CHAPTER 15
Vector Analysis

Section 15.1 Vector Fields

1. All vectors are parallel to y-axis.
   Matches (c)

2. All vectors are parallel to x-axis.
   Matches (d)

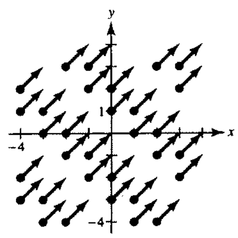
3. All vectors point outward.
   Matches (b)

4. Vectors are in rotational pattern.
   Matches (e)

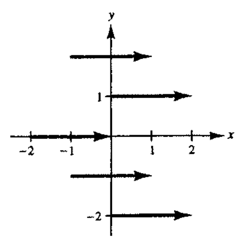
5. Vectors are parallel to x-axis for $y = n\pi$.
   Matches (a)

6. Vectors along x-axis have no x-component.
   Matches (f)

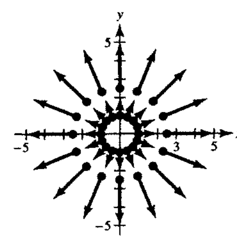
7. $\mathbf{F}(x, y) = \mathbf{i} + \mathbf{j}$
   $\|\mathbf{F}\| = \sqrt{2}$



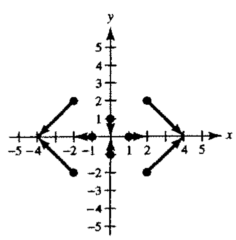
8. $\mathbf{F}(x, y) = 2\mathbf{i}$
   $\|\mathbf{F}\| = 2$



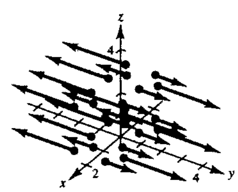
9. $\mathbf{F}(x, y) = x\mathbf{i} + y\mathbf{j}$
   $\|\mathbf{F}\| = \sqrt{x^2 + y^2} = c$
   $x^2 + y^2 = c^2$



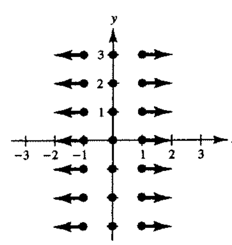
10. $\mathbf{F}(x, y) = x\mathbf{i} - y\mathbf{j}$
    $\|\mathbf{F}\| = \sqrt{x^2 + y^2}$



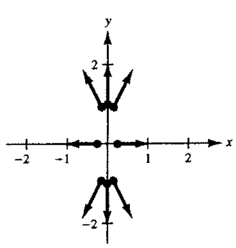
11. $\mathbf{F}(x, y, z) = 3y\mathbf{j}$
    $\|\mathbf{F}\| = 3|y| = c$



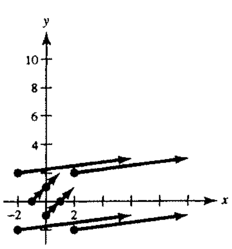
12. $\mathbf{F}(x, y) = x\mathbf{i}$
    $\|\mathbf{F}\| = |x| = c$



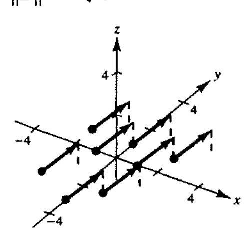
13. $\mathbf{F}(x, y) = 4x\mathbf{i} + y\mathbf{j}$
    $\|\mathbf{F}\| = \sqrt{16x^2 + y^2} = c$
    $$\frac{x^2}{c^2/16} + \frac{y^2}{c^2} = 1$$



14. $\mathbf{F}(x, y) = (x^2 + y^2)\mathbf{i} + \mathbf{j}$
    $\|\mathbf{F}\| = \sqrt{1 + (x^2 + y^2)^2}$



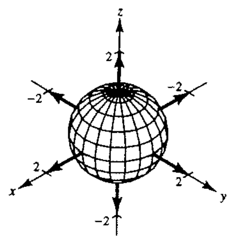
15. $\mathbf{F}(x, y, z) = \mathbf{i} + \mathbf{j} + \mathbf{k}$
    $\|\mathbf{F}\| = \sqrt{3}$



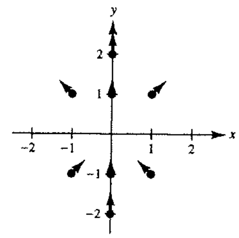
16. $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

$\|\mathbf{F}\| = \sqrt{x^2 + y^2 + z^2} = c$

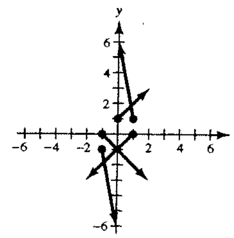
$x^2 + y^2 + z^2 = c^2$



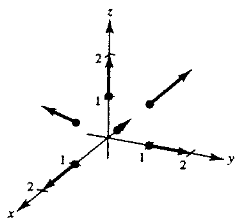
17.



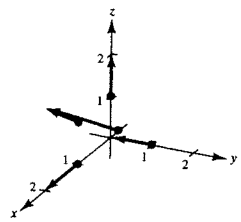
18. $\mathbf{F}(x, y) = (2y - 3x)\mathbf{i} + (2y + 3x)\mathbf{j}$



19.



20. $\mathbf{F}(x, y, z) = x\mathbf{i} - y\mathbf{j} + z\mathbf{k}$



21. $f(x, y) = 5x^2 + 3xy + 10y^2$

$f_x(x, y) = 10x + 3y$

$f_y(x, y) = 3x + 20y$

$\mathbf{F}(x, y) = (10x + 3y)\mathbf{i} + (3x + 20y)\mathbf{j}$

22. $f(x, y) = \sin 3x \cos 4y$

$f_x(x, y) = 3 \cos 3x \cos 4y$

$f_y(x, y) = -4 \sin 3x \sin 4y$

$\mathbf{F}(x, y) = 3 \cos 3x \cos 4y\mathbf{i} - 4 \sin 3x \sin 4y\mathbf{j}$

23. $f(x, y, z) = z - ye^{x^2}$

$f_x(x, y, z) = -2xye^{x^2}$

$f_y(x, y, z) = -e^{x^2}$

$f_z = 1$

$\mathbf{F}(x, y, z) = -2xye^{x^2}\mathbf{i} - e^{x^2}\mathbf{j} + \mathbf{k}$

24. $f(x, y, z) = \dfrac{y}{z} + \dfrac{z}{x} - \dfrac{xz}{y}$

$f_x(x, y, z) = -\dfrac{z}{x^2} - \dfrac{z}{y}$

$f_y(x, y, z) = \dfrac{1}{z} + \dfrac{xz}{y^2}$

$f_z(x, y, z) = -\dfrac{y}{z^2} + \dfrac{1}{x} - \dfrac{x}{y}$

$\mathbf{F}(x, y, z) = \left(-\dfrac{z}{x^2} - \dfrac{z}{y}\right)\mathbf{i} + \left(\dfrac{1}{z} + \dfrac{xz}{y^2}\right)\mathbf{j} + \left(-\dfrac{y}{z^2} + \dfrac{1}{x} - \dfrac{x}{y}\right)\mathbf{k}$

25. $g(x, y, z) = xy \ln(x + y)$

$g_x(x, y, z) = y \ln(x + y) + \dfrac{xy}{x + y}$

$g_y(x, y, z) = x \ln(x + y) + \dfrac{xy}{x + y}$

$g_z(x, y, z) = 0$

$\mathbf{G}(x, y, z) = \left[\dfrac{xy}{x + y} + y \ln(x + y)\right]\mathbf{i} + \left[\dfrac{xy}{x + y} + x \ln(x + y)\right]\mathbf{j}$

26. $g(x, y, z) = x \arcsin yz$

$g_x(x, y, z) = \arcsin yz$

$g_y(x, y, z) = \dfrac{xz}{\sqrt{1 - y^2z^2}}$

$g_z(x, y, z) = \dfrac{xy}{\sqrt{1 - y^2z^2}}$

$\mathbf{G}(x, y, z) = (\arcsin yz)\mathbf{i} + \dfrac{xz}{\sqrt{1 - y^2z^2}}\mathbf{j} + \dfrac{xy}{\sqrt{1 - y^2z^2}}\mathbf{k}$

27. $\mathbf{F}(x, y) = 12xy\mathbf{i} + 6(x^2 + y)\mathbf{j}$

$M = 12xy$ and $N = 6(x^2 + y)$ have continuous first partial derivatives.

$\dfrac{\partial N}{\partial x} = 12x = \dfrac{\partial M}{\partial y} \implies \mathbf{F}$ is conservative.

28. $\mathbf{F}(x, y) = \dfrac{1}{x^2}(y\mathbf{i} - x\mathbf{j}) = \dfrac{y}{x^2}\mathbf{i} - \dfrac{1}{x}\mathbf{j}$

$M = y/x^2$ and $N = -(1/x)$ have continuous first partial derivatives for all $x \neq 0$.

$\dfrac{\partial N}{\partial x} = \dfrac{1}{x^2} = \dfrac{\partial M}{\partial y} \implies \mathbf{F}$ is conservative.

29. $\mathbf{F}(x, y) = \sin y\mathbf{i} + x\cos y\mathbf{j}$

$M = \sin y$ and $N = x\cos y$ have continuous first partial derivatives.

$\dfrac{\partial N}{\partial x} = \cos y = \dfrac{\partial M}{\partial y} \implies \mathbf{F}$ is conservative.

30. $\mathbf{F}(x, y) = \dfrac{1}{xy}(y\mathbf{i} - x\mathbf{j}) = \dfrac{1}{x}\mathbf{i} - \dfrac{1}{y}\mathbf{j}$

$M = 1/x$ and $N = -1/y$ have continuous first partial derivatives for all $x, y \neq 0$.

$\dfrac{\partial N}{\partial x} = 0 = \dfrac{\partial M}{\partial y} \implies \mathbf{F}$ is conservative.

31. $M = 15y^3, N = -5xy^2$

$\dfrac{\partial N}{\partial x} = -5y^2 \neq \dfrac{\partial M}{\partial y} = 45y^2 \implies$ Not conservative

32. $M = \dfrac{x}{\sqrt{x^2 + y^2}}, N = \dfrac{y}{\sqrt{x^2 + y^2}}$

$\dfrac{\partial N}{\partial x} = +\dfrac{-xy}{(x^2 + y^2)^{3/2}} = \dfrac{\partial M}{\partial y} \implies$ Conservative

33. $M = \dfrac{2}{y}e^{2x/y}, N = \dfrac{-2x}{y^2}e^{2x/y}$

$\dfrac{\partial N}{\partial x} = \dfrac{-2(y + 2x)}{y^3}e^{2x/y} = \dfrac{\partial M}{\partial y} \implies$ Conservative

34. $M = \dfrac{y}{\sqrt{1 - x^2y^2}}, N = \dfrac{-x}{\sqrt{1 - x^2y^2}}$

$\dfrac{\partial N}{\partial x} = +\dfrac{-1}{(1 - x^2y^2)^{3/2}} \neq \dfrac{\partial M}{\partial y} = \dfrac{1}{(1 - x^2y^2)^{3/2}}$

$\implies$ Not conservative

35. $\mathbf{F}(x, y) = 2xy\mathbf{i} + x^2\mathbf{j}$

$\dfrac{\partial}{\partial y}[2xy] = 2x$

$\dfrac{\partial}{\partial x}[x^2] = 2x$

Conservative

$f_x(x, y) = 2xy, f_y(x, y) = x^2, f(x, y) = x^2y + K$

36. $\mathbf{F}(x, y) = \dfrac{1}{y^2}(y\mathbf{i} - 2x\mathbf{j})$

$= \dfrac{1}{y}\mathbf{i} - \dfrac{2x}{y^2}\mathbf{j}$

$\dfrac{\partial}{\partial y}\left[\dfrac{1}{y}\right] = -\dfrac{1}{y^2}$

$\dfrac{\partial}{\partial x}\left[-\dfrac{2x}{y^2}\right] = -\dfrac{2}{y^2}$

Not conservative

37. $\mathbf{F}(x, y) = xe^{x^2y}(2y\mathbf{i} + x\mathbf{j})$

$\dfrac{\partial}{\partial y}[2xye^{x^2y}] = 2xe^{x^2y} + 2x^3ye^{x^2y}$

$\dfrac{\partial}{\partial x}[x^2e^{x^2y}] = 2xe^{x^2y} + 2x^3ye^{x^2y}$

Conservative

$f_x(x, y) = 2xye^{x^2y}$

$f_y(x, y) = x^2e^{x^2y}$

$f(x, y) = e^{x^2y} + K$

38. $\mathbf{F}(x, y) = 3x^2y^2\mathbf{i} + 2x^3y\mathbf{j}$

$\dfrac{\partial}{\partial y}[3x^2y^2] = 6x^2y$

$\dfrac{\partial}{\partial x}[2x^3y] = 6x^2y$

Conservative

$f_x(x, y) = 3x^2y^2$

$f_y(x, y) = 2x^3y$

$f(x, y) = x^3y^2 + K$

39. $\mathbf{F}(x, y) = \dfrac{x}{x^2 + y^2}\mathbf{i} + \dfrac{y}{x^2 + y^2}\mathbf{j}$

$$\frac{\partial}{\partial y}\left[\frac{x}{x^2 + y^2}\right] = -\frac{2xy}{(x^2 + y^2)^2}$$

$$\frac{\partial}{\partial x}\left[\frac{y}{x^2 + y^2}\right] = -\frac{2xy}{(x^2 + y^2)^2}$$

