11.4

11.4 - The Cross Product ($\mathbb{R}^3$)

Finds us a vector that is orthogonal to $\vec{u}$ and $\vec{v}$

Cross product of vectors

$$\vec{u} \times \vec{v} = (u_2v_3 - u_3v_2)\hat{i} - (u_1v_3 - u_3v_1)\hat{j} + (u_1v_2 - u_2v_1)\hat{k}$$

This is derived from the determinate of the matrix

$$\begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix}$$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

find the product of $\hat{i}$
- ignore it's row and column ($\hat{j}$, $\hat{k}$, u_1, and v_1)

... and $\hat{j}$

... and $\hat{k}$

start at upper left

$$= \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}\hat{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}\hat{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\hat{k}$$

$$= (u_2v_3 - u_3v_2)\hat{i} - (u_1v_3 - u_3v_1)\hat{j} + (u_1v_2 - u_2v_1)\hat{k}$$

Look at pg 791 first example
look at website for HW

Example

$\vec{u} = 2\hat{i} + \hat{j} - 3\hat{k}$

$\vec{v} = 3\hat{i} - \hat{j} + 5\hat{k}$

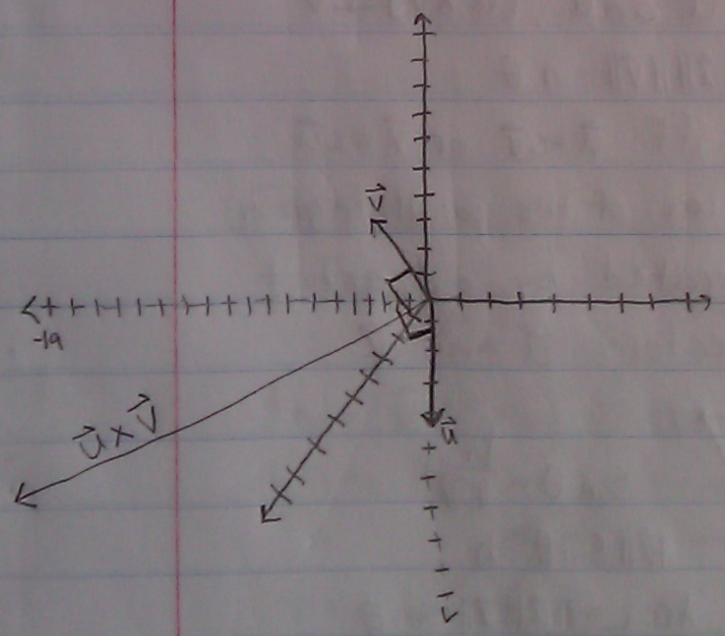
$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -3 \\ 3 & -1 & 5 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -3 \\ -1 & 5 \end{vmatrix} \hat{i} - \begin{vmatrix} 2 & -3 \\ 3 & 5 \end{vmatrix} \hat{j} + \begin{vmatrix} 2 & 1 \\ 3 & -1 \end{vmatrix} \hat{k}$$

$$= \Big(1(5) - (-3)(-1)\Big)\hat{i} - \Big(2(5) - (-3)3\Big)\hat{j} + \Big(2(-1) - (1)3\Big)\hat{k}$$

$$= 2\hat{i} - 19\hat{j} - 5\hat{k}$$

$$= \langle 2, -19, -5 \rangle$$



Theorem - Algebra of Cross-Products

Let $\vec{u}, \vec{v}$ and $\vec{w}$ be vectors in $\mathbb{R}^3$ and c be a scalar

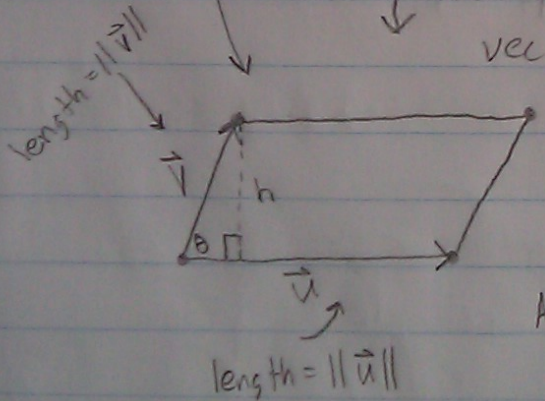
1. $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$
2. $\vec{u} \times (\vec{v} + \vec{w}) = (\vec{u} \times \vec{v}) + (\vec{u} \times \vec{w})$
3. $c(\vec{u} \times \vec{v}) = (c\vec{u}) \times \vec{v} = \vec{u} \times (c\vec{v})$
4. $\vec{u} \times \vec{0} = \vec{0} \times \vec{u} = \vec{0}$
5. $\vec{u} \times \vec{u} = \vec{0}$
6. $\vec{u} \cdot (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \cdot \vec{w}$

