13.4 Differentials

In $\mathbb{R}^2$, if $y=f(x)$ then its derivative is $\frac{dy}{dx}=f'(x)$ and then the differential is $dy=f'(x)\,dx$

In $\mathbb{R}^3$, we must consider $z=f(x,y)$ where both x and y are independant. The increments of x and y are Δx and Δy respectively and then <u>the increment of z</u> is $\Delta z = f(x+\Delta x, y+\Delta y) - f(x,y)$

Then the differentials of x and y are denoted

$$\Delta x = dx \quad \text{and} \quad \Delta y = dy$$

the total differential of z is

$$dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy$$

$$= f_x(x,y)\,dx + f_y(x,y)\,dy$$

<u>Example</u> Find the total differential of

① $z = e^{x^2+y^2}(2x+3y)$

$f(x,y) = z = e^{x^2+y^2}(2x+3y)$

$f_x(x,y) = \left[e^{x^2+y^2}(2x+0)\right](2x+3y) + e^{x^2+y^2}\left[2+0\right]$

$= e^{x^2+y^2}(4x^2+6xy+2)$

$f_y(x,y) = \left[e^{x^2+y^2}(0+2y)\right](2x+3y) + e^{x^2+y^2}\left[0+3\right]$

$= e^{x^2+y^2}(4xy+6y^2+3)$

$dz = e^{x^2+y^2}(4x^2+6xy+2)\,dx + e^{x^2+y^2}(4xy+6y^2+3)$

③ $f(x,y) = \sin(xy) + x^2 + \ln(2y)$ → (there's no x, so it's constant)

$f_x(x,y) = \cos(xy)\, y + 2x + 0$ ←

$= y\cos(xy) + 2x$

$f_y(x,y) = \cos(xy)\, x + 0 + \frac{1}{2y}\, 2$

$= x\cos(xy) + \frac{1}{y}$

$dz = (y\cos(xy) + 2x)\,dx + (x\cos(xy) + \frac{1}{y})\,dy$

A function $z = f(x,y)$ is differentiable at (x_0, y_0) if Δz can be written in the form

$$\Delta z = f_x(x_0,y_0)\,\Delta x + f_y(x_0,y_0)\,\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

and $\varepsilon_1, \varepsilon_2 \to 0$ while $(x_0, y_0) \to (0,0)$

Example Verify that $f(x,y) = y^2 - 2x$ is differentiable

we know: $\Delta z = f(x+\Delta x, y+\Delta y) - f(x,y)$

$= \left[(y+\Delta y)^2 - 2(x+\Delta x)\right] - \left[y^2 - 2x\right]$

$= (y+\Delta y)(y+\Delta y) - 2x - 2\Delta x - y^2 + 2x$

$= y^2 + 2y\Delta y + \Delta y^2 - 2x - 2\Delta x - y^2 + 2x$

$= 2y\Delta y + \Delta y^2 - 2\Delta x$

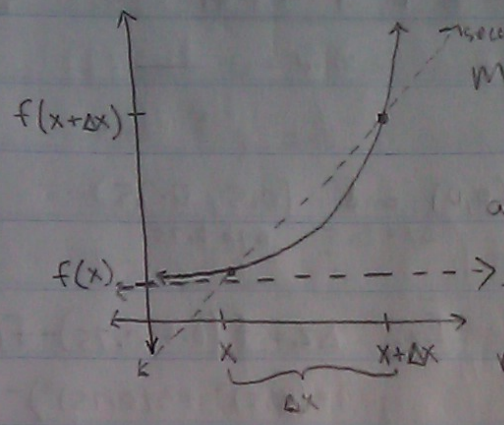
$= -2\Delta x + 2y\Delta y + 0\Delta x + \Delta y\,\Delta y$

$= f_x(x,y)\Delta x + f_y(x,y)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$

* Since $f(x,y) = y^2 - 2x$
  $f_x(x,y) = -2$
  $f_y(x,y) = 2y$

Since we can write it in this form, it's differentiable everywhere

Approximating Rates of Change



$m_{secant} = \frac{f(x+\Delta x)-f(x)}{\Delta x}$ ← approximated rate of change

$m_{tangent} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$ ← Actual rate of change

In $\mathbb{R}^3$ let z be a function of x and y, be continuous, and differentiable for all x and y

Let (x_1, y_1) and (x_2, y_2) be two points in the domain

Then $\Delta x = x_2 - x_1$

$\Delta y = y_2 - y_1$

Then $\Delta z \approx f_x(x_1, y_1)\,\Delta x + f_y(x_1, y_1)\,\Delta y$

$\Delta z = f(x_2, y_2) - f(x_1, y_1)$

Example Find the Approximate and Exact Rate of Change

① $f(x,y) = (x^2+y^2)^{3/2}$ at $(1,2)$ and $(1.05, 2.10)$
① ②

Apx: $f_x(x,y) = \frac{3}{2}(x^2+y^2)^{\frac{1}{2}}(2x) = 3x\sqrt{x^2+y^2}$

$f_x(1,2) = 3(1)\sqrt{(1)+(4)} \approx 6.708203932$

$f_y(x,y) = \frac{3}{2}(x^2+y^2)^{\frac{1}{2}}(2y) = 3y\sqrt{x^2+y^2}$

$f_y(1,2) = 3(2)\sqrt{5} = 6\sqrt{5} \approx 13.41640786$

$\Delta x = 1.05 - 1 = 0.05$ $\Delta y = 2.1 - 2 = 0.10$ So...

$\Delta z \approx (6.708\ldots)(0.05) + (13.416\ldots)(0.10) = 1.677050983$ (continued...)

$\Delta z = f(1.05, 2.1) - f(1,2) = ((1.05)^2 + (2.1)^2)^{3/2} - (1^2 + 2^2)^{3/2}$

≈ 1.762301075

(error $\approx .0852500918$)

② $f(x,y) = x^2 + 3y^3$ $(0,0)$ and $(0.5, 0.75)$

$\Delta x = 0.5$ $\Delta y = 0.75$

Apx: $f_x(x,y) = 2x$

$f_x(0,0) = 0$

$f_y(x,y) = 9y^2$

$f_y(0,0) = 0$

$\Delta z \approx 0$

Ex. $\Delta z = f(0.5, 0.75) - f(0,0)$

$= ((0.5)^2 + 3(0.75)^3) - 0$

$= 1.515625$

Bonus math nerd stuff:

Look for a function that does not exist a point, but it's derivative does...