Conservative

$$f_x(x, y) = \frac{x}{x^2 + y^2}$$

$$f_y(x, y) = \frac{y}{x^2 + y^2}$$

$$f(x, y) = \frac{1}{2}\ln(x^2 + y^2) + K$$

40. $\mathbf{F}(x, y) = \dfrac{2y}{x}\mathbf{i} - \dfrac{x^2}{y^2}\mathbf{j}$

$$\frac{\partial}{\partial y}\left[\frac{2y}{x}\right] = \frac{2}{x}$$

$$\frac{\partial}{\partial x}\left[-\frac{x^2}{y^2}\right] = -\frac{2x}{y^2}$$

Not conservative

41. $\mathbf{F}(x, y) = e^x(\cos y\mathbf{i} + \sin y\mathbf{j})$

$$\frac{\partial}{\partial y}[e^x \cos y] = -e^x \sin y$$

$$\frac{\partial}{\partial x}[e^x \sin y] = e^x \sin y$$

Not conservative

42. $\mathbf{F}(x, y) = \dfrac{2x}{(x^2 + y^2)^2}\mathbf{i} + \dfrac{2y}{(x^2 + y^2)^2}\mathbf{j}$

$$\frac{\partial}{\partial y}\left[\frac{2x}{(x^2 + y^2)^2}\right] = -\frac{8xy}{(x^2 + y^2)^3}$$

$$\frac{\partial}{\partial x}\left[\frac{2y}{(x^2 + y^2)^2}\right] = -\frac{8xy}{(x^2 + y^2)^3}$$

Conservative

$$f_x(x, y) = \frac{2x}{(x^2 + y^2)^2}$$

$$f_y(x, y) = \frac{2y}{(x^2 + y^2)^2}$$

$$f(x, y) = -\frac{1}{x^2 + y^2} + K$$

43. $\mathbf{F}(x, y, z) = xyz\mathbf{i} + y\mathbf{j} + z\mathbf{k},\ (1, 2, 1)$

$$\mathbf{curl\ F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xyz & y & z \end{vmatrix} = xy\mathbf{j} - xz\mathbf{k}$$

$\mathbf{curl\ F}\,(1, 2, 1) = 2\mathbf{j} - \mathbf{k}$

44. $\mathbf{F}(x, y, z) = x^2z\mathbf{i} - 2xz\mathbf{j} + yz\mathbf{k},\ (2, -1, 3)$

$$\mathbf{curl\ F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2z & -2xz & yz \end{vmatrix}$$

$$= (z + 2x)\mathbf{i} - (0 - x^2)\mathbf{j} + (-2z - 0)\mathbf{k}$$

$$= (z + 2x)\mathbf{i} + x^2\mathbf{j} - 2z\mathbf{k}$$

$\mathbf{curl\ F}\,(2, -1, 3) = 7\mathbf{i} + 4\mathbf{j} - 6\mathbf{k}$

45. $\mathbf{F}(x, y, z) = e^x \sin y\mathbf{i} - e^x \cos y\mathbf{j},\ (0, 0, 3)$

$$\mathbf{curl\ F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ e^x \sin y & -e^x \cos y & 0 \end{vmatrix} = -2e^x \cos y\mathbf{k}$$

$\mathbf{curl\ F}\,(0, 0, 3) = -2\mathbf{k}$

46. $\mathbf{F}(x, y, z) = e^{-xyz}(\mathbf{i} + \mathbf{j} + \mathbf{k}),\ (3, 2, 0)$

$$\mathbf{curl\ F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ e^{-xyz} & e^{-xyz} & e^{-xyz} \end{vmatrix} = (-xz + xy)e^{-xyz}\mathbf{i} - (-yz + xy)e^{-xyz}\mathbf{j} + (-yz + xz)e^{-xyz}\mathbf{k}$$

$\mathbf{curl\ F}\,(3, 2, 0) = 6\mathbf{i} - 6\mathbf{j}$

47. $\mathbf{F}(x, y, z) = \arctan\left(\frac{x}{y}\right)\mathbf{i} + \ln\sqrt{x^2 + y^2}\,\mathbf{j} + \mathbf{k}$

$$\text{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \arctan\left(\frac{x}{y}\right) & \frac{1}{2}\ln(x^2 + y^2) & 1 \end{vmatrix} = \left[\frac{x}{x^2 + y^2} - \frac{(-x/y^2)}{1 + (x/y)^2}\right]\mathbf{k} = \frac{2x}{x^2 + y^2}\mathbf{k}$$

48. $\mathbf{F}(x, y, z) = \frac{yz}{y - z}\mathbf{i} + \frac{xz}{x - z}\mathbf{j} + \frac{xy}{x - y}\mathbf{k}$

$$\text{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{yz}{y - z} & \frac{xz}{x - z} & \frac{xy}{x - y} \end{vmatrix}$$

$$= \left[\frac{x^2}{(x - y)^2} - \frac{x^2}{(x - z)^2}\right]\mathbf{i} - \left[\frac{-y^2}{(x - y)^2} - \frac{y^2}{(y - z)^2}\right]\mathbf{j} + \left[\frac{-z^2}{(x - z)^2} - \frac{-z^2}{(y - z)^2}\right]\mathbf{k}$$

$$= x^2\left[\frac{1}{(x - y)^2} - \frac{1}{(x - z)^2}\right]\mathbf{i} + y^2\left[\frac{1}{(x - y)^2} + \frac{1}{(y - z)^2}\right]\mathbf{j} + z^2\left[\frac{1}{(y - z)^2} - \frac{1}{(x - z)^2}\right]\mathbf{k}$$

49. $\mathbf{F}(x, y, z) = \sin(x - y)\mathbf{i} + \sin(y - z)\mathbf{j} + \sin(z - x)\mathbf{k}$

$$\text{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin(x - y) & \sin(y - z) & \sin(z - x) \end{vmatrix} = \cos(y - z)\mathbf{i} + \cos(z - x)\mathbf{j} + \cos(x - y)\mathbf{k}$$

50. $\mathbf{F}(x, y, z) = \sqrt{x^2 + y^2 + z^2}(\mathbf{i} + \mathbf{j} + \mathbf{k})$

$$\text{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sqrt{x^2 + y^2 + z^2} & \sqrt{x^2 + y^2 + z^2} & \sqrt{x^2 + y^2 + z^2} \end{vmatrix} = \frac{(y - z)\mathbf{i} + (z - x)\mathbf{j} + (x - y)\mathbf{k}}{\sqrt{x^2 + y^2 + z^2}}$$

51. $\mathbf{F}(x, y, z) = \sin y\,\mathbf{i} - x\cos y\,\mathbf{j} + \mathbf{k}$

$$\text{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin y & -x\cos y & 1 \end{vmatrix} = -2\cos y\,\mathbf{k} \neq \mathbf{0}$$

Not conservative

52. $\mathbf{F}(x, y, z) = e^z(y\mathbf{i} + x\mathbf{j} + \mathbf{k})$

$$\text{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ye^z & xe^z & e^z \end{vmatrix} = -xe^z\mathbf{i} + ye^z\mathbf{j} \neq \mathbf{0}$$

Not conservative

53. $\mathbf{F}(x, y, z) = e^z(y\mathbf{i} + x\mathbf{j} + xy\mathbf{k})$

$$\text{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ye^z & xe^z & xye^z \end{vmatrix} = \mathbf{0}$$

Conservative

$f_x(x, y, z) = ye^z$

$f_y(x, y, z) = xe^z$

$f_z(x, y, z) = xye^z$

$f(x, y, z) = xye^z + K$

54. $\mathbf{F}(x, y, z) = y^2z^3\mathbf{i} + 2xyz^3\mathbf{j} + 3xy^2z^2\mathbf{k}$

$$\text{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2z^3 & 2xyz^3 & 3xy^2z^2 \end{vmatrix} = \mathbf{0}$$

Conservative

$f(x, y, z) = xy^2z^3 + K$

55. $\mathbf{F}(x, y, z) = \frac{1}{y}\mathbf{i} - \frac{x}{y^2}\mathbf{j} + (2z - 1)\mathbf{k}$

$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{1}{y} & -\frac{x}{y^2} & 2z - 1 \end{vmatrix} = \mathbf{0}$$

Conservative

$$f_x(x, y, z) = \frac{1}{y}$$

$$f_y(x, y, z) = -\frac{x}{y^2}$$

$$f_z(x, y, z) = 2z - 1$$

$$f(x, y, z) = \int \frac{1}{y}\,dx = \frac{x}{y} + g(y, z) + K_1$$

$$f(x, y, z) = \int -\frac{x}{y^2}\,dy = \frac{x}{y} + h(x, z) + K_2$$

$$f(x, y, z) = \int (2z - 1)\,dz$$

$$= z^2 - z + p(x, y) + K_3$$

$$f(x, y, z) = \frac{x}{y} + z^2 - z + K$$

56. $\mathbf{F}(x, y, z) = \frac{x}{x^2 + y^2}\mathbf{i} + \frac{y}{x^2 + y^2}\mathbf{j} + \mathbf{k}$

$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{x^2 + y^2} & \frac{y}{x^2 + y^2} & 1 \end{vmatrix} = \mathbf{0}$$

Conservative

$$f_x(x, y, z) = \frac{x}{x^2 + y^2}$$

$$f_y(x, y, z) = \frac{y}{x^2 + y^2}$$

$$f_z(x, y, z) = 1$$

$$f(x, y, z) = \int \frac{x}{x^2 + y^2}\,dx$$

$$= \frac{1}{2}\ln(x^2 + y^2) + g(y, z) + K_1$$

$$f(x, y, z) = \int \frac{y}{x^2 + y^2}\,dy$$

$$= \frac{1}{2}\ln(x^2 + y^2) + h(x, z) + K_2$$

$$f(x, y, z) = \int dz = z + p(x, y) + K_3$$

$$f(x, y, z) = \frac{1}{2}\ln(x^2 + y^2) + z + K$$

57. $\mathbf{F}(x, y, z) = 6x^2\mathbf{i} - xy^2\mathbf{j}$

$$\text{div } \mathbf{F}(x, y, z) = \frac{\partial}{\partial x}[6x^2] + \frac{\partial}{\partial y}[-xy^2]$$

$$= 12x - 2xy$$

58. $\mathbf{F}(x, y) = xe^x\mathbf{i} + ye^y\mathbf{j}$

$$\text{div } \mathbf{F}(x, y) = \frac{\partial}{\partial x}[xe^x] + \frac{\partial}{\partial y}[ye^y]$$

$$= xe^x + e^x + ye^y + e^y$$

$$= e^x(x + 1) + e^y(y + 1)$$

59. $\mathbf{F}(x, y, z) = \sin x\,\mathbf{i} + \cos y\,\mathbf{j} + z^2\mathbf{k}$

$$\text{div } \mathbf{F}(x, y, z) = \frac{\partial}{\partial x}[\sin x] + \frac{\partial}{\partial y}[\cos y] + \frac{\partial}{\partial z}[z^2] = \cos x - \sin y + 2z$$

60. $\mathbf{F}(x, y, z) = \ln(x^2 + y^2)\mathbf{i} + xy\mathbf{j} + \ln(y^2 + z^2)\mathbf{k}$

$$\text{div } \mathbf{F}(x, y, z) = \frac{\partial}{\partial x}[\ln(x^2 + y^2)] + \frac{\partial}{\partial y}[xy] + \frac{\partial}{\partial z}[\ln(y^2 + z^2)] = \frac{2x}{x^2 + y^2} + x + \frac{2z}{y^2 + z^2}$$

61. $\mathbf{F}(x, y, z) = xyz\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

$$\text{div } \mathbf{F}(x, y, z) = yz + 1 + 1 = yz + 2$$

$$\text{div } \mathbf{F}(1, 2, 1) = 4$$

62. $\mathbf{F}(x, y, z) = x^2z\mathbf{i} - 2xz\mathbf{j} + yz\mathbf{k}$

$$\text{div } \mathbf{F}(x, y, z) = 2xz + y$$

$$\text{div } \mathbf{F}(2, -1, 3) = 11$$

63. $\mathbf{F}(x, y, z) = e^x \sin y\mathbf{i} - e^x \cos y\mathbf{j}$

$\text{div } \mathbf{F}(x, y, z) = e^x \sin y + e^x \sin y$

$\text{div } \mathbf{F}(0, 0, 3) = 0$

64. $\mathbf{F}(x, y, z) = \ln(xyz)(\mathbf{i} + \mathbf{j} + \mathbf{k})$

$\text{div } \mathbf{F}(x, y, z) = \dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z}$

$\text{div } \mathbf{F}(3, 2, 1) = \dfrac{1}{3} + \dfrac{1}{2} + 1 = \dfrac{11}{6}$

65. See the definition, page 1054. Examples include velocity fields, gravitational fields and magnetic fields.

66. See the definition of Conservative Vector Field on page 1057. To test for a conservative vector field, see Theorem 15.1 and 15.2.

