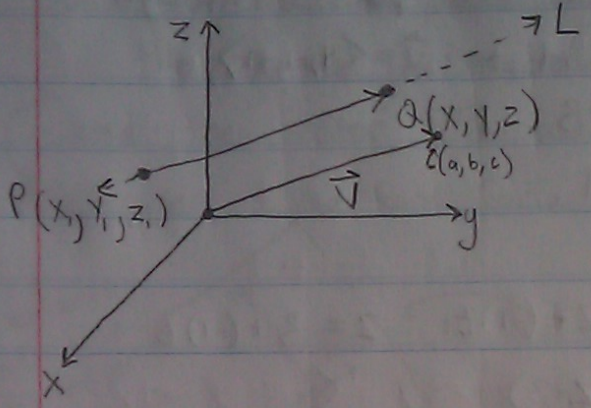
11.5 - Lines and Planes in Space



Let L be a line in $\mathbb{R}^3$ that passes through the points P and Q and is parallel to the vector $\vec{v}$. Then $\vec{v}$ is what we call the direction vector

$\vec{v} = \langle a, b, c \rangle$

direction numbers

Since $\vec{v} \parallel \overrightarrow{PQ}$ and $\vec{v}$ is in the standard position, then $\overrightarrow{PQ} = t\vec{v}$ where t is a scalar

$\langle x - x_1, y - y_1, z - z_1 \rangle = t\langle a, b, c \rangle$

$= \langle at, bt, ct \rangle$

So $x - x_1 = at$

$y - y_1 = bt$

$z - z_1 = ct$

Theorem - Parametric Equations of Lines in space

A line $L \parallel \vec{v} = \langle a, b, c \rangle$ and passing through $P(x_1, y_1, z_1)$ is represented by the system of parametric equations:

$$\begin{cases} x = x_1 + at \\ y = y_1 + bt \\ z = z_1 + ct \end{cases}$$

or the symmetric equations:

$$\left(\frac{x - x_1}{a}\right) = \left(\frac{y - y_1}{b}\right) = \left(\frac{z - z_1}{c}\right)$$

Example Find the parametric and symmetric equations of the line that passes through $(-1, 2, 3)$ and is parallel to $\vec{v} = \langle 4, -1, -1 \rangle$

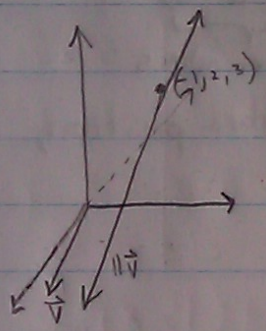
$x_1 = -1 \quad y_1 = 2 \quad z_1 = 3$

$a = 4 \quad b = -1 \quad c = -1$

Parametric Equations:

$x = -1 + 4t \qquad y = 2 + (-1)t \qquad z = 3 + (-1)t$

$$\frac{(x-(-1))}{4} = \frac{(y-2)}{-1} = \frac{(z-3)}{-1}$$



$\underbrace{(x_1, y_1, z_1)}$

Example Pass through $(-1, 3, 6)$ and $(4, -2, -1)$

$\vec{v} = \langle 4-(-1), -2-3, -1-6 \rangle$

$= \langle 5, -5, -7 \rangle$

$a = 5 \quad b = -5 \quad c = -7$

Parametric:

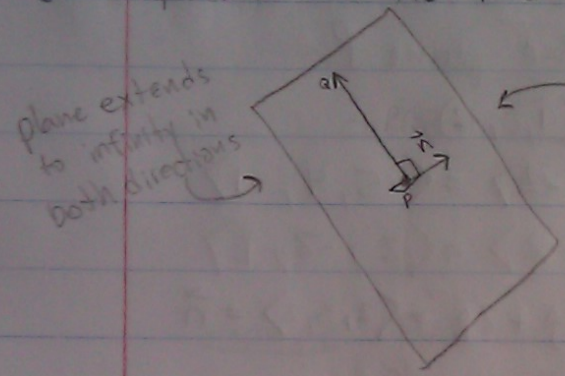
$x = -1 + 5t \qquad y = 3 - 5t \qquad z = 6 - 7t$

Symmetric:

$$\frac{(x+1)}{5} = \frac{(y-3)}{-5} = \frac{(z-6)}{-7}$$

An infinite number of parallel lines in space will give us a plane, whose direction is determined by the normal (orthogonal) vector $\vec{n} = \langle a, b, c \rangle$ and the vector $\overrightarrow{PQ}$ falls on the plane

(Basically: where is the face facing?)



($\vec{n}$ is perpendicular to the face of the plane)

Since $\vec{n} \perp \overrightarrow{PQ}$ we have:

$\vec{n} \cdot \overrightarrow{PQ} = 0$

$\langle a, b, c \rangle \cdot \langle x - x_1, y - y_1, z - z_1 \rangle = 0$

$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$

Standard form of a plane in space $\mathbb{R}^3$

$ax - ax_1 + by - by_1 + cz - cz_1 = 0$

$\boxed{ax + by + cz + d = 0 \text{ General Form}}$

Example Find the equation of the plane passing through $(-1, 4, -3)$ with the normal vector $\langle 2, 4, -1 \rangle$.

