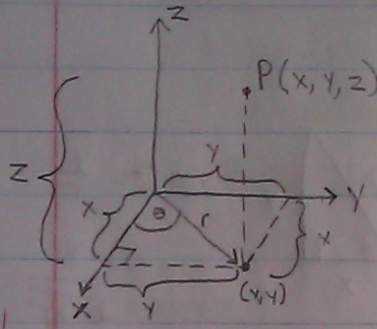
11.7 - Cylindrical and Spherical Coordinates (CCS)

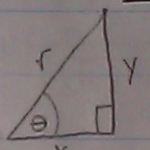
CCS

Le $P(r, \theta, z)$ be a point in the CCS

1) (r, θ) is the polarized (x, y)
(the projection of P in the xy plane)

2) z is the distance from (r, θ) to P





From cylindrical to rectangular

$\cos\theta = \frac{x}{r} \Rightarrow x = r\cos\theta$

$\sin\theta = \frac{y}{r} \Rightarrow y = r\sin\theta$

z remains unchanged

From rectangular to cylindrical

$r = \sqrt{x^2 + y^2}$

$\tan\theta = \frac{y}{x} \Rightarrow \arctan\left(\frac{y}{x}\right) = \theta$

z again remains unchanged

Examples - Convert...

① $(x, y, z) = (-2, 4, 1)$ to cylindrical

$r = \sqrt{(-2)^2 + (4)^2} = \sqrt{20} = 2\sqrt{5}$

$\theta = \arctan\left(\frac{y}{-2}\right)$

$= \arctan(-2)$

$\approx -63.4°$

$z = 1 \Rightarrow z = 1$

$r = 2\sqrt{5} \quad \theta \approx -63.4° \quad z = 1$ → $(2\sqrt{5}, -63.4, 1)$

② $(r, \theta, z) = (3, \frac{\pi}{6}, 2)$ to rectangular

$x = 3\cos\left(\frac{\pi}{6}\right) = 3\frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2}$

$y = 3\sin\left(\frac{\pi}{6}\right) = 3\left(\frac{1}{2}\right) = \frac{3}{2}$

$z = 2$

$\left(\frac{3\sqrt{3}}{2}, \frac{3}{2}, 2\right)$

③ $x^2+y^2+z^2=49$ to cylindrical

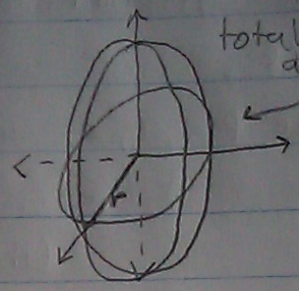
$x=r\cos\theta \quad y=r\sin\theta$

$(r\cos\theta)^2+(r\sin\theta)^2+z^2=49$

$r^2\cos^2\theta+r^2\sin^2\theta+z^2=49$

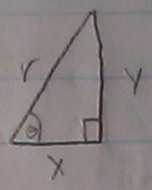
$r^2(\cos^2\theta+\sin^2\theta)+z^2=49$

$r^2+z^2=49$ (sphere w/ radius 7)



④ $r=4\sin\theta$ to rec.

$r=\sqrt{x^2+y^2} \quad \theta=\arctan\left(\frac{y}{x}\right)$



$\sin\theta=\frac{y}{r}$

$\sin\theta=\frac{y}{\sqrt{x^2+y^2}}$

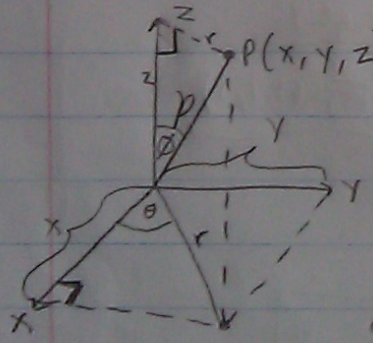
$\sqrt{x^2+y^2}=\frac{4y}{\sqrt{x^2+y^2}}$ ← (multiply both sides by $\sqrt{x^2+y^2}$)

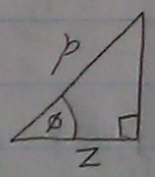
$x^2+y^2=4y$

Spherical Coordinate System (SCS)

Let $P(\rho,\theta,\phi)$ be a point in the SCS

1. ρ is the distance from P to the origin
2. θ is the same as in the CCS
3. ϕ is the angle between the positive z-axis and the line segment containing P





$\sin\phi=\frac{r}{\rho}\Rightarrow r=\rho\sin\phi$

So...

$x=r\cos\theta=\rho\sin\phi\cos\theta$

$y=r\sin\theta=\rho\sin\phi\sin\theta$

$\cos\phi=\frac{z}{\rho}\Rightarrow z=\rho\cos\phi$

From Spherical to Rectangular

$x = \rho \sin\phi \cos\theta$

$y = \rho \sin\phi \sin\theta$

$z = \rho \cos\phi$

From Rectangular to Spherical

$\rho = \sqrt{x^2+y^2+z^2}$

$\theta = \arctan\left(\frac{y}{x}\right)$

$\phi = \arccos\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$

Examples

① Rec $(-1, 3, 2\sqrt{3}) \rightarrow$ sph.

$x = -1$

$y = 3$

$z = 2\sqrt{3}$

$\rho = \sqrt{(-1)^2+(3)^2+(2\sqrt{3})^2} = \sqrt{1+9+4(3)} = \sqrt{22}$

$\theta = \arctan\left(\frac{3}{-1}\right) = \arctan(-3) \approx -71.6°$

$\phi = \arccos\left(\frac{2\sqrt{3}}{\sqrt{22}}\right) \approx 42.4°$