67. See the definition on page 1060.

68. See the definition on page 1062.

69. $\mathbf{F}(x, y, z) = \mathbf{i} + 2x\mathbf{j} + 3y\mathbf{k}$

$\mathbf{G}(x, y, z) = x\mathbf{i} - y\mathbf{j} + z\mathbf{k}$

$$\mathbf{F} \times \mathbf{G} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2x & 3y \\ x & -y & z \end{vmatrix} = (2xz + 3y^2)\mathbf{i} - (z - 3xy)\mathbf{j} + (-y - 2x^2)\mathbf{k}$$

$$\mathbf{curl}\,(\mathbf{F} \times \mathbf{G}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2xz + 3y^2 & 3xy - z & -y - 2x^2 \end{vmatrix} = (-1 + 1)\mathbf{i} - (-4x - 2x)\mathbf{j} + (3y - 6y)\mathbf{k} = 6x\mathbf{j} - 3y\mathbf{k}$$

70. $\mathbf{F}(x, y, z) = x\mathbf{i} - z\mathbf{k}$

$\mathbf{G}(x, y, z) = x^2\mathbf{i} + y\mathbf{j} + z^2\mathbf{k}$

$$\mathbf{F} \times \mathbf{G} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & 0 & -z \\ x^2 & y & z^2 \end{vmatrix} = yz\mathbf{i} - (xz^2 + x^2z)\mathbf{j} + xy\mathbf{k}$$

$$\mathbf{curl}(\mathbf{F} \times \mathbf{G}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ yz & -xz^2 - x^2z & xy \end{vmatrix}$$

$$= (x + 2xz + x^2)\mathbf{i} - (y - y)\mathbf{j} + (-z^2 - 2xz - z)\mathbf{k}$$

$$= x(x + 2z + 1)\mathbf{i} - z(z + 2x + 1)\mathbf{k}$$

71. $\mathbf{F}(x, y, z) = xyz\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

$$\mathbf{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xyz & y & z \end{vmatrix} = xy\mathbf{j} - xz\mathbf{k}$$

$$\mathbf{curl}(\mathbf{curl}\,\mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 0 & xy & -xz \end{vmatrix} = z\mathbf{j} + y\mathbf{k}$$

72. $\mathbf{F}(x, y, z) = x^2z\mathbf{i} - 2xz\mathbf{j} + yz\mathbf{k}$

$$\mathbf{curl}\,\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2z & -2xz & yz \end{vmatrix} = (z + 2x)\mathbf{i} + x^2\mathbf{j} - 2z\mathbf{k}$$

$$\mathbf{curl}(\mathbf{curl}\,\mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ z + 2x & x^2 & -2z \end{vmatrix} = \mathbf{j} + 2x\mathbf{k}$$

73. $\mathbf{F}(x, y, z) = \mathbf{i} + 2x\mathbf{j} + 3y\mathbf{k}$

$\mathbf{G}(x, y, z) = x\mathbf{i} - y\mathbf{j} + z\mathbf{k}$

$$\mathbf{F} \times \mathbf{G} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2x & 3y \\ x & -y & z \end{vmatrix}$$

$$= (2xz + 3y^2)\mathbf{i} - (z - 3xy)\mathbf{j} + (-y - 2x^2)\mathbf{k}$$

$\operatorname{div}(\mathbf{F} \times \mathbf{G}) = 2z + 3x$

74. $\mathbf{F}(x, y, z) = x\mathbf{i} - z\mathbf{k}$

$\mathbf{G}(x, y, z) = x^2\mathbf{i} + y\mathbf{j} + z^2\mathbf{k}$

$$\mathbf{F} \times \mathbf{G} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & 0 & -z \\ x^2 & y & z^2 \end{vmatrix} = yz\mathbf{i} - (xz^2 + x^2z)\mathbf{j} + xy\mathbf{k}$$

$\operatorname{div}(\mathbf{F} \times \mathbf{G}) = 0$

75. $\mathbf{F}(x, y, z) = xyz\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

$$\mathbf{curl\,F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xyz & y & z \end{vmatrix} = xy\mathbf{j} - xz\mathbf{k}$$

$\operatorname{div}(\mathbf{curl\,F}) = x - x = 0$

76. $\mathbf{F}(x, y, z) = x^2z\mathbf{i} - 2xz\mathbf{j} + yz\mathbf{k}$

$$\mathbf{curl\,F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2z & -2xz & yz \end{vmatrix} = (z + 2x)\mathbf{i} + x^2\mathbf{j} - 2z\mathbf{k}$$

$\operatorname{div}(\mathbf{curl\,F}) = 2 - 2 = 0$

77. Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ and $\mathbf{G} = Q\mathbf{i} + R\mathbf{j} + S\mathbf{k}$ where $M, N, P, Q, R,$ and S have continuous partial derivatives.

$\mathbf{F} + \mathbf{G} = (M + Q)\mathbf{i} + (N + R)\mathbf{j} + (P + S)\mathbf{k}$

$$\mathbf{curl}(\mathbf{F} + \mathbf{G}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ M + Q & N + R & P + S \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(P + S) - \frac{\partial}{\partial z}(N + R)\right]\mathbf{i} - \left[\frac{\partial}{\partial x}(P + S) - \frac{\partial}{\partial z}(M + Q)\right]\mathbf{j} + \left[\frac{\partial}{\partial x}(N + R) - \frac{\partial}{\partial y}(M + Q)\right]\mathbf{k}$$

$$= \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\mathbf{i} - \left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z}\right)\mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k} + \left(\frac{\partial S}{\partial y} - \frac{\partial R}{\partial z}\right)\mathbf{i} - \left(\frac{\partial S}{\partial x} - \frac{\partial Q}{\partial z}\right)\mathbf{j} + \left(\frac{\partial R}{\partial x} - \frac{\partial Q}{\partial y}\right)\mathbf{k}$$

$$= \mathbf{curl\,F} + \mathbf{curl\,G}$$

78. Let $f(x, y, z)$ be a scalar function whose second partial derivatives are continuous.

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

$$\mathbf{curl}(\nabla f) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} & \dfrac{\partial f}{\partial z} \end{vmatrix} = \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right)\mathbf{i} - \left(\frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial z \partial x}\right)\mathbf{j} + \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right)\mathbf{k} = \mathbf{0}$$

79. Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ and $\mathbf{G} = R\mathbf{i} + S\mathbf{j} + T\mathbf{k}$.

$$\operatorname{div}(\mathbf{F} + \mathbf{G}) = \frac{\partial}{\partial x}(M + R) + \frac{\partial}{\partial y}(N + S) + \frac{\partial}{\partial z}(P + T) = \frac{\partial M}{\partial x} + \frac{\partial R}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial S}{\partial y} + \frac{\partial P}{\partial z} + \frac{\partial T}{\partial z}$$

$$= \left[\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}\right] + \left[\frac{\partial R}{\partial x} + \frac{\partial S}{\partial y} + \frac{\partial T}{\partial z}\right]$$

$$= \operatorname{div}\mathbf{F} + \operatorname{div}\mathbf{G}$$

80. Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ and $\mathbf{G} = R\mathbf{i} + S\mathbf{j} + T\mathbf{k}$.

$$\mathbf{F} \times \mathbf{G} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ M & N & P \\ R & S & T \end{vmatrix} = (NT - PS)\mathbf{i} - (MT - PR)\mathbf{j} + (MS - NR)\mathbf{k}$$

$$\begin{aligned}\operatorname{div}(\mathbf{F} \times \mathbf{G}) &= \frac{\partial}{\partial x}(NT - PS) + \frac{\partial}{\partial y}(PR - MT) + \frac{\partial}{\partial z}(MS - NR) \\ &= N\frac{\partial T}{\partial x} + T\frac{\partial N}{\partial x} - P\frac{\partial S}{\partial x} - S\frac{\partial P}{\partial x} + P\frac{\partial R}{\partial y} + R\frac{\partial P}{\partial y} - M\frac{\partial T}{\partial y} - T\frac{\partial M}{\partial y} + M\frac{\partial S}{\partial z} + S\frac{\partial M}{\partial z} - N\frac{\partial R}{\partial z} - R\frac{\partial N}{\partial z} \\ &= \left[\left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)R + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right)S + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)T\right] - \left[M\left(\frac{\partial T}{\partial y} - \frac{\partial S}{\partial z}\right) + N\left(\frac{\partial R}{\partial z} - \frac{\partial T}{\partial x}\right) + P\left(\frac{\partial S}{\partial x} - \frac{\partial R}{\partial y}\right)\right] \\ &= (\mathbf{curl\ F}) \cdot \mathbf{G} - \mathbf{F} \cdot (\mathbf{curl\ G})\end{aligned}$$

81. $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$

$$\begin{aligned}\nabla \times [\nabla f + (\nabla \times \mathbf{F})] &= \mathbf{curl}(\nabla f + (\nabla \times \mathbf{F})) \\ &= \mathbf{curl}(\nabla f) + \mathbf{curl}(\nabla \times \mathbf{F}) \qquad \text{(Exercise 77)} \\ &= \mathbf{curl}(\nabla \times \mathbf{F}) \qquad \text{(Exercise 78)} \\ &= \nabla \times (\nabla \times \mathbf{F})\end{aligned}$$

82. Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$.

$$\begin{aligned}\nabla \times (f\mathbf{F}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ fM & fN & fP \end{vmatrix} \\ &= \left(\frac{\partial f}{\partial y}P + f\frac{\partial P}{\partial y} - \frac{\partial f}{\partial z}N - f\frac{\partial N}{\partial z}\right)\mathbf{i} - \left(\frac{\partial f}{\partial x}P + f\frac{\partial P}{\partial x} - \frac{\partial f}{\partial z}M - f\frac{\partial M}{\partial z}\right)\mathbf{j} + \left(\frac{\partial f}{\partial x}N + f\frac{\partial N}{\partial x} - \frac{\partial f}{\partial y}M - f\frac{\partial M}{\partial y}\right)\mathbf{k} \\ &= f\left[\left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\mathbf{i} - \left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z}\right)\mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k}\right] + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} & \dfrac{\partial f}{\partial z} \\ M & N & P \end{vmatrix} = f[\nabla \times \mathbf{F}] + (\nabla f) \times \mathbf{F}\end{aligned}$$

83. Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$, then $f\mathbf{F} = fM\mathbf{i} + fN\mathbf{j} + fP\mathbf{k}$.

$$\begin{aligned}\operatorname{div}(f\mathbf{F}) &= \frac{\partial}{\partial x}(fM) + \frac{\partial}{\partial y}(fN) + \frac{\partial}{\partial z}(fP) = f\frac{\partial M}{\partial x} + M\frac{\partial f}{\partial x} + f\frac{\partial N}{\partial y} + N\frac{\partial f}{\partial y} + f\frac{\partial P}{\partial z} + P\frac{\partial f}{\partial z} \\ &= f\left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial N}{\partial z}\right) + \left(\frac{\partial f}{\partial x}M + \frac{\partial f}{\partial y}N + \frac{\partial f}{\partial z}P\right) \\ &= f \operatorname{div} \mathbf{F} + \nabla f \cdot \mathbf{F}\end{aligned}$$

84. Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$.

$$\mathbf{curl\ F} = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\mathbf{i} - \left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z}\right)\mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k}$$

$$\begin{aligned}\operatorname{div}(\mathbf{curl\ F}) &= \frac{\partial}{\partial x}\left[\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right] - \frac{\partial}{\partial y}\left[\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z}\right] + \frac{\partial}{\partial z}\left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] \\ &= \frac{\partial^2 P}{\partial x \partial y} - \frac{\partial^2 N}{\partial x \partial z} - \frac{\partial^2 P}{\partial y \partial x} + \frac{\partial^2 M}{\partial y \partial z} + \frac{\partial^2 N}{\partial z \partial x} - \frac{\partial^2 M}{\partial z \partial y} = 0 \qquad \text{(since the mixed partials are equal)}\end{aligned}$$

In Exercises 85-88, $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $f(x, y, z) = \|\mathbf{F}(x, y, z)\| = \sqrt{x^2 + y^2 + z^2}$.

85. $\ln f = \frac{1}{2}\ln(x^2 + y^2 + z^2)$

$$\nabla(\ln f) = \frac{x}{x^2 + y^2 + z^2}\mathbf{i} + \frac{y}{x^2 + y^2 + z^2}\mathbf{j} + \frac{z}{x^2 + y^2 + z^2}\mathbf{k} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{x^2 + y^2 + z^2} = \frac{\mathbf{F}}{f^2}$$

86. $\frac{1}{f} = \frac{1}{\sqrt{x^2 + y^2 + z^2}}$

$$\nabla\left(\frac{1}{f}\right) = \frac{-x}{(x^2 + y^2 + z^2)^{3/2}}\mathbf{i} + \frac{-y}{(x^2 + y^2 + z^2)^{3/2}}\mathbf{j} + \frac{-z}{(x^2 + y^2 + z^2)^{3/2}}\mathbf{k} = \frac{-(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})}{\left(\sqrt{x^2 + y^2 + z^2}\right)^3} = -\frac{\mathbf{F}}{f^3}$$

87. $f^n = \left(\sqrt{x^2 + y^2 + z^2}\right)^n$

$$\nabla f^n = n\left(\sqrt{x^2 + y^2 + z^2}\right)^{n-1}\frac{x}{\sqrt{x^2 + y^2 + z^2}}\mathbf{i} + n\left(\sqrt{x^2 + y^2 + z^2}\right)^{n-1}\frac{y}{\sqrt{x^2 + y^2 + z^2}}\mathbf{j}$$

$$+ n\left(\sqrt{x^2 + y^2 + z^2}\right)^{n-1}\frac{z}{\sqrt{x^2 + y^2 + z^2}}\mathbf{k}$$

$$= n\left(\sqrt{x^2 + y^2 + z^2}\right)^{n-2}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = nf^{n-2}\mathbf{F}$$

88. $w = \frac{1}{f} = \frac{1}{\sqrt{x^2 + y^2 + z^2}}$

$$\frac{\partial w}{\partial x} = -\frac{x}{(x^2 + y^2 + z^2)^{3/2}}$$

$$\frac{\partial w}{\partial y} = -\frac{y}{(x^2 + y^2 + z^2)^{3/2}}$$

$$\frac{\partial w}{\partial z} = -\frac{z}{(x^2 + y^2 + z^2)^{3/2}}$$

$$\frac{\partial^2 w}{\partial x^2} = \frac{2x^2 - y^2 - z^2}{(x^2 + y^2 + z^2)^{5/2}}$$

$$\frac{\partial^2 w}{\partial y^2} = \frac{2y^2 - x^2 - z^2}{(x^2 + y^2 + z^2)^{5/2}}$$

$$\frac{\partial^2 w}{\partial z^2} = \frac{2z^2 - x^2 - y^2}{(x^2 + y^2 + z^2)^{5/2}}$$

$$\nabla^2 w = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} = 0$$

Therefore $w = \frac{1}{f}$ is harmonic.

