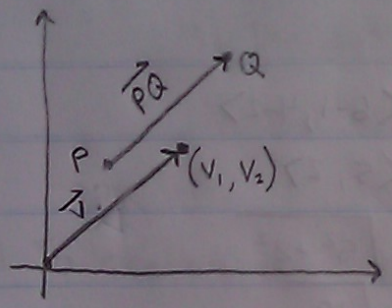
Calculus III

7/6/2009

11.1

11.1 Vectors in the Plane ($\mathbb{R}^2$) (reals in two dimensions)

A vector is a directed line segment (or terminal ray) that is measured by length (magnitude) and direction but NOT position.



Let an initial point $P(p_1, p_2)$ and a terminal point $Q(q_1, q_2)$ be the endpoints of the vector $\vec{PQ}$

A vector in standard position is one whose initial point is at the origin.

$\vec{PQ} = \langle q_1 - p_1, q_2 - p_2 \rangle$

$\vec{v} = \langle v_1, v_2 \rangle$ where $\underbrace{v_1 = q_1 - p_1}_{\text{x-component}}$ and $\underbrace{v_2 = q_2 - p_2}_{\text{y-component}}$

The zero vector

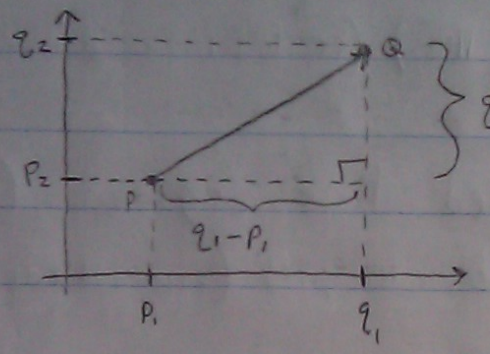
The vector whose initial and terminal points are both at the origin. (thus the length is zero)

$\vec{0} = \langle 0, 0 \rangle$

Magnitude (length)

Magnitude (or norm) of the vector $\vec{v}$ is given by

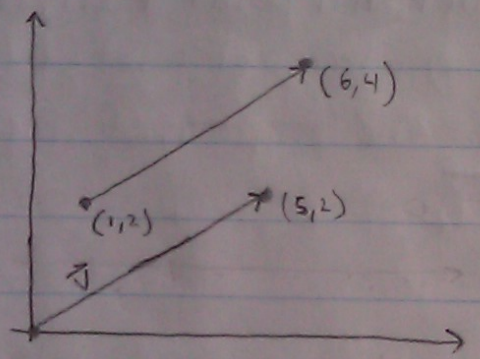
$$\|\vec{v}\| = \sqrt{(q_1-p_1)^2+(q_2-p_2)^2} = \sqrt{v_1^2+v_2^2}$$



$$\|\vec{v}\|^2 = (q_1-p_1)^2+(q_2-p_2)^2$$

$$\|\vec{v}\| = \sqrt{(q_1-p_1)^2+(q_2-p_2)^2}$$

Example



$$\vec{v} = \langle 6-1, 4-2\rangle$$
$$= \langle 5, 2\rangle$$
$$\|v\| = \sqrt{5^2+2^2}$$
$$= \sqrt{29}$$

Equivalent Vectors

Two vectors $\vec{u}$ and $\vec{v}$ are said to be equivalent iff $u_1 = v_1$, $u_2 = v_2$ and $\|\vec{u}\| = \|\vec{v}\|$ and we say $\vec{u} \equiv \vec{v}$

Example

$\vec{u}$: (3,2), (5,6) $\Rightarrow \vec{u} = \langle 5-3, 6-2\rangle = \langle 2,4\rangle$

