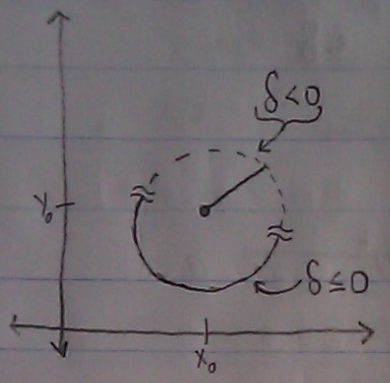
# 13.2 - Limits and Continuity of Functions of Two Vars

Consider a point $(x_0, y_0)$ in $\mathbb{R}^2$ with a $\delta$ neighborhood where $\delta > 0$ and very small



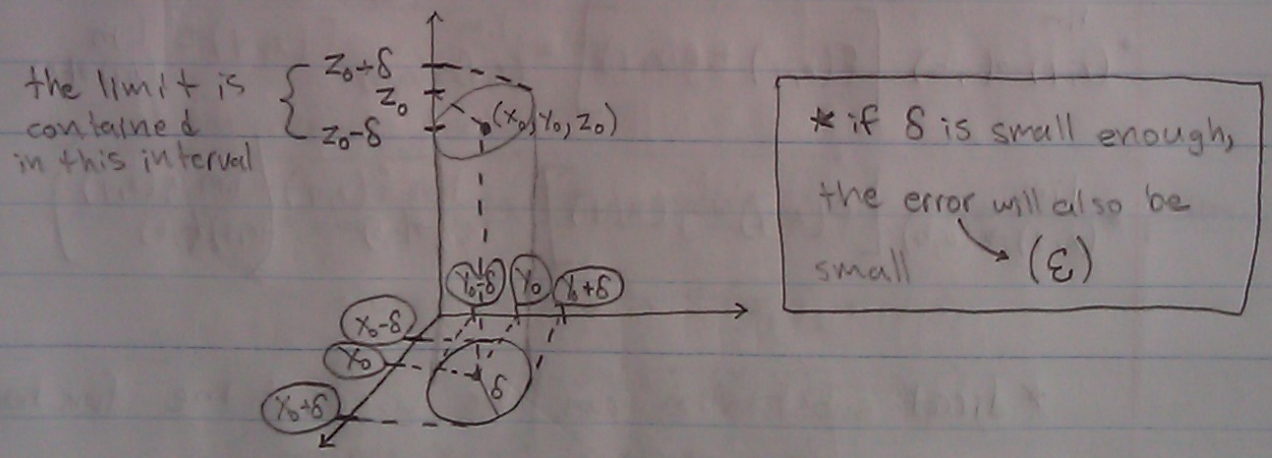

A point $(x, y)$ that lies on the boundary is called an <u>exterior point</u>.

A point $(x, y)$ that lies within the region is called an <u>interior point</u>.

From the distance formula:

$$0 \leq \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2} \leq \delta$$

If we extend this into $\mathbb{R}^3$, we can find a z-value as the limit of the function




$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L \quad \leftarrow (z \text{ value})$$

if $|f(x,y) - L| < \varepsilon$ whenever $0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$

**Example** Verify that $\lim_{(x,y)\to(3,4)} y = 4$

Let $f(x,y) = y$ and $L = 4$ and $(x_0, y_0) = (3,4)$

if $|f(x,y) - L| < \varepsilon$ whenever $0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$

we know that $0 < \sqrt{(x-3)^2 + (y-4)^2} < \delta$

also we want $|y-4| < \varepsilon$

$$|y-4| = \sqrt{(y-4)^2}$$
$$\leq \sqrt{(x-3)^2 + (y-4)^2}$$
$$< \delta = \varepsilon$$
$$< \varepsilon \text{ ///}$$

## Properties of Limits

- $\lim_{(x,y)\to(a,b)} c = c$ (constant)
- $\lim_{(x,y)\to(a,b)} cf(x,y) = c \lim_{(x,y)\to(a,b)} f(x,y)$
- $\lim_{(x,y)\to(a,b)} [f(x,y) \pm g(x,y)] = \lim_{(x,y)\to(a,b)} f(x,y) \pm \lim_{(x,y)\to(a,b)} g(x,y)$
- $\lim_{(x,y)\to(a,b)} [f(x,y)\ g(x,y)] = \lim_{(x,y)\to(a,b)} f(x,y) \left( \lim_{(x,y)\to(a,b)} g(x,y) \right)$

* direct substitution can be used if the function is continuous at $(a,b)$

## Continuity

A function is continuous on it's own domain.

A function $z = f(x,y)$ is continuous at $(a,b)$ if

1. $f(a,b)$ exists
2. $\lim_{(x,y)\to(a,b)} f(x,y)$ exists
3. $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$

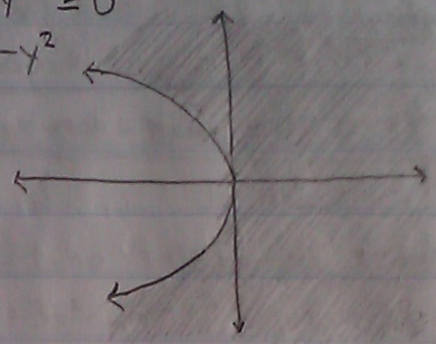
$(A - \text{round}(a,0)) \cdot \text{rand} + \text{round}(\text{rand} \cdot A, 0) - \text{round}(\text{rand} \cdot A, 0)$

## Examples Find the limit and continuity

① $\lim_{(x,y)\to(3,4)} \sqrt{x+y^2} = \sqrt{3+4^2} = \sqrt{19}$

$x + y^2 \geq 0$

$x \geq -y^2$



② $\lim_{(x,y)\to(-1,-2)} \frac{xy}{x^2+y^2} = \frac{(-1)(-2)}{(-1)^2+(-2)^2} = \frac{2}{1+4} = \frac{2}{5}$

$x^2 + y^2 \neq 0$

$(x,y) \neq (0,0)$

(cannot be the origin)

## Indeterminate form

If $\lim_{(x,y)\to(a,b)} f(x,y) = \frac{0}{0}$, consider two of the following paths...

- along the x-axis : $(x,y) \to (x,0)$
- along the y-axis : $(x,y) \to (0,y)$
- along $y = x$ : $(x,y) \to (x,x)$

or

$(x,y) \to (y,y)$

If the two paths you choose are not equal then the limit Does Not Exist

**Example:** $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{2x+3y} = \frac{0-0}{0+0} = \frac{0}{0}$

1: $\lim_{(x,y)\to(x,0)} \frac{x^2-0}{2x+0} = \lim_{(x,y)\to(x,0)} \frac{x^2}{2x} = \frac{x}{2}$

2: $\lim_{(x,y)\to(x,x)} \frac{x^2-x^2}{2x-3x} = 0$

$0 \neq \frac{x}{2}$ so the limit does not exist