89. True. $\|\mathbf{F}(x + y)\| = \sqrt{16x^2 + y^4} \to 0$ as $(x, y) \to (0, 0)$.

90. True. If (x, y) is on the positive y-axis, then $x = 0$ and $y > 0$. Thus, $\mathbf{F}(x, y) = \mathbf{F}(0, y) = -y^2\mathbf{j}$.

91. False. Curl is defined on vector fields, not scalar fields.

92. False. See Example 7.

Section 15.2 Line Integrals

1.
$$x^2 + y^2 = 9$$
$$\frac{x^2}{9} + \frac{y^2}{9} = 1$$
$$\cos^2 t + \sin^2 t = 1$$
$$\cos^2 t = \frac{x^2}{9}$$
$$\sin^2 t = \frac{y^2}{9}$$
$$x = 3\cos t$$
$$y = 3\sin t$$
$$\mathbf{r}(t) = 3\cos t\mathbf{i} + 3\sin t\mathbf{j}$$
$$0 \le t \le 2\pi$$

2.
$$\frac{x^2}{16} + \frac{y^2}{9} = 1$$
$$\cos^2 t + \sin^2 t = 1$$
$$\cos^2 t = \frac{x^2}{16}$$
$$\sin^2 t = \frac{y^2}{9}$$
$$x = 4\cos t$$
$$y = 3\sin t$$
$$\mathbf{r}(t) = 4\cos t\mathbf{i} + 3\sin t\mathbf{j}$$
$$0 \le t \le 2\pi$$

3. $\mathbf{r}(t) = \begin{cases} t\mathbf{i}, & 0 \le t \le 3 \\ 3\mathbf{i} + (t-3)\mathbf{j}, & 3 \le t \le 6 \\ (9-t)\mathbf{i} + 3\mathbf{j}, & 6 \le t \le 9 \\ (12-t)\mathbf{j}, & 9 \le t \le 12 \end{cases}$

4. $\mathbf{r}(t) = \begin{cases} t\mathbf{i} + \frac{4}{5}t\mathbf{j}, & 0 \le t \le 5 \\ 5\mathbf{i} + (9-t)\mathbf{j}, & 5 \le t \le 9 \\ (14-t)\mathbf{i}, & 9 \le t \le 14 \end{cases}$

5. $\mathbf{r}(t) = \begin{cases} t\mathbf{i} + \sqrt{t}\mathbf{j}, & 0 \le t \le 1 \\ (2-t)\mathbf{i} + (2-t)\mathbf{j}, & 1 \le t \le 2 \end{cases}$

6. $\mathbf{r}(t) = \begin{cases} t\mathbf{i} + t^2\mathbf{j}, & 0 \le t \le 2 \\ (4-t)\mathbf{i} + 4\mathbf{j}, & 2 \le t \le 4 \\ (8-t)\mathbf{j}, & 4 \le t \le 8 \end{cases}$

7. $\mathbf{r}(t) = 4t\mathbf{i} + 3t\mathbf{j},\ 0 \le t \le 2;\ \mathbf{r}'(t) = 4\mathbf{i} + 3\mathbf{j}$

$$\int_C (x-y)\,ds = \int_0^2 (4t - 3t)\sqrt{(4)^2 + (3)^2}\,dt = \int_0^2 5t\,dt = \left[\frac{5t^2}{2}\right]_0^2 = 10$$

8. $\mathbf{r}(t) = t\mathbf{i} + (2-t)\mathbf{j},\ 0 \le t \le 2;\ \mathbf{r}'(t) = \mathbf{i} - \mathbf{j}$

$$\int_C 4xy\,ds = \int_0^2 4t(2-t)\sqrt{1+1}\,dt = 4\sqrt{2}\int_0^2 (2t - t^2)\,dt = 4\sqrt{2}\left[t^2 - \frac{t^3}{3}\right]_0^2 = 4\sqrt{2}\left(4 - \frac{8}{3}\right) = \frac{16\sqrt{2}}{3}$$

9. $\mathbf{r}(t) = \sin t\mathbf{i} + \cos t\mathbf{j} + 8t\mathbf{k},\ 0 \le t \le \frac{\pi}{2};\ \mathbf{r}'(t) = \cos t\mathbf{i} - \sin t\mathbf{j} + 8\mathbf{k}$

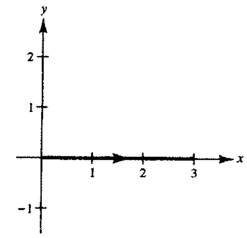
$$\int_C (x^2 + y^2 + z^2)\,ds = \int_0^{\pi/2} (\sin^2 t + \cos^2 t + 64t^2)\sqrt{(\cos t)^2 + (-\sin t)^2 + 64}\,dt$$
$$= \int_0^{\pi/2} \sqrt{65}(1 + 64t^2)\,dt = \left[\sqrt{65}\left(t + \frac{64t^3}{3}\right)\right]_0^{\pi/2} = \sqrt{65}\left(\frac{\pi}{2} + \frac{8\pi^3}{3}\right) = \frac{\sqrt{65}\pi}{6}(3 + 16\pi^2)$$

10. $\mathbf{r}(t) = 12t\mathbf{i} + 5t\mathbf{j} + 3\mathbf{k},\ 0 \le t \le 2;\ \mathbf{r}'(t) = 12\mathbf{i} + 5\mathbf{j}$

$$\int_C 8xyz\,ds = \int_0^2 8(12t)(5t)(3)\sqrt{12^2 + 5^2 + 0^2}\,dt = \int_0^2 18{,}720t^2\,dt = 18{,}720\left[\frac{t^3}{3}\right]_0^2 = 49{,}920$$

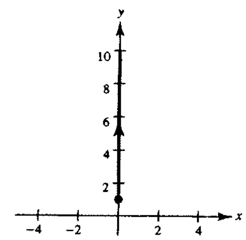
11. $\mathbf{r}(t) = t\mathbf{i},\ 0 \le t \le 3$

$$\int_C (x^2 + y^2)\,ds = \int_0^3 [t^2 + 0^2]\sqrt{1 + 0}\,dt$$
$$= \int_0^3 t^2\,dt$$
$$= \left[\frac{1}{3}t^3\right]_0^3 = 9$$



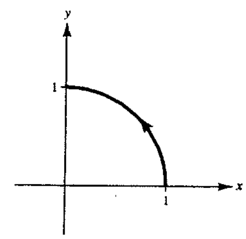
12. $\mathbf{r}(t) = t\mathbf{j},\ 1 \le t \le 10$

$$\int_C (x^2 + y^2)\,ds = \int_1^{10} [0 + t^2]\sqrt{0 + 1}\,dt$$
$$= \int_1^{10} t^2\,dt$$
$$= \left[\frac{1}{3}t^3\right]_1^{10} = 333$$



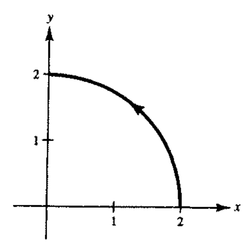
13. $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j},\ 0 \le t \le \dfrac{\pi}{2}$

$$\int_C (x^2 + y^2)\,ds = \int_0^{\pi/2} [\cos^2 t + \sin^2 t]\sqrt{(-\sin t)^2 + (\cos t)^2}\,dt$$
$$= \int_0^{\pi/2} dt = \frac{\pi}{2}$$



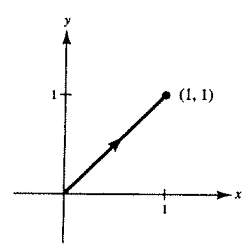
14. $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j},\ 0 \le t \le \dfrac{\pi}{2}$

$$\int_C (x^2 + y^2)\,ds = \int_0^{\pi/2} [4\cos^2 t + 4\sin^2 t]\sqrt{(-2\sin t)^2 + (2\cos t)^2}\,dt$$
$$= \int_0^{\pi/2} 8\,dt = 4\pi$$



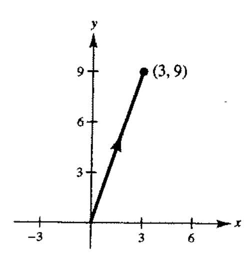
15. $\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j},\ 0 \le t \le 1$

$$\int_C \left(x + 4\sqrt{y}\right) ds = \int_0^1 \left(t + 4\sqrt{t}\right)\sqrt{1 + 1}\,dt$$
$$= \left[\sqrt{2}\left(\frac{t^2}{2} + \frac{8}{3}t^{3/2}\right)\right]_0^1 = \frac{19\sqrt{2}}{6}$$

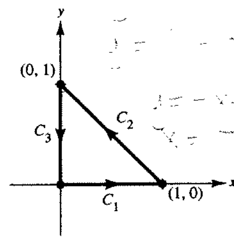


16. $\mathbf{r}(t) = t\mathbf{i} + 3t\mathbf{j},\ 0 \le t \le 3$

$$\int_C \left(x + 4\sqrt{y}\right) ds = \int_0^3 \left(t + 4\sqrt{3t}\right)\sqrt{1 + 9}\,dt$$
$$= \left[\sqrt{10}\left(\frac{t^2}{2} + \frac{8\sqrt{3}}{3}t^{3/2}\right)\right]_0^3$$
$$= \frac{\sqrt{10}}{6}(27 + 144) = \frac{57\sqrt{10}}{2}$$



17. $\mathbf{r}(t) = \begin{cases} t\mathbf{i}, & 0 \le t \le 1 \\ (2-t)\mathbf{i} + (t-1)\mathbf{j}, & 1 \le t \le 2 \\ (3-t)\mathbf{j}, & 2 \le t \le 3 \end{cases}$



$$\int_{C_1} (x + 4\sqrt{y})\, ds = \int_0^1 t\, dt = \frac{1}{2}$$

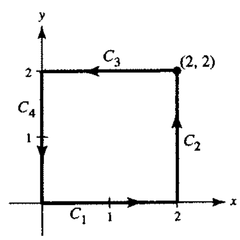
$$\int_{C_2} (x + 4\sqrt{y})\, ds = \int_1^2 \left[(2-t) + 4\sqrt{t-1}\right]\sqrt{1+1}\, dt$$

$$= \sqrt{2}\left[2t - \frac{t^2}{2} + \frac{8}{3}(t-1)^{3/2}\right]_1^2 = \frac{19\sqrt{2}}{6}$$

$$\int_{C_3} (x + 4\sqrt{y})\, ds = \int_2^3 4\sqrt{3-t}\, dt = \left[-\frac{8}{3}(3-t)^{3/2}\right]_2^3 = \frac{8}{3}$$

$$\int_C (x + 4\sqrt{y})\, ds = \frac{1}{2} + \frac{19\sqrt{2}}{6} + \frac{8}{3} = \frac{19 + 19\sqrt{2}}{6} = \frac{19(1+\sqrt{2})}{6}$$

18. $\mathbf{r}(t) = \begin{cases} t\mathbf{i}, & 0 \le t \le 2 \\ 2\mathbf{i} + (t-2)\mathbf{j}, & 2 \le t \le 4 \\ (6-t)\mathbf{i} + 2\mathbf{j}, & 4 \le t \le 6 \\ (8-t)\mathbf{j}, & 6 \le t \le 8 \end{cases}$



$$\int_{C_1} (x + 4\sqrt{y})\, ds = \int_0^2 t\, dt = 2$$

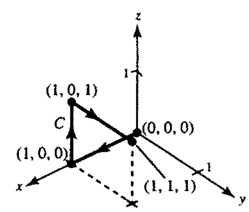
$$\int_{C_2} (x + 4\sqrt{y})\, ds = \int_2^4 \left(2 + 4\sqrt{t-2}\right) dt = 4 + \frac{16\sqrt{2}}{3}$$

$$\int_{C_3} (x + 4\sqrt{y})\, ds = \int_4^6 \left((6-t) + 4\sqrt{2}\right) dt = 2 + 8\sqrt{2}$$

$$\int_{C_4} (x + 4\sqrt{y})\, ds = \int_6^8 4\sqrt{8-t}\, dt = \frac{16\sqrt{2}}{3}$$

$$\int_C (x + 4\sqrt{y})\, ds = 2 + 4 + \frac{16\sqrt{2}}{3} + 2 + 8\sqrt{2} + \frac{16\sqrt{2}}{3} = 8 + \frac{56}{3}\sqrt{2}$$

19. C_1: $(0, 0, 0)$ to $(1, 0, 0)$: $\mathbf{r}(t) = t\mathbf{i}, 0 \le t \le 1, \mathbf{r}'(t) = \mathbf{i}, \|\mathbf{r}'(t)\| = 1$



$$\int_{C_1} (2x + y^2 - z)\, ds = \int_0^1 2t\, dt = t^2\Big]_0^1 = 1$$

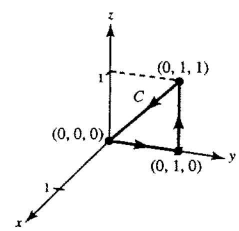
C_2: $(1, 0, 0)$ to $(1, 0, 1)$: $\mathbf{r}(t) = \mathbf{i} + t\mathbf{k}, 0 \le t \le 1, \mathbf{r}'(t) = \mathbf{k}, \|\mathbf{r}'(t)\| = 1$

$$\int_{C_2} (2x + y^2 - z)\, ds = \int_0^1 (2 - t)\, dt = \left[2t - \frac{t^2}{2}\right]_0^1 = \frac{3}{2}$$

C_3: $(1, 0, 1)$ to $(1, 1, 1)$: $\mathbf{r}(t) = \mathbf{i} + t\mathbf{j} + \mathbf{k}, 0 \le t \le 1, \mathbf{r}'(t) = \mathbf{j}, \|\mathbf{r}'(t)\| = 1$

$$\int_{C_3} (2x + y^2 - z)\, ds = \int_0^1 (2 + t^2 - 1)\, dt = \left[t + \frac{t^3}{3}\right]_0^1 = \frac{4}{3}$$

