13.3 – Partial Derivatives

If $z = f(x,y)$, then the first partial derivative of f with respect to x is given as:

$$f_x(x,y) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$$

with respect to y:

$$f_y(x,y) = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}$$

Example Find the first partial derivatives:

$$f(x,y) = 3x^2 + 2xy - y^2$$

$$f(x+\Delta x, y) = 3(x+\Delta x)^2 + 2(x+\Delta x)y - y^2$$

$$= 3(x^2 + 2x\Delta x + \Delta x^2) + 2yx + 2\Delta x y - y^2$$

$$= 3x^2 + 6x\Delta x + 3(\Delta x)^2 + 2xy + 2\Delta x y - y^2$$

$$f_x(x,y) = \lim_{\Delta x \to 0} \frac{\left[3x^2 + 6x\Delta x + 3(\Delta x)^2 + 2xy + 2\Delta x y - y^2\right] - \left[3x^2 + 2xy - y^2\right]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{6x\Delta x + 3(\Delta x)^2 + 2\Delta x y}{\Delta x}$$

$$= \lim_{\Delta x \to 0} 6x + 3\Delta x + 2y$$

$$= \boxed{6x + 2y}$$

$$f(x, y+\Delta y) = 3x^2 + 2x(y+\Delta y) - (y+\Delta y)^2 = 3x^2 + 2xy + 2x\Delta y - (y^2 + 2y\Delta y + \Delta y^2)$$

$$= 3x^2 + 2xy + 2x\Delta y - y^2 - 2y\Delta y - \Delta y^2$$

$$f_y(x,y) = \lim_{\Delta y \to 0} \frac{\left[3x^2 + 2xy + 2x\Delta y - y^2 - 2y\Delta y - \Delta y^2\right] - \left[3x^2 + 2xy - y^2\right]}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{2x\Delta y - 2y\Delta y - \Delta y^2}{\Delta y} = \lim_{\Delta y \to 0} 2x - 2y - \Delta y = \boxed{2x - 2y}$$

Notation

$$f_x(x,y) = f_x = Z_x = \frac{\partial}{\partial x}\left[f(x,y)\right] = \frac{\partial z}{\partial x}$$

* When taking the partial derivative w.r.t. x, treat y as a constant and vise-versa

Examples (using rules)

① $f(x,y) = 6xy + x^2 - 3xy^3$

w.r.t. x ← | → w.r.t. y

$f(x,y) = (6y)x + x^2 - (3y^3)x$

$f_x(x,y) = (6y)1 + 2x - (3y^3)1$

$= 6y + 2x - 3y^3$

$f(x,y) = (6x)y + x^2 - (3x)y^3$

$f_y(x,y) = (6x)1 + 0 - (3x)3y^2$ (no y)

$= 6x + 0 - 3x3y^2$

② $f(x,y) = \sqrt{x^3y + 5xy^2 - xy} = (x^3y + 5xy^2 - xy)^{1/2}$

$f_x(x,y) = \frac{1}{2}(x^3y + 5xy^2 - xy)^{-1/2}(3x^2y + 5(y)^2 - y)$

$= \frac{3x^2y + 5y^2 - y}{2\sqrt{x^3y + 5xy^2 - xy}}$

$f_y(x,y) = \frac{1}{2}(x^3y + 5xy^2 - xy)^{-1/2}(x^3 + 10xy - x)$

$= \frac{x^3 + 10xy - x}{2\sqrt{x^3y + 5xy^2 - xy}}$

Slope (Rate of change) at a point (a,b):

$$m_x = \left.\frac{\partial z}{\partial x}\right|_{(a,b)} = f_x(a,b)$$ in the x direction (parallel to the xz plane)

$$m_y = \left.\frac{\partial z}{\partial y}\right|_{(a,b)} = f_y(a,b)$$ in the y direction (parallel to the yz plane)

Example Find the slope...

$$f(x,y) = \ln(2x - x^2 + y^3) \text{ at } (0,1)$$

m_x: $f_x(x,y) = \frac{1}{2x-x^2+y^3}(2-2x+0) = \frac{2-2x}{2x-x^2+y^3}$ → evaluate at $(0,1)$ → $\frac{2-2(0)}{2(0)-(0)^2+(1)^3} = \boxed{2}$

m_y: $f_y(x,y) = \frac{1}{2x-x^2+y^3}(3y^2) = \frac{3y^2}{2x-x^2+y^3}$ → evaluate at $(0,1)$ → $\frac{3(1)^2}{2(0)-(0)^2+(1)^3} = \boxed{3}$

Higher Order Derivatives

The second partial derivatives:

- $f_{xx}(x,y) = z_{xx} = \frac{\partial}{\partial x}\left[\frac{\partial z}{\partial x}\right] = \frac{\partial^2 z}{\partial x^2}$
- $f_{yy}(x,y) = z_{yy} = \frac{\partial}{\partial y}\left[\frac{\partial z}{\partial y}\right] = \frac{\partial^2 z}{\partial y^2}$

The mixed partial derivatives:

- $f_{xy}(x,y) = z_{xy} = \frac{\partial}{\partial y}\left[\frac{\partial z}{\partial x}\right] = \frac{\partial^2 z}{\partial x \partial y}$
- $f_{yx}(x,y) = z_{yx} = \frac{\partial}{\partial x}\left[\frac{\partial z}{\partial y}\right] = \frac{\partial^2 z}{\partial y \partial x}$

The equality of mixed partials theorem

If $f(x,y)$ is a function such that it's mixed partials f_{xy} and f_{yx} are continuous on some region R then $f_{xy} = f_{yx}$

Example: Find all second partials.

① $f(x,y) = 3x^2 + 6xy - 4y^2$

$f_x(x,y) = 6x + 6y$ $\quad$ $f_y(x,y) = 6x - 8y$

$f_{xx}(x,y) = 6$ $\quad$ $f_{yy}(x,y) = -8$

$f_{xy}(x,y) = 6$ $\quad$ $f_{yx}(x,y) = 6$

since this function is continuous on $[-\infty, \infty]$, its second derivative(s) are also continuous on some interval

② $f(x,y) = \frac{1}{x} + \frac{1}{xy} - \frac{1}{y^2} = x^{-1} + (xy)^{-1} - y^{-2}$

$f_x = -x^{-2} + -(xy)^{-2}(y)$

product rule → $-x^{-2} - [(xy)^{-2} y]$

$f_{xx} = 2x^{-3} - \left[-2(xy)^{-3}(y)y + (xy)^{-2}(0)\right] = \frac{2}{x^3} + \frac{2y^2}{(xy)^3}$

$= \frac{2}{x^3} + \frac{2y^2}{x^3y^3} = \boxed{\frac{2}{x^3} + \frac{2}{x^3y}}$

$f_y = 0 - (xy)^{-2}(x) + 2y^{-3}$

$f_{yy} = -2(xy)^{-3}(x)(x) - 6y^{-4} = \frac{2x^2}{(xy)^3} - \frac{6}{y^4} = \boxed{\frac{2}{xy^3} - \frac{6}{y^4}}$

* do the mixed partials and comparison for homework