$\vec{v}$: (-1,4), (1,8) $\Rightarrow \vec{v} = \langle 1--1, 8-4\rangle = \langle 2,4\rangle$

$$\|\vec{u}\| = \|\vec{v}\| = \sqrt{2^2+4^2}$$
$$= \sqrt{20}$$
$$= 2\sqrt{5}$$

* Reduce final answer values *

Vector Addition and Scalar Multiplication

Let $\vec{u}$ and $\vec{v}$ be vectors and let c be a scalar

1) The vector sum of $\vec{u}+\vec{v} = \langle u_1+v_1, u_2+v_2 \rangle$

2) The scalar multiple of $\vec{u}c = c\langle u_1, u_2 \rangle$

$= \langle cu_1, cu_2 \rangle$

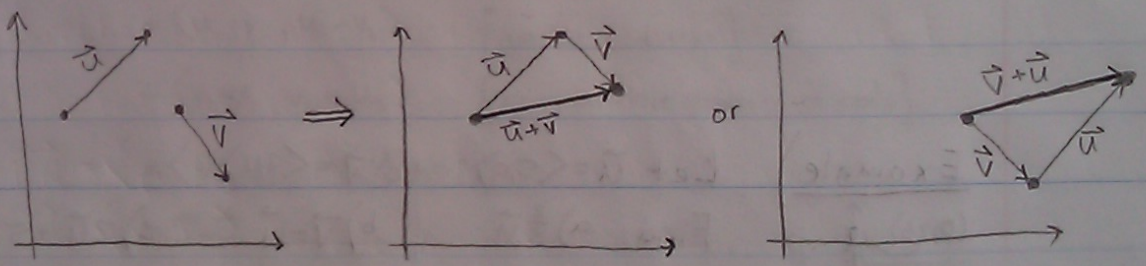
3) The negation of $\vec{v}$ is $-\vec{v} = (-1)\vec{v} = (-1)\langle v_1, v_2 \rangle$

$= \langle -v_1, -v_2 \rangle$

4) The difference of $\vec{u}-\vec{v} = \vec{u}+-\vec{v} = \langle u_1+-v_1, u_2+-v_2 \rangle$

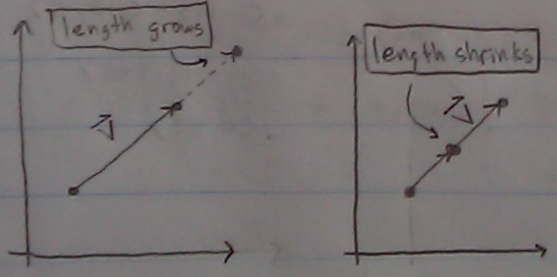
$= \langle u_1-v_1, u_2-v_2 \rangle$

1) To sketch $\vec{u}+\vec{v}$ we must place the initial point of one vector at the terminal point of the other.



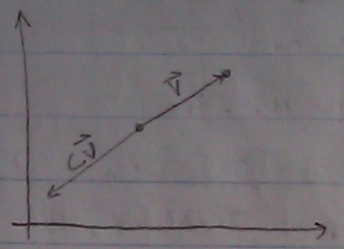
moving $\vec{v}$ moving $\vec{u}$

2) The scalar multiple of a vector changes it's magnitude, but not direction unless c is negative.



$c>1$ $0<c<1$

3) To negate a vector is to switch it's direction

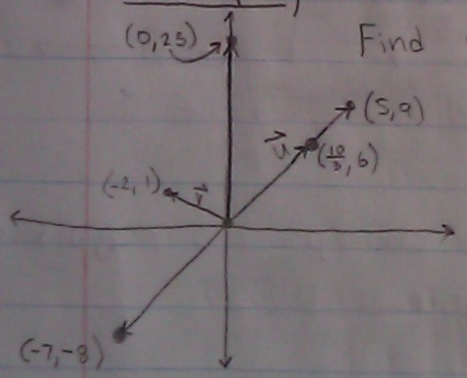


$c<0$

4)