Combining, $\displaystyle\int_C (2x + y^2 - z)\, ds = 1 + \frac{3}{2} + \frac{4}{3} = \frac{23}{6}$



20. C_1: (0, 0, 0) to (0, 1, 0): $\mathbf{r}(t) = t\mathbf{j},\ 0 \le t \le 1,\ \mathbf{r}'(t) = \mathbf{j},\ \|\mathbf{r}'(t)\| = 1$

$$\int_{C_1} (2x + y^2 - z)\,ds = \int_0^1 t^2\,dt = \frac{t^3}{3}\Big]_0^1 = \frac{1}{3}$$

C_2: (0, 1, 0) to (0, 1, 1): $\mathbf{r}(t) = \mathbf{j} + t\mathbf{k},\ 0 \le t \le 1,\ \mathbf{r}'(t) = \mathbf{k},\ \|\mathbf{r}'(t)\| = 1$

$$\int_{C_2} (2x + y^2 - z)\,ds = \int_0^1 (1 - t)\,dt = \left[t - \frac{t^2}{2}\right]_0^1 = \frac{1}{2}$$

C_3: (0, 1, 1) to (0, 0, 0): $\mathbf{r}(t) = (1 - t)\mathbf{j} + (1 - t)\mathbf{k},\ 0 \le t \le 1,\ \mathbf{r}'(t) = -\mathbf{j} - \mathbf{k},\ \|\mathbf{r}'(t)\| = \sqrt{2}$

$$\int_{C_3} (2x + y^2 - z)\,ds = \int_0^1 [(1 - t)^2 - (1 - t)]\sqrt{2}\,dt = \int_0^1 (t^2 - t)\sqrt{2}\,dt = \sqrt{2}\left[\frac{t^3}{3} - \frac{t^2}{2}\right]_0^1 = \frac{-\sqrt{2}}{6}$$

Combining, $\displaystyle\int_C (2x + y^2 - z)\,ds = \frac{1}{3} + \frac{1}{2} - \frac{\sqrt{2}}{6} = \frac{5 - \sqrt{2}}{6}$

21. $\rho(x, y, z) = \dfrac{1}{2}(x^2 + y^2 + z^2)$

$$\mathbf{r}(t) = 3\cos t\,\mathbf{i} + 3\sin t\,\mathbf{j} + 2t\,\mathbf{k},\ 0 \le t \le 4\pi$$

$$\mathbf{r}'(t) = -3\sin t\,\mathbf{i} + 3\cos t\,\mathbf{j} + 2\mathbf{k}$$

$$\|\mathbf{r}'(t)\| = \sqrt{(-3\sin t)^2 + (3\cos t)^2 + (2)^2} = \sqrt{13}$$

$$\text{Mass} = \int_C \rho(x, y, z)\,ds = \int_0^{4\pi} \frac{1}{2}[(3\cos t)^2 + (3\sin t)^2 + (2t)^2]\sqrt{13}\,dt$$

$$= \frac{\sqrt{13}}{2}\int_0^{4\pi} (9 + 4t^2)\,dt = \left[\frac{\sqrt{13}}{2}\left(9t + \frac{4t^3}{3}\right)\right]_0^{4\pi}$$

$$= \frac{2\sqrt{13}\pi}{3}(27 + 64\pi^2) \approx 4973.8$$

22. $\rho(x, y, z) = z$

$$\mathbf{r}(t) = 3\cos t\,\mathbf{i} + 3\sin t\,\mathbf{j} + 2t\,\mathbf{k},\ 0 \le t \le 4\pi$$

$$\mathbf{r}'(t) = -3\sin t\,\mathbf{i} + 3\cos t\,\mathbf{j} + 2\mathbf{k}$$

$$\|\mathbf{r}'(t)\| = \sqrt{(-3\sin t)^2 + (3\cos t)^2 + (2)^2} = \sqrt{13}$$

$$\text{Mass} = \int_C \rho(x, y, z)\,ds = \int_0^{4\pi} 2t\sqrt{13}\,dt = 16\pi^2\sqrt{13}$$

23. $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j},\ 0 \le t \le \pi$

$$\mathbf{r}'(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j},\ \|\mathbf{r}'(t)\| = 1$$

$$\text{Mass} = \int_C \rho(x, y)\,ds = \int_C (x + y)\,ds$$

$$= \int_0^{\pi} (\cos t + \sin t)\,dt$$

$$= \Big[\sin t - \cos t\Big]_0^{\pi}$$

$$= 1 + 1 = 2$$

24. $\mathbf{r}(t) = t^2\mathbf{i} + 2t\mathbf{j},\ 0 \le t \le 1$

$$\mathbf{r}'(t) = 2t\mathbf{i} + 2\mathbf{j},\ \|\mathbf{r}'(t)\| = \sqrt{4t^2 + 4}$$

$$\text{Mass} = \int_C \rho(x, y)\,ds = \int_C \frac{3}{4}y\,ds$$

$$= \int_0^1 \frac{3}{4}(2t)\sqrt{4t^2 + 4}\,dt$$

$$= \int_0^1 3t(t^2 + 1)^{1/2}\,dt$$

$$= (t^2 + 1)^{3/2}\Big]_0^1$$

$$= 2\sqrt{2} - 1$$

25. $\mathbf{r}(t) = t^2\mathbf{i} + 2t\mathbf{j} + t\mathbf{k},\ 1 \le t \le 3$

$$\mathbf{r}'(t) = 2t\mathbf{i} + 2\mathbf{j} + \mathbf{k},\ \|\mathbf{r}'(t)\| = \sqrt{4t^2 + 5}$$

$$\text{Mass} = \int_C \rho(x, y, z)\,ds = \int_C kz\,ds$$

$$= \int_1^3 kt\sqrt{4t^2 + 5}\,dt$$

$$= \frac{k(4t^2 + 5)^{3/2}}{12}\Big]_1^3$$

$$= \frac{k}{12}\left[41\sqrt{41} - 27\right]$$

26. $\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j} + 3t\mathbf{k},\ 0 \le t \le 2\pi$

$\mathbf{r}'(t) = -2\sin t\mathbf{i} + 2\cos t\mathbf{j} + 3\mathbf{k}$

$\|\mathbf{r}'(t)\| = \sqrt{4+9} = \sqrt{13}$

$$\text{Mass} = \int_C \rho(x, y, z)\,ds = \int_C (k + z)\,ds$$
$$= \int_0^{2\pi} (k + 3t)\sqrt{13}\,dt$$
$$= \sqrt{13}\left(kt + \frac{3t^2}{2}\right)\Bigg]_0^{2\pi}$$
$$= \sqrt{13}(2\pi k + 6\pi^2)$$

27. $\mathbf{F}(x, y) = xy\mathbf{i} + y\mathbf{j}$

C: $\mathbf{r}(t) = 4t\mathbf{i} + t\mathbf{j},\ 0 \le t \le 1$

$\mathbf{F}(t) = 4t^2\mathbf{i} + t\mathbf{j}$

$\mathbf{r}'(t) = 4\mathbf{i} + \mathbf{j}$

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 (16t^2 + t)\,dt$$
$$= \left[\frac{16}{3}t^3 + \frac{1}{2}t^2\right]_0^1 = \frac{35}{6}$$

28. $\mathbf{F}(x, y) = xy\mathbf{i} + y\mathbf{j}$

C: $\mathbf{r}(t) = 4\cos t\mathbf{i} + 4\sin t\mathbf{j},\ 0 \le t \le \frac{\pi}{2}$

$\mathbf{F}(t) = 16\sin t\cos t\mathbf{i} + 4\sin t\mathbf{j}$

$\mathbf{r}'(t) = -4\sin t\mathbf{i} + 4\cos t\mathbf{j}$

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{\pi/2} (-64\sin^2 t\cos t + 16\sin t\cos t)\,dt$$
$$= \left[-\frac{64}{3}\sin^3 t + 8\sin^2 t\right]_0^{\pi/2} = -\frac{40}{3}$$

29. $\mathbf{F}(x, y) = 3x\mathbf{i} + 4y\mathbf{i}$

C: $\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j},\ 0 \le t \le \frac{\pi}{2}$

$\mathbf{F}(t) = 6\cos t\mathbf{i} + 8\sin t\mathbf{j}$

$\mathbf{r}'(t) = -2\sin t\mathbf{i} + 2\cos t\mathbf{j}$

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{\pi/2} (-12\sin t\cos t + 16\sin t\cos t)\,dt$$
$$= \left[2\sin^2 t\right]_0^{\pi/2} = 2$$

30. $\mathbf{F}(x, y) = 3x\mathbf{i} + 4y\mathbf{j}$

C: $\mathbf{r}(t) = t\mathbf{i} + \sqrt{4 - t^2}\mathbf{j},\ -2 \le t \le 2$

$\mathbf{F}(t) = 3t\mathbf{i} + 4\sqrt{4 - t^2}\mathbf{j}$

$\mathbf{r}'(t) = \mathbf{i} - \dfrac{t}{\sqrt{4 - t^2}}\mathbf{j}$

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_{-2}^{2} (3t - 4t)\,dt = \left[-\frac{t^2}{2}\right]_{-2}^{2} = 0$$

31. $\mathbf{F}(x, y, z) = x^2y\mathbf{i} + (x - z)\mathbf{j} + xyz\mathbf{k}$

C: $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + 2\mathbf{k},\ 0 \le t \le 1$

$\mathbf{F}(t) = t^4\mathbf{i} + (t - 2)\mathbf{j} + 2t^3\mathbf{k}$

$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j}$

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 [t^4 + 2t(t - 2)]\,dt$$
$$= \left[\frac{t^5}{5} + \frac{2t^3}{3} - 2t^2\right]_0^1 = -\frac{17}{15}$$

32. $\mathbf{F}(x, y, z) = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$

C: $\mathbf{r}(t) = 2\sin t\mathbf{i} + 2\cos t\mathbf{j} + \frac{1}{2}t^2\mathbf{k},\ 0 \le t \le \pi$

$\mathbf{F}(t) = 4\sin^2 t\mathbf{i} + 4\cos^2 t\mathbf{j} + \frac{1}{4}t^4\mathbf{k}$

$\mathbf{r}'(t) = 2\cos t\mathbf{i} - 2\sin t\mathbf{j} + t\mathbf{k}$

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{\pi}\left(8\sin^2 t\cos t - 8\cos^2 t\sin t + \frac{1}{4}t^5\right)dt$$
$$= \left[\frac{8}{3}\sin^3 t + \frac{8}{3}\cos^3 t + \frac{t^6}{24}\right]_0^{\pi}$$
$$= -\frac{8}{3} + \frac{\pi^6}{24} - \frac{8}{3} = \frac{\pi^6}{24} - \frac{16}{3}$$

33. $\mathbf{F}(x, y, z) = x^2z\mathbf{i} + 6y\mathbf{j} + yz^2\mathbf{k}$

$\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \ln t\mathbf{k},\ 1 \le t \le 3$

$\mathbf{F}(t) = t^2\ln t\mathbf{i} + 6t^2\mathbf{j} + t^2\ln^2 t\mathbf{k}$

$d\mathbf{r} = \left(\mathbf{i} + 2t\mathbf{j} + \frac{1}{t}\mathbf{k}\right)dt$

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_1^3 [t^2\ln t + 12t^3 + t(\ln t)^2]\,dt$$
$$\approx 249.49$$

34. $\mathbf{F}(x, y, z) = \dfrac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\sqrt{x^2 + y^2 + z^2}}$

$\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + e^t\mathbf{k},\ 0 \le t \le 2$

$\mathbf{F}(t) = \dfrac{t\mathbf{i} + t\mathbf{j} + e^t\mathbf{k}}{\sqrt{2t^2 + e^{2t}}}$

$d\mathbf{r} = (\mathbf{i} + \mathbf{j} + e^t\mathbf{k})\, dt$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^2 \frac{1}{\sqrt{2t^2 + e^{2t}}}(2t + e^{2t})\, dt \approx 6.91$$

35. $\mathbf{F}(x, y) = -x\mathbf{i} - 2y\mathbf{j}$

C: $y = x^3$ from $(0, 0)$ to $(2, 8)$

$\mathbf{r}(t) = t\mathbf{i} + t^3\mathbf{j},\ 0 \le t \le 2$

$\mathbf{r}'(t) = \mathbf{i} + 3t^2\mathbf{j}$

$\mathbf{F}(t) = -t\mathbf{i} - 2t^3\mathbf{j}$

$\mathbf{F} \cdot \mathbf{r}' = -t - 6t^5$

$$\text{Work} = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^2 (-t - 6t^5)\, dt$$

$$= \left[-\frac{1}{2}t^2 - t^6\right]_0^2 = -66$$

36. $\mathbf{F}(x, y) = x^2\mathbf{i} - xy\mathbf{j}$

C: $x = \cos^3 t,\ y = \sin^3 t$ from $(1, 0)$ to $(0, 1)$

$\mathbf{r}(t) = \cos^3 t\mathbf{i} + \sin^3 t\mathbf{j},\ 0 \le t \le \dfrac{\pi}{2}$

$\mathbf{r}'(t) = -3\cos^2 t \sin t\mathbf{i} + 3\sin^2 t \cos t\mathbf{j}$

$\mathbf{F}(t) = \cos^6 t\mathbf{i} - \cos^3 t \sin^3 t\mathbf{j}$

$$\begin{aligned}
\mathbf{F} \cdot \mathbf{r}' &= -3\cos^8 t \sin t - 3\cos^4 t \sin^5 t \\
&= -3\cos^4 t \sin t(\cos^4 t + \sin^4 t) \\
&= -3\cos^4 t \sin t[\cos^4 t + (1 - \cos^2 t)^2] \\
&= -3\cos^4 t \sin t(2\cos^4 t - 2\cos^2 t + 1) \\
&= -6\cos^8 t \sin t + 6\cos^6 t \sin t - 3\cos^4 t \sin t
\end{aligned}$$

$$\text{Work} = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{\pi/2} [-6\cos^8 t \sin t + 6\cos^6 t \sin t - 3\cos^4 t \sin t]\, dt$$

$$= \left[\frac{2\cos^9 t}{3} - \frac{6\cos^7 t}{7} + \frac{3\cos^5 t}{5}\right]_0^{\pi/2} = -\frac{43}{105}$$