* Read the proof in the book

Theorem - Geometric Properties of the Cross Product

Let $\vec{u}$ and $\vec{v}$ be non-zero vectors in $\mathbb{R}^3$ and let θ be the angle between $\vec{u}$ and $\vec{v}$

1. $(\vec{u} \times \vec{v}) \perp \vec{u}$ and $(\vec{u} \times \vec{v}) \perp \vec{v}$
2. $\|\vec{u} \times \vec{v}\| = \|\vec{u}\| \|\vec{v}\| \sin\theta$
3. $\vec{u} \times \vec{v} = 0$ iff $\vec{u} = c\vec{v}$ or $\vec{v} = c\vec{u}$
4. $\|\vec{u} \times \vec{v}\|$ = area of the parallelogram created by adjacent vectors $\vec{u}$ and $\vec{v}$



$\sin\theta = \frac{h}{\|\vec{v}\|}$

$\|\vec{v}\| \sin\theta = h$

Area = $\|\vec{u}\| \|\vec{v}\| \sin\theta$

base, height

cool.

Example Find the area of a parallelogram created by the vectors $\vec{u} = \langle 1, 1, 1 \rangle$ and $\vec{v} = \langle 0, 1, 1 \rangle$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} \hat{i} - \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} \hat{j} + \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} \hat{k}$$

$$= (1-1)\hat{i} - (1-0)\hat{j} + (1-0)\hat{k}$$

$$= 0\hat{i} - (1)\hat{j} + (1)\hat{k}$$

$$= \langle 0, -1, 1 \rangle$$

$\|\vec{u}\| = \sqrt{3}$

$\|\vec{v}\| = \sqrt{2}$ $\quad \|\vec{u} \times \vec{v}\| = \boxed{\sqrt{2}}$

Connection: $\sqrt{2} = \sqrt{3}\sqrt{2} \sin\theta$

$= \sqrt{6} \sin\theta$

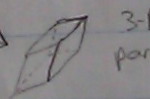
$\sin\theta = \frac{\sqrt{2}}{\sqrt{6}}$

Theorem - Triple Scalar Product

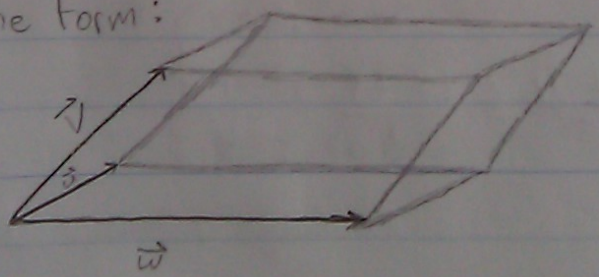
For $\vec{u}$, $\vec{v}$ and $\vec{w}$ the triple scalar product is

$$\vec{u} \cdot (\vec{v} \times \vec{w})$$

And $|\vec{u} \cdot (\vec{v} \times \vec{w})|$ = volume of the parallelepipid



of the form:



Example

$\vec{u} = \langle 1,3,1 \rangle \quad \vec{v} = \langle 0,6,6 \rangle \quad \vec{w} = \langle -4,0,-4 \rangle$

$$\vec{v} \times \vec{w} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 6 & 6 \\ -4 & 0 & -4 \end{vmatrix} = \begin{vmatrix} 6 & 6 \\ 0 & -4 \end{vmatrix} \hat{i} - \begin{vmatrix} 0 & 6 \\ -4 & -4 \end{vmatrix} \hat{j} + \begin{vmatrix} 0 & 6 \\ -4 & 0 \end{vmatrix} \hat{k}$$

$$= -24\hat{i} - 24\hat{j} + 24\hat{k}$$

$$= \langle -24, -24, 24 \rangle$$

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = \langle 1,3,1 \rangle \cdot \langle -24, -24, 24 \rangle$$

$$= -24 - 72 + 24$$

$$= -72$$

$$\boxed{V = 72}$$