Example) Let $\vec{u}=\langle 5,9\rangle$ and $\vec{v}=\langle -2,1\rangle$

Find a) $\frac{2}{3}\vec{u}$ b) $\vec{v}-\vec{u}$ c) $2\vec{u}+5\vec{v}$



a) $\frac{2}{3}\vec{u}=\frac{2}{3}\langle 5,9\rangle=\langle \frac{10}{3},6\rangle$

b) $\vec{v}-\vec{u}=\langle -2-5,1-9\rangle=\langle -7,-8\rangle$

c) $2\vec{u}+5\vec{v}=2\langle 5,9\rangle+5\langle -2,1\rangle$

$=\langle 10,18\rangle+\langle -10,5\rangle$

$=\langle 10+-10,18+5\rangle$

$=\langle 0,23\rangle$

Theorem - The properties of vectors

Let $\vec{u}, \vec{v}$, and $\vec{w}$ be vectors and c and d be scalars

1) $\vec{u}+\vec{v} = \vec{v}+\vec{u}$ [commutative]

2) $(\vec{u}+\vec{v})+\vec{w} = \vec{u}+(\vec{v}+\vec{w})$ [associative]

3) $\vec{u} + \vec{0} = \vec{u}$ [additive identity]

4) $\vec{u} + -\vec{u} = \vec{0}$ [additive inverse]

5) $c(d\vec{u}) = (cd)\vec{u}$

6) $(c+d)\vec{u} = c\vec{u} + d\vec{u}$ [distributive]

7) $c(\vec{u}+\vec{v}) = c\vec{u} + c\vec{v}$ [distributive]

8) $\vec{u}(1) = \vec{u}$ [multiplicative]

Proof (1)

$\vec{u}+\vec{v} = \langle u_1, u_2 \rangle + \langle v_1, v_2 \rangle$ [component form]

$= \langle u_1 + v_1, u_2 + v_2 \rangle$ [vector addition]

$= \langle v_1 + u_1, v_2 + u_2 \rangle$ [commutative property of reals]

$= \langle v_1, v_2 \rangle + \langle u_1, u_2 \rangle$

$= \vec{v} + \vec{u}$ ///

Proof (4)

$\vec{u} + (-\vec{u}) = \vec{u} + (-1)\vec{u}$

$= \langle u_1, u_2 \rangle + (-1)\langle u_1, u_2 \rangle$

$= \langle u_1, u_2 \rangle + \langle -u_1, -u_2 \rangle$

$= \langle u_1 - u_1, u_2 - u_2 \rangle$

$= \langle 0, 0 \rangle$

$= \vec{0}$ ///

Theorem - Length of a Scalar Multiple

Let $\vec{v}$ be a vector in $\mathbb{R}^2$ and c be scalar. $\|c\vec{v}\| = |c|\,\|\vec{v}\|$

Proof

$$\|c\vec{v}\| = \|\langle cv_1, cv_2\rangle\|$$
$$= \sqrt{(cv_1)^2 + (cv_2)^2}$$
$$= \sqrt{c^2v_1^2 + c^2v_2^2}$$
$$= c\sqrt{v_1^2 + v_2^2}$$
$$= |c|\,\|\vec{v}\| \quad ///$$

Theorem - Unit Vector in the Direction of $\vec{v}$

Let $\vec{v}$ be a vector in $\mathbb{R}^2$ and $\vec{v} \neq \vec{0}$

$\vec{u} = \underbrace{\frac{1}{\|\vec{v}\|}}_{\text{scalar}} \cdot \vec{v}$ is called the unit vector of $\vec{v}$, it has a magnitude of one and is in the same direction as $\vec{v}$

Proof

(1) Direction $\|\vec{v}\| > 0$

Then $\frac{1}{\|\vec{v}\|} > 0$ A positive scalar multiple of $\vec{v}$ will maintain it's direction

(2) Magnitude = 1

$$\|\vec{u}\| = \left\|\frac{1}{\|\vec{v}\|}\vec{v}\right\|$$
$$= \left|\frac{1}{\|\vec{v}\|}\right| \|\vec{v}\| \quad \text{[length of scalar multiple]}$$
$$= \frac{1}{\|\vec{v}\|}\|\vec{v}\|$$
$$= 1$$