37. $\mathbf{F}(x, y) = 2x\mathbf{i} + y\mathbf{j}$

C: counterclockwise around the triangle whose vertices are $(0, 0)$, $(1, 0)$, $(1, 1)$

$$\mathbf{r}(t) = \begin{cases} t\mathbf{i}, & 0 \le t \le 1 \\ \mathbf{i} + (t - 1)\mathbf{j}, & 1 \le t \le 2 \\ (3 - t)\mathbf{i} + (3 - t)\mathbf{j}, & 2 \le t \le 3 \end{cases}$$

On C_1: $\mathbf{F}(t) = 2t\mathbf{i},\ \mathbf{r}'(t) = \mathbf{i}$

$$\text{Work} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 2t\, dt = 1$$

On C_2: $\mathbf{F}(t) = 2\mathbf{i} + (t - 1)\mathbf{j},\ \mathbf{r}'(t) = \mathbf{j}$

$$\text{Work} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_1^2 (t - 1)\, dt = \frac{1}{2}$$

On C_3: $\mathbf{F}(t) = 2(3 - t)\mathbf{i} + (3 - t)\mathbf{j},\ \mathbf{r}'(t) = -\mathbf{i} - \mathbf{j}$

$$\text{Work} = \int_{C_3} \mathbf{F} \cdot d\mathbf{r} = \int_2^3 [-2(3 - t) - (3 - t)]\, dt = -\frac{3}{2}$$

$$\text{Total work} = \int_C \mathbf{F} \cdot d\mathbf{r} = 1 + \frac{1}{2} - \frac{3}{2} = 0$$

38. $\mathbf{F}(x, y) = -y\mathbf{i} - x\mathbf{j}$

C: counterclockwise along the semicircle $y = \sqrt{4 - x^2}$ from $(2, 0)$ to $(-2, 0)$

$$\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j}, \quad 0 \le t \le \pi$$

$$\mathbf{r}'(t) = -2\sin t\mathbf{i} + 2\cos t\mathbf{j}$$

$$\mathbf{F}(t) = -2\sin t\mathbf{i} - 2\cos t\mathbf{j}$$

$$\mathbf{F} \cdot \mathbf{r}' = 4\sin^2 t - 4\cos^2 t = -4\cos 2t$$

$$\text{Work} = \int_C \mathbf{F} \cdot d\mathbf{r} = -4\int_0^{\pi} \cos 2t\,dt = \Big[-2\sin 2t\Big]_0^{\pi} = 0$$

39. $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} - 5z\mathbf{k}$

C: $\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j} + t\mathbf{k}, \ 0 \le t \le 2\pi$

$$\mathbf{r}'(t) = -2\sin t\mathbf{i} + 2\cos t\mathbf{j} + \mathbf{k}$$

$$\mathbf{F}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j} - 5t\mathbf{k}$$

$$\mathbf{F} \cdot \mathbf{r}' = -5t$$

$$\text{Work} = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} -5t\,dt = -10\pi^2$$

40. $\mathbf{F}(x, y, z) = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$

C: line from $(0, 0, 0)$ to $(5, 3, 2)$

$$\mathbf{r}(t) = 5t\mathbf{i} + 3t\mathbf{j} + 2t\mathbf{k}, \quad 0 \le t \le 1$$

$$\mathbf{r}'(t) = 5\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$$

$$\mathbf{F}(t) = 6t^2\mathbf{i} + 10t^2\mathbf{j} + 15t^2\mathbf{k}$$

$$\mathbf{F} \cdot \mathbf{r}' = 90t^2$$

$$\text{Work} = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 90t^2\,dt = 30$$

41. $\mathbf{r}(t) = 3\sin t\mathbf{i} + 3\cos t\mathbf{j} + \dfrac{10}{2\pi}t\mathbf{k}, \ 0 \le t \le 2\pi$

$$\mathbf{F} = 150\mathbf{k}$$

$$d\mathbf{r} = \left(3\cos t\mathbf{i} - 3\sin t\mathbf{j} + \frac{10}{2\pi}\mathbf{k}\right)dt$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \frac{1500}{2\pi}\,dt = \left[\frac{1500}{2\pi}t\right]_0^{2\pi} = 1500 \text{ ft} \cdot \text{lb}$$

42. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}, \ 0 \le t \le 1$

$$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j}$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} \approx \frac{1 - 0}{3(4)}[5 + 4(4) + 2(4) + 4(6) + 11] = \frac{16}{3}$$

(x, y)	$(0, 0)$	$\left(\frac{1}{4}, \frac{1}{16}\right)$	$\left(\frac{1}{2}, \frac{1}{4}\right)$	$\left(\frac{3}{4}, \frac{9}{16}\right)$	$(1, 1)$
$\mathbf{F}(x, y)$	$5\mathbf{i}$	$3.5\mathbf{i} + \mathbf{j}$	$2\mathbf{i} + 2\mathbf{j}$	$1.5\mathbf{i} + 3\mathbf{j}$	$\mathbf{i} + 5\mathbf{j}$
$\mathbf{r}'(t)$	$\mathbf{i}$	$\mathbf{i} + 0.5\mathbf{j}$	$\mathbf{i} + \mathbf{j}$	$\mathbf{i} + 1.5\mathbf{j}$	$\mathbf{i} + 2\mathbf{j}$
$\mathbf{F} \cdot \mathbf{r}'$	5	4	4	6	11

43. $\mathbf{F}(x, y) = x^2\mathbf{i} + xy\mathbf{j}$

(a) $\mathbf{r}_1(t) = 2t\mathbf{i} + (t - 1)\mathbf{j}, 1 \le t \le 3$

$$\mathbf{r}_1'(t) = 2\mathbf{i} + \mathbf{j}$$

$$\mathbf{F}(t) = 4t^2\mathbf{i} + 2t(t - 1)\mathbf{j}$$

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_1^3 (8t^2 + 2t(t - 1))\,dt = \frac{236}{3}$$

Both paths join $(2, 0)$ and $(6, 2)$. The integrals are negatives of each other because the orientations are different.

(b) $\mathbf{r}_2(t) = 2(3 - t)\mathbf{i} + (2 - t)\mathbf{j}, 0 \le t \le 2$

$$\mathbf{r}_2'(t) = -2\mathbf{i} - \mathbf{j}$$

$$\mathbf{F}(t) = 4(3 - t)^2\mathbf{i} + 2(3 - t)(2 - t)\mathbf{j}$$

$$\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_0^2 [-8(3 - t)^2 - 2(3 - t)(2 - t)]\,dt = -\frac{236}{3}$$

44. $\mathbf{F}(x, y) = x^2y\mathbf{i} + xy^{3/2}\mathbf{j}$

(a) $\mathbf{r}_1(t) = (t + 1)\mathbf{i} + t^2\mathbf{j}, 0 \le t \le 2$

$$\mathbf{r}_1'(t) = \mathbf{i} + 2t\mathbf{j}$$

$$\mathbf{F}(t) = (t + 1)^2t^2\mathbf{i} + (t + 1)t^3\mathbf{j}$$

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_0^2 [(t + 1)^2t^2 + 2t^4(t + 1)]\,dt = \frac{256}{5}$$

—CONTINUED—

44. —CONTINUED—

(b) $\mathbf{r}_2(t) = (1 + 2\cos t)\mathbf{i} + 4\cos^2 t\mathbf{j},\ 0 \le t \le \frac{\pi}{2}$

$\mathbf{r}_2'(t) = -2\sin t\mathbf{i} - 8\cos t\sin t\mathbf{j}$

$\mathbf{F}(t) = (1 + 2\cos t)^2(4\cos^2 t)\mathbf{i} + (1 + 2\cos t)(8\cos^3 t)\mathbf{j}$

$$\int_{C_2} \mathbf{F}\cdot d\mathbf{r} = \int_0^{\pi/2}\Big[(1 + 2\cos t)^2(4\cos^2 t)(-2\sin t) - 8\cos t\sin t(1 + 2\cos t)(8\cos^3 t)\,dt = -\frac{256}{5}$$

Both paths join (1, 0) and (3, 4). The integrals are negatives of each other because the orientations are different.

45. $\mathbf{F}(x, y) = y\mathbf{i} - x\mathbf{j}$

C: $\mathbf{r}(t) = t\mathbf{i} - 2t\mathbf{j}$

$\mathbf{r}'(t) = \mathbf{i} - 2\mathbf{j}$

$\mathbf{F}(t) = -2t\mathbf{i} - t\mathbf{j}$

$\mathbf{F}\cdot\mathbf{r}' = -2t + 2t = 0$

Thus, $\int_C \mathbf{F}\cdot d\mathbf{r} = 0.$

46. $\mathbf{F}(x, y) = -3y\mathbf{i} + x\mathbf{j}$

C: $\mathbf{r}(t) = t\mathbf{i} - t^3\mathbf{j}$

$\mathbf{r}'(t) = \mathbf{i} - 3t^2\mathbf{j}$

$\mathbf{F}(t) = 3t^3\mathbf{i} + t\mathbf{j}$

$\mathbf{F}\cdot\mathbf{r}' = 3t^3 - 3t^3 = 0$

Thus, $\int_C \mathbf{F}\cdot d\mathbf{r} = 0.$

47. $\mathbf{F}(x, y) = (x^3 - 2x^2)\mathbf{i} + \left(x - \frac{y}{2}\right)\mathbf{j}$

C: $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}$

$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j}$

$\mathbf{F}(t) = (t^3 - 2t^2)\mathbf{i} + \left(t - \frac{t^2}{2}\right)\mathbf{j}$

$\mathbf{F}\cdot\mathbf{r}' = (t^3 - 2t^2) + 2t\left(t - \frac{t^2}{2}\right) = 0$

Thus, $\int_C \mathbf{F}\cdot d\mathbf{r} = 0.$

48. $\mathbf{F}(x, y) = x\mathbf{i} + y\mathbf{j}$

C: $\mathbf{r}(t) = 3\sin t\mathbf{i} + 3\cos t\mathbf{j}$

$\mathbf{r}'(t) = 3\cos t\mathbf{i} - 3\sin t\mathbf{j}$

$\mathbf{F}(t) = 3\sin t\mathbf{i} + 3\cos t\mathbf{j}$

$\mathbf{F}\cdot\mathbf{r}' = 9\sin t\cos t - 9\sin t\cos t = 0$

Thus, $\int_C \mathbf{F}\cdot d\mathbf{r} = 0.$

49. $x = 2t,\ y = 10t,\ 0 \le t \le 1 \implies y = 5x$ or $x = \frac{y}{5},\ 0 \le y \le 10$

$$\int_C (x + 3y^2)\,dy = \int_0^{10}\left(\frac{y}{5} + 3y^2\right)dy = \left[\frac{y^2}{10} + y^3\right]_0^{10} = 1010$$

50. $x = 2t,\ y = 10t,\ 0 \le t \le 1 \implies y = 5x,\ 0 \le x \le 2$

$$\int_C (x + 3y^2)\,dx = \int_0^2 (x + 75x^2)\,dx = \left[\frac{x^2}{2} + 25x^3\right]_0^2 = 202$$

51. $x = 2t,\ y = 10t,\ 0 \le t \le 1 \implies x = \frac{y}{5},\ 0 \le y \le 10,\ dx = \frac{1}{5}\,dy$

$$\int_C xy\,dx + y\,dy = \int_0^{10}\left(\frac{y^2}{25} + y\right)dy = \left[\frac{y^3}{75} + \frac{y^2}{2}\right]_0^{10} = \frac{190}{3} \quad \text{OR}$$

$y = 5x,\ dy = 5\,dx,\ 0 \le x \le 2$

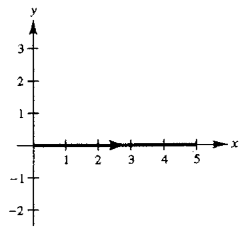
$$\int_C xy\,dx + y\,dy = \int_0^2 (5x^2 + 25x)\,dx = \left[\frac{5x^3}{3} + \frac{25x^2}{2}\right]_0^2 = \frac{190}{3}$$

52. $x = 2t,\ y = 10t,\ 0 \le t \le 1 \implies y = 5x,\ dy = 5\,dx,\ 0 \le x \le 2$

$$\int_C (3y - x)\,dx + y^2\,dy = \int_0^2 (3(5x) - x)\,dx + (5x)^2 5\,dx = \int_0^2 (14x + 125x^2)\,dx$$

$$= \left[7x^2 + \frac{125}{3}x^3\right]_0^2 = 28 + \frac{125}{3}(8) = \frac{1084}{3}$$

53. $\mathbf{r}(t) = t\mathbf{i},\ 0 \le t \le 5$

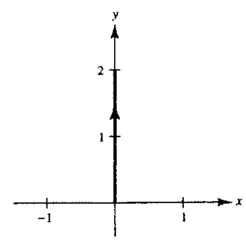


$x(t) = t,\quad y(t) = 0$

$dx = dt,\quad dy = 0$

$$\int_C (2x - y)\,dx + (x + 3y)\,dy = \int_0^5 2t\,dt = 25$$

54. $\mathbf{r}(t) = t\mathbf{j},\ 0 \le t \le 2$

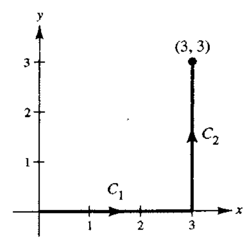


$x(t) = 0,\ y(t) = t$

$dx = 0,\ dy = dt$

$$\int_C (2x - y)\,dx + (x + 3y)\,dy = \int_0^2 3t\,dt = \left[\frac{3}{2}t^2\right]_0^2 = 6$$