$a = 2 \quad b = 4 \quad c = -1$

$x_1 = -1 \quad y_1 = 4 \quad z_1 = -3$

Standard Form:

$2(x - -1) + 4(y - 4) + -1(z - -3) = 0$

$2x + 2 + 4y - 16 - z - 3 = 0$

$\boxed{2x + 4y - z - 17 = 0}$

Example Find the equation of the plane passing through $(-1,4,-3)$, $(6,-2,-1)$, and $(4,0,-2)$

*we need a normal vector $\vec{n}$, so if we find two vectors $\vec{u}$ and $\vec{v}$ on the plane P the vector $\vec{u} \times \vec{v}$ is orthogonal to both, and therefore normal to the plane P.

Pick our initial point: $P=(4,0,-2)$

$$\vec{u} = \langle -1-4, 4-0, -3-2 \rangle = \langle -5, 4, -1 \rangle$$

$$\vec{v} = \langle 6-4, -2-0, -1-2 \rangle = \langle 2, -2, 1 \rangle$$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -5 & 4 & -1 \\ 2 & -2 & 1 \end{vmatrix} = 2\hat{i} + 3\hat{j} + 2\hat{k} = \langle 2,3,2 \rangle = \vec{n}$$

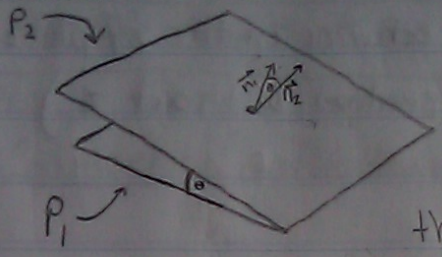
$a=2$ $b=3$ $c=2$ $x_1=4$ $y_1=0$ $z_1=-2$

$$\boxed{2(x-4)+3(y-0)+2(z+2)=0}$$

$$2x-8+3y+2z+4=0$$

$$\boxed{2x+3y+2z-4=0}$$

The Angle Between two planes



$\vec{n}_1$ is normal to P_1

$\vec{n}_2$ is normal to P_2

the angle between both P_1 and P_2 as well as $\vec{n}_1$ and $\vec{n}_2$ is θ

$$\cos\theta = \frac{|\vec{n}_1 \cdot \vec{n}_2|}{\|\vec{n}_1\|\|\vec{n}_2\|}$$

Example Find the angle between:

$P_1: 2x - 5y + 4z = 6$

$P_2: 3x + y - 2z = 4$

$\vec{n}_1 = \langle 2, -5, 4 \rangle$

$\vec{n}_2 = \langle 3, 1, -2 \rangle$

$\vec{n}_1 \cdot \vec{n}_2 = 6 - 5 - 8 = -7$

$\|\vec{n}_1\| = \sqrt{2^2 + (-5)^2 + 4^2} = \sqrt{45} = 3\sqrt{5}$

$\|\vec{n}_2\| = \sqrt{3^2 + 1^2 + (-2)^2} = \sqrt{14}$

$$\cos\theta = \frac{|-7|}{3\sqrt{5}\sqrt{14}} = \frac{7}{3\sqrt{70}}$$

$$\theta = \cos^{-1}\left(\frac{7}{3\sqrt{70}}\right) \approx 73.806^\circ$$

Sketching a Plane in $\mathbb{R}^3$

- xy trace ($z=0$) (the behavior on the xy plane)

(the portions of the plane equation without z)

- xz trace ($y=0$)
- yz trace ($x=0$)

Example $3x+y-2z=6$

xy trace

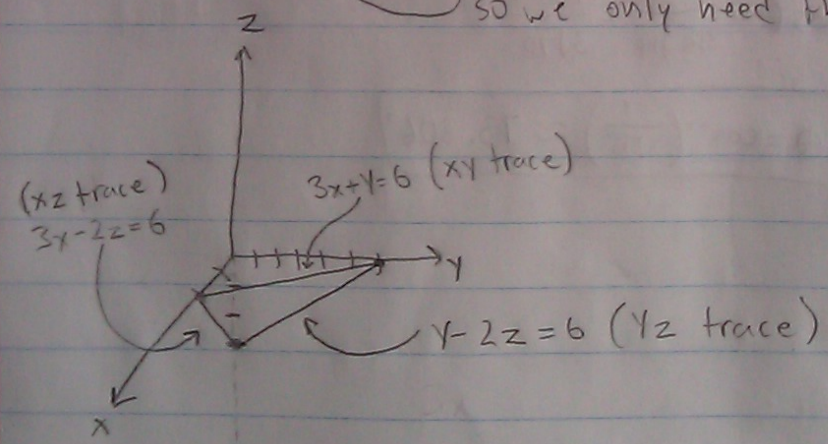
$3x+y=6$ $\quad y=0: x=2$ $\quad x=0: y=6$

Line: $(2,0,0)$ $(0,6,0)$

yz trace

$y-2z=6$ $\quad y=0: z=-3$

$(0,0,-3)$

xz trace

$3x-2z=6$ (we found all the other intercepts, so we only need the equation)