Example

$\vec{u} = \langle 5, 10 \rangle$

$\|\vec{u}\| = \sqrt{5^2 + 10^2}$

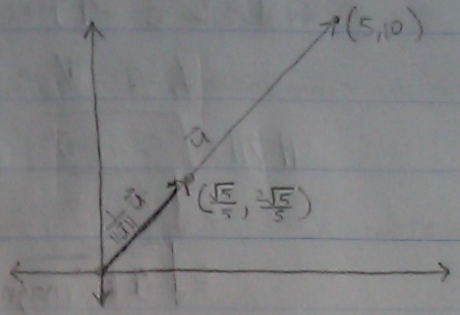
$= \sqrt{125}$

$= 5\sqrt{5}$

$\frac{1}{\|\vec{u}\|}\vec{u} = \frac{1}{5\sqrt{5}}\langle 5, 10 \rangle$

$= \langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \rangle$ ← rationalized

$= \langle \frac{\sqrt{5}}{5}, \frac{2\sqrt{5}}{5} \rangle$ unit vector



$\sqrt{4} < \sqrt{5} < \sqrt{9}$

perfect squares nearest...

$2 < \sqrt{5} < 3$

CHECK: $\sqrt{\left(\frac{1}{\sqrt{5}}\right)^2 + \left(\frac{2}{\sqrt{5}}\right)^2}$

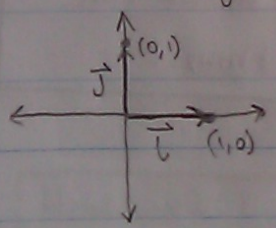
$= \sqrt{\frac{1}{5} + \frac{4}{5}}$

$= \sqrt{1}$

$= 1$

The Standard Unit Vectors in $\mathbb{R}^2$

$\vec{i} = \langle 1, 0 \rangle$ $\quad \vec{j} = \langle 0, 1 \rangle$



The vector $\vec{v} = \langle v_1, v_2 \rangle$ can be written as

$\vec{v} = v_1\vec{i} + v_2\vec{j}$

$v_1\vec{i} = v_1\langle 1, 0 \rangle$

$= \langle v_1, 0 \rangle$

$v_2\vec{j} = v_2\langle 0, 1 \rangle$

$= \langle 0, v_2 \rangle$

$\vec{v} = \langle v_1, 0 \rangle + \langle 0, v_2 \rangle$

$= \langle v_1 + 0, 0 + v_2 \rangle$

$= \langle v_1, v_2 \rangle$ ✓

$\vec{i}$ is the same as $\hat{i}$

$\vec{j}$ is also the same as $\hat{j}$

Example

$\vec{v} = 5\vec{i} - 4\vec{j}$

$V_1 = 5 \quad V_2 = -4$

$\|\vec{v}\| = \sqrt{5^2 + (-4)^2}$

$= \sqrt{25 + 16}$

$= \sqrt{41}$ magnitude

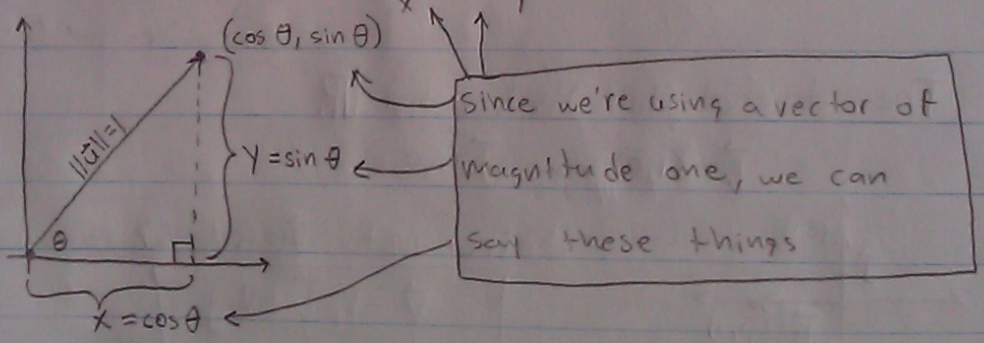
$\frac{1}{\|\vec{v}\|}\vec{v} = \frac{1}{\sqrt{41}}(5\vec{i} - 4\vec{j})$