55. $\mathbf{r}(t) = \begin{cases} t\mathbf{i}, & 0 \le t \le 3 \\ 3\mathbf{i} + (t - 3)\mathbf{j}, & 3 \le t \le 6 \end{cases}$



C_1: $x(t) = t,\ y(t) = 0,$

$dx = dt,\ dy = 0$

$$\int_{C_1} (2x - y)\,dx + (x + 3y)\,dy = \int_0^3 2t\,dt = 9$$

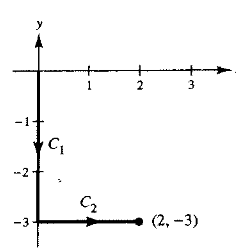
C_2: $x(t) = 3,\ y(t) = t - 3$

$dx = 0,\ dy = dt$

$$\int_{C_2} (2x - y)\,dx + (x + 3y)\,dy = \int_3^6 [3 + 3(t - 3)]\,dt = \left[\frac{3t^2}{2} - 6t\right]_3^6 = \frac{45}{2}$$

$$\int_C (2x - y)\,dx + (x + 3y)\,dy = 9 + \frac{45}{2} = \frac{63}{2}$$

56. $\mathbf{r}(t) = \begin{cases} -t\mathbf{j}, & 0 \le t \le 3 \\ (t - 3)\mathbf{i} - 3\mathbf{j}, & 3 \le t \le 5 \end{cases}$



C_1: $x(t) = 0,\ y(t) = -t$

$dx = 0,\ dy = -dt$

$$\int_{C_1} (2x - y)\,dx + (x + 3y)\,dy = \int_0^3 3t\,dt = \frac{27}{2}$$

C_2: $x(t) = t - 3,\ y(t) = -3$

$dx = dt,\ dy = 0$

$$\int_{C_2} (2x - y)\,dx + (x + 3y)\,dy = \int_3^5 [2(t - 3) + 3]\,dt = \left[(t - 3)^2 + 3t\right]_3^5 = 10$$

$$\int_C (2x - y)\,dx + (x + 3y)\,dy = \frac{27}{2} + 10 = \frac{47}{2}$$

57. $x(t) = t,\ y(t) = 1 - t^2,\ 0 \le t \le 1,\ dx = dt,\ dy = -2t\,dt$

$$\int_C (2x - y)\,dx + (x + 3y)\,dy = \int_0^1 [(2t - 1 + t^2) + (t + 3 - 3t^2)(-2t)]\,dt$$

$$= \int_0^1 (6t^3 - t^2 - 4t - 1)\,dt = \left[\frac{3t^4}{2} - \frac{t^3}{3} - 2t^2 - t\right]_0^1 = -\frac{11}{6}$$

58. $x(t) = t,\ y(t) = t^{3/2},\ 0 \le t \le 4,\ dx = dt,\ dy = \frac{3}{2}t^{1/2}\,dt$

$$\int_C (2x - y)\,dx + (x + 3y)\,dy = \int_0^4 \left[(2t - t^{3/2}) + (t + 3t^{3/2})\left(\frac{3}{2}t^{1/2}\right)\right]dt$$

$$= \int_0^4 \left(\frac{9}{2}t^2 + \frac{1}{2}t^{3/2} + 2t\right)dt = \left[\frac{3}{2}t^3 + \frac{1}{5}t^{5/2} + t^2\right]_0^4 = 96 + \frac{1}{5}(32) + 16 = \frac{592}{5}$$

59. $x(t) = t,\ y(t) = 2t^2,\ 0 \le t \le 2$

$dx = dt,\ dy = 4t\,dt$

$$\int_C (2x - y)\,dx + (x + 3y)\,dy = \int_0^2 (2t - 2t^2)\,dt + (t + 6t^2)4t\,dt$$

$$= \int_0^2 (24t^3 + 2t^2 + 2t)\,dt = \left[6t^4 + \frac{2}{3}t^3 + t^2\right]_0^2 = \frac{316}{3}$$

60. $x(t) = 4\sin t,\ y(t) = 3\cos t,\ 0 \le t \le \frac{\pi}{2}$

$dx = 4\cos t\,dt,\ dy = -3\sin t\,dt$

$$\int_C (2x - y)\,dx + (x + 3y)\,dy = \int_0^{\pi/2} (8\sin t - 3\cos t)(4\cos t)\,dt + (4\sin t + 9\cos t)(-3\sin t)\,dt$$

$$= \int_0^{\pi/2} (5\sin t\cos t - 12\cos^2 t - 12\sin^2 t)\,dt$$

$$= \left[\frac{5}{2}\sin^2 t - 12t\right]_0^{\pi/2} = \frac{5}{2} - 6\pi$$

61. $f(x, y) = h$

C: line from $(0, 0)$ to $(3, 4)$

$\mathbf{r} = 3t\mathbf{i} + 4t\mathbf{j},\ 0 \le t \le 1$

$\mathbf{r}'(t) = 3\mathbf{i} + 4\mathbf{j}$

$\|\mathbf{r}'(t)\| = 5$

Lateral surface area:

$$\int_C f(x, y)\,ds = \int_0^1 5h\,dt = 5h$$

62. $f(x, y) = y$

C: line from $(0, 0)$ to $(4, 4)$

$\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j},\ 0 \le t \le 4$

$\mathbf{r}'(t) = \mathbf{i} + \mathbf{j}$

$\|\mathbf{r}'(t)\| = \sqrt{2}$

Lateral surface area:

$$\int_C f(x, y)\,ds = \int_0^4 t\left(\sqrt{2}\right)dt = 8\sqrt{2}$$

63. $f(x, y) = xy$

C: $x^2 + y^2 = 1$ from $(1, 0)$ to $(0, 1)$

$$\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j},\ 0 \le t \le \frac{\pi}{2}$$

$$\mathbf{r}'(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j}$$

$$\|\mathbf{r}'(t)\| = 1$$

Lateral surface area:

$$\int_C f(x, y)\,ds = \int_0^{\pi/2} \cos t \sin t\,dt = \left[\frac{\sin^2 t}{2}\right]_0^{\pi/2} = \frac{1}{2}$$

64. $f(x, y) = x + y$

C: $x^2 + y^2 = 1$ from $(1, 0)$ to $(0, 1)$

$$\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j},\ 0 \le t \le \frac{\pi}{2}$$

$$\mathbf{r}'(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j}$$

$$\|\mathbf{r}'(t)\| = 1$$

Lateral surface area:

$$\int_C f(x, y)\,ds = \int_0^{\pi/2} (\cos t + \sin t)\,dt = \Big[\sin t - \cos t\Big]_0^{\pi/2} = 2$$

65. $f(x, y) = h$

C: $y = 1 - x^2$ from $(1, 0)$ to $(0, 1)$

$$\mathbf{r}(t) = (1 - t)\mathbf{i} + [1 - (1 - t)^2]\mathbf{j},\ 0 \le t \le 1$$

$$\mathbf{r}'(t) = -\mathbf{i} + 2(1 - t)\mathbf{j}$$

$$\|\mathbf{r}'(t)\| = \sqrt{1 + 4(1 - t)^2}$$

Lateral surface area:

$$\int_C f(x, y)\,ds = \int_0^1 h\sqrt{1 + 4(1 - t)^2}\,dt$$

$$= -\frac{h}{4}\left[2(1 - t)\sqrt{1 + 4(1 - t)^2} + \ln\left|2(1 - t) + \sqrt{1 + 4(1 - t)^2}\right|\right]_0^1$$

$$= \frac{h}{4}\left[2\sqrt{5} + \ln\left(2 + \sqrt{5}\right)\right] \approx 1.4789h$$

66. $f(x, y) = y + 1$

C: $y = 1 - x^2$ from $(1, 0)$ to $(0, 1)$

$$\mathbf{r}(t) = (1 - t)\mathbf{i} + [1 - (1 - t)^2]\mathbf{j},\ 0 \le t \le 1$$

$$\mathbf{r}'(t) = -\mathbf{i} + 2(1 - t)\mathbf{j}$$

$$\|\mathbf{r}'(t)\| = \sqrt{1 + 4(1 - t)^2}$$

Lateral surface area:

$$\int_C f(x, y)\,ds = \int_0^1 [2 - (1 - t)^2]\sqrt{1 + 4(1 - t)^2}\,dt$$

$$= 2\int_0^1 \sqrt{1 + 4(1 - t)^2}\,dt - \int_0^1 (1 - t)^2\sqrt{1 + 4(1 - t)^2}\,dt$$

$$= -\frac{1}{2}\left[2(1 - t)\sqrt{1 + 4(1 - t)^2} + \ln\left|2(1 - t) + \sqrt{1 + 4(1 - t)^2}\right|\right]_0^1$$

$$+ \frac{1}{64}\left[2(1 - t)[2(4)(1 - t)^2 + 1]\sqrt{1 + 4(1 - t)^2} - \ln\left|2(1 - t) + \sqrt{1 + 4(1 - t)^2}\right|\right]_0^1$$

$$= \frac{1}{2}\left[2\sqrt{5} + \ln\left(2 + \sqrt{5}\right)\right] - \frac{1}{64}\left[18\sqrt{5} - \ln\left(2 + \sqrt{5}\right)\right]$$

$$= \frac{23}{32}\sqrt{5} + \frac{33}{64}\ln\left(2 + \sqrt{5}\right) = \frac{1}{64}\left[46\sqrt{5} + 33\ln\left(2 + \sqrt{5}\right)\right] \approx 2.3515$$

67. $f(x, y) = xy$

C: $y = 1 - x^2$ from $(1, 0)$ to $(0, 1)$

You could parameterize the curve C as in Exercises 65 and 66. Alternatively, let $x = \cos t$, then:

$$y = 1 - \cos^2 t = \sin^2 t$$

$$\mathbf{r}(t) = \cos t\mathbf{i} + \sin^2 t\mathbf{j},\ 0 \le t \le \frac{\pi}{2}$$

$$\mathbf{r}'(t) = -\sin t\mathbf{i} + 2\sin t\cos t\mathbf{j}$$

$$\|\mathbf{r}'(t)\| = \sqrt{\sin^2 t + 4\sin^2 t\cos^2 t} = \sin t\sqrt{1 + 4\cos^2 t}$$

Lateral surface area:

$$\int_C f(x, y)\,ds = \int_0^{\pi/2} \cos t\sin^2 t\left(\sin t\sqrt{1 + 4\cos^2 t}\right) dt = \int_0^{\pi/2} \sin^2 t\left[(1 + 4\cos^2 t)^{1/2}\sin t\cos t\right] dt$$

Let $u = \sin^2 t$ and $dv = (1 + 4\cos^2 t)^{1/2}\sin t\cos t$, then $du = 2\sin t\cos t\,dt$ and $v = -\frac{1}{12}(1 + 4\cos^2 t)^{3/2}$.

$$\int_C f(x, y)\,ds = \left[-\frac{1}{12}\sin^2 t(1 + 4\cos^2 t)^{3/2}\right]_0^{\pi/2} + \frac{1}{6}\int_0^{\pi/2}(1 + 4\cos^2 t)^{3/2}\sin t\cos t\,dt$$

$$= \left[-\frac{1}{12}\sin^2 t(1 + 4\cos^2 t)^{3/2} - \frac{1}{120}(1 + 4\cos^2 t)^{5/2}\right]_0^{\pi/2}$$

$$= \left(-\frac{1}{12} - \frac{1}{120}\right) + \frac{1}{120}(5)^{5/2} = \frac{1}{120}\left(25\sqrt{5} - 11\right) \approx 0.3742$$

68. $f(x, y) = x^2 - y^2 + 4$

C: $x^2 + y^2 = 4$

$$\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j},\ 0 \le t \le 2\pi$$

$$\mathbf{r}'(t) = -2\sin t\mathbf{i} + 2\cos t\mathbf{j}$$

$$\|\mathbf{r}'(t)\| = 2$$

Lateral surface area:

$$\int_C f(x, y)\,ds = \int_0^{2\pi}(4\cos^2 t - 4\sin^2 t + 4)(2)\,dt = 8\int_0^{2\pi}(1 + \cos 2t)\,dt = \left[8\left(t + \frac{1}{2}\sin 2t\right)\right]_0^{2\pi} = 16\pi$$

69. (a) $f(x, y) = 1 + y^2$

$$\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j},\ 0 \le t \le 2\pi$$

$$\mathbf{r}'(t) = -2\sin t\mathbf{i} + 2\cos t\mathbf{j}$$

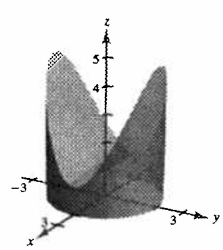
$$\|\mathbf{r}'(t)\| = 2$$

$$S = \int_C f(x, y)\,ds = \int_0^{2\pi}(1 + 4\sin^2 t)(2)\,dt$$

$$= \Big[2t + 4(t - \sin t\cos t)\Big]_0^{2\pi} = 12\pi \approx 37.70\text{ cm}^2$$

(b) $0.2(12\pi) = \dfrac{12\pi}{5} \approx 7.54\text{ cm}^3$

(c)



70. $f(x, y) = 20 + \dfrac{1}{4}x$

C: $y = x^{3/2}$, $0 \le x \le 40$

$\mathbf{r}(t) = t\mathbf{i} + t^{3/2}\mathbf{j}$, $0 \le t \le 40$

$\mathbf{r}'(t) = \mathbf{i} + \dfrac{3}{2}t^{1/2}\mathbf{j}$

$\|\mathbf{r}'(t)\| = \sqrt{1 + \left(\dfrac{9}{4}\right)t}$

Lateral surface area: $\displaystyle\int_C f(x, y)\, ds = \int_0^{40}\left(20 + \frac{1}{4}t\right)\sqrt{1 + \left(\frac{9}{4}\right)t}\, dt$

