s + 64t + 6t^2}$

$\frac{\partial w}{\partial t} = (6x+2y)(4) + (2x)(2t) = 24x + 8y + 4xt$

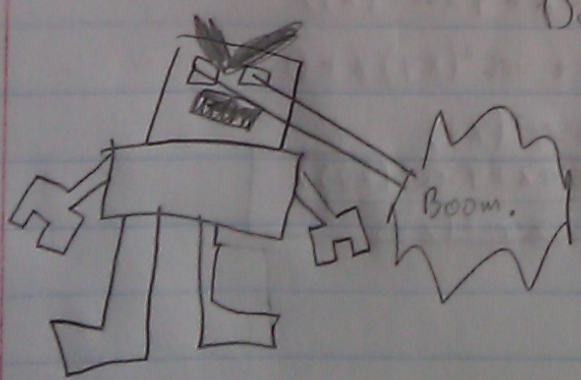
$= 24(3s+4t) + 8(t^2 - s) + 4(3s+4t)t = 72s + 96t + 8t^2 - 8s + 12st + 16t^2$

$= \boxed{64s + 96t + 12st + 24t^2}$

Do by substitution for homework

answer: $\frac{\partial w}{\partial s} = 42s + 64t + 6t^2$

dun. same as before $\frac{\partial w}{\partial t} = 64s + 96t + 12st + 24t^2$



Demo arigato Mr. Robota!!

Kilroy

Implicit functions

If $y = f(x) \Rightarrow 0 = f(x) - y \Rightarrow F(x,y) = 0$

(Implicit – while a function of two variables, only one is independant)

If $z = f(x,y) \Rightarrow 0 = f(x,y) - z \Rightarrow F(x,y,z) = 0$

Theorem - Implicit differentiation

If $F(x,y)=0$ defines $y=f(x)$ then

$$\frac{dy}{dx} = \frac{-F_x(x,y)}{F_y(x,y)}$$

If $F(x,y,z)=0$ defines $z=f(x,y)$ then

$$\frac{\partial z}{\partial x} = \frac{-F_x(x,y,z)}{F_y(x,y,z)}$$

$$\frac{\partial z}{\partial y} = \frac{-F_y(x,y,z)}{F_z(x,y,z)}$$

Examples

#28) $\cos x + \tan(xy) + 5 = 0 = F(x,y)$

$$F_x(x,y) = -\sin x + \sec^2(xy)(y) + 0$$
$$= -\sin x + y\sec^2(xy)$$

$$F_y(x,y) = 0 + \sec^2(xy)\,x + 0$$
$$= x\sec^2(xy)$$

$$\frac{dy}{dx} = \boxed{\frac{-(-\sin x + y\sec^2(xy))}{x\sec^2(xy)}}$$

34) $z = e^x \sin(y+z)$ $\quad 0 = e^x \sin(y+z) - z$

$F(x,y,z) = e^x \sin(y+z) - z$

$F_x = e^x \sin(y+z) - 0$

$F_y = e^x \cos(y+z)(1) - 0$

$F_z = e^x \cos(y+z)(1) - 1$

$$\frac{\partial z}{\partial x} = \frac{-e^x \sin(y+z)}{e^x \cos(y+z) - 1}$$

$$\frac{\partial z}{\partial y} = \frac{-e^x \cos(y+z)}{e^x \cos(y+z) - 1}$$