$= \frac{5}{\sqrt{41}}\vec{i} - \frac{4}{\sqrt{41}}\vec{j}$

$= \frac{5\sqrt{41}}{41}\vec{i} - \frac{4\sqrt{41}}{41}\vec{j}$ unit vector in the direction of $\vec{u}$

Direction of a vector in $\mathbb{R}^2$

If $\vec{u}$ is a unit vector (mag. = 1) then it's component form can be written as $\langle \underbrace{\cos\theta}_{x}, \underbrace{\sin\theta}_{y} \rangle$



Example

If $\vec{u}$ is a unit vector formed by $\theta = \frac{\pi}{3}$, find $\vec{u}$

$\vec{u} = \langle \cos(\frac{\pi}{3}), \sin(\frac{\pi}{3}) \rangle$

$= \langle \frac{1}{2}, \frac{\sqrt{3}}{2} \rangle$

If $\|\vec{u}\| \neq 1$, but is the length $\|\vec{v}\|$ then

$$\vec{u} = \|\vec{v}\| \langle \cos\theta, \sin\theta \rangle \quad \text{(component form)}$$

or

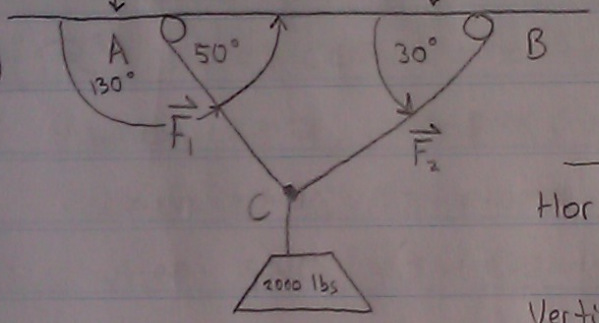
$$\vec{u} = \|\vec{v}\| \cos\theta \,\hat{\imath} + \|\vec{v}\| \sin\theta \,\hat{\jmath} \quad \text{(standard unit vector form)}$$

<u>Example</u> Find the vector $\vec{u}$ of length $\|\vec{v}\| = 3$ which is created by the angle $30°$.

$$\vec{u} = 3\cos 30° \,\hat{\imath} + 3\sin 30° \,\hat{\jmath}$$
$$= 3\left(\frac{\sqrt{3}}{2}\right)\hat{\imath} + 3\left(\frac{1}{2}\right)\hat{\jmath}$$
$$= \frac{3\sqrt{3}}{2}\,\hat{\imath} + \frac{3}{2}\,\hat{\jmath}$$

#89)

(we need both angles on the same side)



$$\vec{F}_1 = \|\vec{F}_1\|\left(\cos(130°)\,\hat{\imath} + \sin(130°)\,\hat{\jmath}\right)$$
$$\vec{F}_2 = \|\vec{F}_2\|\left(\cos(30°)\,\hat{\imath} + \sin(30°)\,\hat{\jmath}\right)$$

Horizontal component:

$$0 = \|\vec{F}_1\|\cos(130°)\,\hat{\imath} + \|\vec{F}_2\|\cos(30°)\,\hat{\imath}$$

Vertical component:

$$2000 = \|\vec{F}_1\|\sin(130°)\,\hat{\jmath} + \|\vec{F}_2\|\sin(30°)\,\hat{\jmath}$$

$$\|\vec{F}_1\|\cos(130°) = -\|\vec{F}_2\|\cos(30°)$$
$$\|\vec{F}_1\| = -\|\vec{F}_2\|\frac{\cos(30°)}{\cos(130°)}$$
$$2000 = -\|\vec{F}_2\|\frac{\cos(30°)}{\cos(130°)}\sin(130°)\,\hat{\jmath} + \|\vec{F}_2\|\sin(30°)\,\hat{\jmath}$$

Homework:

pg 769 - 1 - 57 (odd)