Let $u = \sqrt{1 + \left(\frac{9}{4}\right)t}$, then $t = \frac{4}{9}(u^2 - 1)$ and $dt = \frac{8}{9}u\, du$.

$$\int_0^{40}\left(20 + \frac{1}{4}t\right)\sqrt{1 + \left(\frac{9}{4}\right)t}\, dt = \int_1^{\sqrt{91}}\left[20 + \frac{1}{9}(u^2 - 1)\right](u)\left(\frac{8}{9}u\right) du = \frac{8}{81}\int_1^{\sqrt{91}}(u^4 + 179u^2)\, du$$

$$= \frac{8}{81}\left[\frac{u^5}{5} + \frac{179u^3}{3}\right]_1^{\sqrt{91}} = \frac{850{,}304\sqrt{91} - 7184}{1215} \approx 6670.12$$

71. $\mathbf{r}(t) = a\cos t\mathbf{i} + a\sin t\mathbf{j}$, $0 \le t \le 2\pi$

$\mathbf{r}'(t) = -a\sin t\mathbf{i} + a\cos t\mathbf{j}$, $\|\mathbf{r}'(t)\| = a$

$$I_x = \int_C y^2\rho(x, y)\, ds = \int_0^{2\pi}(a^2\sin^2 t)(1)a\, dt = a^3\int_0^{2\pi}\sin^2 t\, dt = a^3\pi$$

$$I_y = \int_C x^2\rho(x, y)\, ds = \int_0^{2\pi}(a^2\cos^2 t)(1)a\, dt = a^3\int_0^{2\pi}\cos^2 t\, dt = a^3\pi$$

72. $\mathbf{r}(t) = a\cos t\mathbf{i} + a\sin t\mathbf{j}$, $0 \le t \le 2\pi$

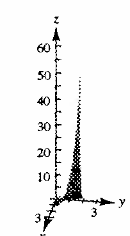
$\mathbf{r}'(t) = -a\sin t\mathbf{i} + a\cos t\mathbf{j}$, $\|\mathbf{r}'(t)\| = a$

$$I_x = \int_C y^2\rho(x, y)\, ds = \int_0^{2\pi}(a^2\sin^2 t)(\sin t)a^2\, dt = a^4\int_0^{2\pi}\sin^3 t\, dt = 0$$

$$I_y = \int_C x^2\rho(x, y)\, ds = \int_0^{2\pi}(a^2\cos^2 t)(\sin t)a^2\, dt = a^4\int_0^{2\pi}\cos^2 t\sin t\, dt = 0$$

73. $S \approx 25$

Matches (b)

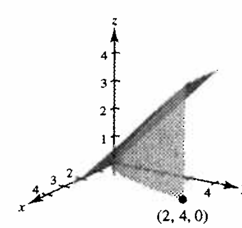


74. $f(x, y) = y$

C: $y = x^2$ from $(0, 0)$ to $(2, 4)$

$S \approx 8$

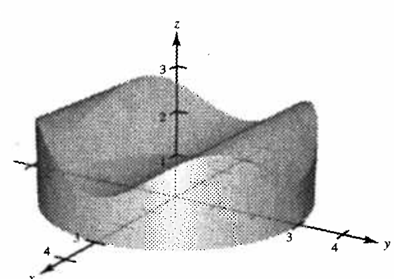
Matches (c)



75. (a) Graph of: $\mathbf{r}(t) = 3\cos t\mathbf{i} + 3\sin t\mathbf{j} + (1 + \sin^2 2t)\mathbf{k}$, $0 \le t \le 2\pi$

For $y = b$ constant, $3\sin t = b \implies \sin t = \dfrac{b}{3}$ and

$$1 + \sin^2 2t = 1 + (2\sin t\cos t)^2 = 1 + 4\sin^2 t\cos^2 t = 1 + 4\sin^2 t(1 - \sin^2 t) = 1 + \frac{4}{9}b^2\left(1 - \frac{b^2}{9}\right).$$



—CONTINUED—

75. —CONTINUED—

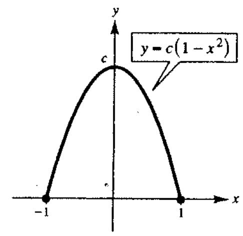
(b) Consider the portion of the surface in the first quadrant. The curve $z = 1 + \sin^2 2t$ is over the curve $\mathbf{r}_1(t) = 3\cos t\mathbf{i} + 3\sin t\mathbf{j},\ 0 \le t \le \pi/2$. Hence, the total lateral surface area is

$$4\int_C f(x, y)\, ds = 4\int_0^{\pi/2} (1 + \sin^2 2t)3\, dt = 12\left(\frac{3\pi}{4}\right) = 9\pi \text{ sq. cm.}$$

(c) The cross sections parallel to the xz-plane are rectangles of height $1 + 4(y/3)^2(1 - y^2/9)$ and base $2\sqrt{9 - y^2}$. Hence,

$$\text{Volume} = 2\int_0^3 2\sqrt{9 - y^2}\left(1 + 4\frac{y^2}{9}\left(1 - \frac{y^2}{9}\right)\right) dy = \frac{27\pi}{2} \approx 42.412 \text{ cm}^3.$$

76. $W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C M\,dx + N\,dy$



$M = 15(4 - x^2y) = 60 - 15x^2(c - cx^2)$

$N = -15xy = -15x(c - cx^2)$

$dx = dx,\ dy = -2cx\,dx$

$$W = \int_{-1}^{1} [60 - 15x^2(c - cx^2) + (-15x(c - cx^2))(-2\,cx)]\, dx$$

$= 120 - 4c + 8c^2$ (parabola)

$w' = 16c - 4 = 0 \implies c = \frac{1}{4}$ yields the minimum work, 119.5. Along the straight line path, $y = 0$, the work is 120.

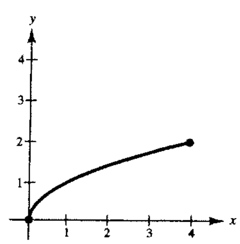
77. See the definition of Line Integral, page 1066.

See Theorem 15.4.

78. See the definition, page 1070.

79. The greater the height of the surface over the curve, the greater the lateral surface area. Hence,

$z_3 < z_1 < z_2 < z_4$.



80. (a) Work = 0

(b) Work is negative, since against force field.

(c) Work is positive, since with force field.

81. False

$$\int_C xy\, ds = \sqrt{2}\int_0^1 t^2\, dt$$

82. False, the orientation of C does not affect the form

$$\int_C f(x, y)\, ds.$$

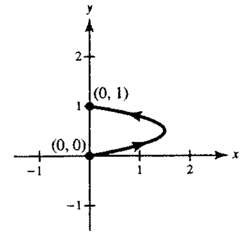
83. False, the orientations are different.

84. False. For example, see Exercise 32.

85. $\mathbf{F}(x, y) = (y - x)\mathbf{i} + xy\mathbf{j}$

$\mathbf{r}(t) = kt(1 - t)\mathbf{i} + t\mathbf{j}, \quad 0 \le t \le 1$

$\mathbf{r}'(t) = k(1 - 2t)\mathbf{i} + \mathbf{j}$



$$\text{Work} = 1 = \int_C \mathbf{F} \cdot d\mathbf{r}$$

$$= \int_0^1 [(t - kt(1 - t))\mathbf{i} + kt^2(1 - t)\mathbf{j}] \cdot [k(1 - 2t)\mathbf{i} + \mathbf{j}]\, dt$$

$$= \int_0^1 [(t - kt(1 - t))k(1 - 2t) + kt^2(1 - t)]\, dt$$

$$= \int_0^1 (-2k^2t^3 - kt^3 - kt^2 + 3k^2t^2 - k^2t + kt)\, dt$$

$$= \frac{-k}{12}$$

$$k = -12$$

Section 15.3 Conservative Vector Fields and Independence of Path

1. $\mathbf{F}(x, y) = x^2\mathbf{i} + xy\mathbf{j}$

(a) $\mathbf{r}_1(t) = t\mathbf{i} + t^2\mathbf{j},\ 0 \le t \le 1$

$\mathbf{r}_1'(t) = \mathbf{i} + 2t\,\mathbf{j}$

$\mathbf{F}(t) = t^2\mathbf{i} + t^3\mathbf{j}$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (t^2 + 2t^4)\, dt = \frac{11}{15}$$

(b) $\mathbf{r}_2(\theta) = \sin\theta\mathbf{i} + \sin^2\theta\mathbf{j},\ 0 \le \theta \le \dfrac{\pi}{2}$

$\mathbf{r}_2'(\theta) = \cos\theta\mathbf{i} + 2\sin\theta\cos\theta\mathbf{j}$

$\mathbf{F}(t) = \sin^2\theta\mathbf{i} + \sin^3\theta\mathbf{j}$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{\pi/2} (\sin^2\theta\cos\theta + 2\sin^4\theta\cos\theta)\, d\theta$$

$$= \left[\frac{\sin^3\theta}{3} + \frac{2\sin^5\theta}{5}\right]_0^{\pi/2} = \frac{11}{15}$$

2. $\mathbf{F}(x, y) = (x^2 + y^2)\mathbf{i} - x\mathbf{j}$

(a) $\mathbf{r}_1(t) = t\mathbf{i} + \sqrt{t}\mathbf{j},\ 0 \le t \le 4$

$\mathbf{r}_1'(t) = \mathbf{i} + \dfrac{1}{2\sqrt{t}}\mathbf{j}$

$\mathbf{F}(t) = (t^2 + t)\mathbf{i} - t\mathbf{j}$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^4 \left(t^2 + t - \frac{1}{2}\sqrt{t}\right) dt$$

$$= \left[\frac{t^3}{3} + \frac{t^2}{2} - \frac{t^{3/2}}{3}\right]_0^4 = \frac{80}{3}$$

(b) $\mathbf{r}_2(w) = w^2\mathbf{i} + w\mathbf{j},\ 0 \le w \le 2$

$\mathbf{r}_2'(w) = 2w\mathbf{i} + \mathbf{j}$

$\mathbf{F}(w) = (w^4 + w^2)\mathbf{i} - w^2\mathbf{j}$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^2 [2w(w^4 + w^2) - w^2]\, dw$$

$$= \left[\frac{w^6}{3} + \frac{w^4}{2} - \frac{w^3}{3}\right]_0^2 = \frac{80}{3}$$

3. $\mathbf{F}(x, y) = y\mathbf{i} - x\mathbf{j}$

(a) $\mathbf{r}_1(\theta) = \sec\theta\mathbf{i} + \tan\theta\mathbf{j},\ 0 \le \theta \le \frac{\pi}{3}$

$\mathbf{r}_1'(\theta) = \sec\theta\tan\theta\mathbf{i} + \sec^2\theta\mathbf{j}$

$\mathbf{F}(\theta) = \tan\theta\mathbf{i} - \sec\theta\mathbf{j}$

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{\pi/3} (\sec\theta\tan^2\theta - \sec^3\theta)\,d\theta = \int_0^{\pi/3} [\sec\theta(\sec^2\theta - 1) - \sec^3\theta]\,d\theta$$

$$= -\int_0^{\pi/3} \sec\theta\,d\theta = \Big[-\ln|\sec\theta + \tan\theta|\Big]_0^{\pi/3} = -\ln\left(2 + \sqrt{3}\right) \approx -1.317$$

(b) $\mathbf{r}_2(t) = \sqrt{t+1}\,\mathbf{i} + \sqrt{t}\,\mathbf{j},\quad 0 \le t \le 3$

$\mathbf{r}_2'(t) = \dfrac{1}{2\sqrt{t+1}}\mathbf{i} + \dfrac{1}{2\sqrt{t}}\mathbf{j}$

$\mathbf{F}(t) = \sqrt{t}\,\mathbf{i} - \sqrt{t+1}\,\mathbf{j}$

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^3 \left[\frac{\sqrt{t}}{2\sqrt{t+1}} - \frac{\sqrt{t+1}}{2\sqrt{t}}\right] dt = -\frac{1}{2}\int_0^3 \frac{1}{\sqrt{t}\sqrt{t+1}}\,dt = -\frac{1}{2}\int_0^3 \frac{1}{\sqrt{t^2 + t + (1/4) - (1/4)}}\,dt$$

$$= -\frac{1}{2}\int_0^3 \frac{1}{\sqrt{[t + (1/2)]^2 - (1/4)}}\,dt = \left[-\frac{1}{2}\ln\left|\left(t + \frac{1}{2}\right) + \sqrt{t^2 + t}\right|\right]_0^3$$

$$= -\frac{1}{2}\left[\ln\left(\frac{7}{2} + 2\sqrt{3}\right) - \ln\left(\frac{1}{2}\right)\right] = -\frac{1}{2}\ln\left(7 + 4\sqrt{3}\right) \approx -1.317$$

4. $\mathbf{F}(x, y) = y\mathbf{i} + x^2\mathbf{j}$

(a) $\mathbf{r}_1(t) = (2 + t)\mathbf{i} + (3 - t)\mathbf{j},\ 0 \le t \le 3$

$\mathbf{r}_1'(t) = \mathbf{i} - \mathbf{j}$

$\mathbf{F}(t) = (3 - t)\mathbf{i} + (2 + t)^2\mathbf{j}$

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^3 [(3 - t) - (2 + t)^2]\,dt = \left[-\frac{(3-t)^2}{2} - \frac{(2+t)^3}{3}\right]_0^3 = -\frac{69}{2}$$

(b) $\mathbf{r}_2(w) = (2 + \ln w)\mathbf{i} + (3 - \ln w)\mathbf{j},\ 1 \le w \le e^3$

$\mathbf{r}_2'(w) = \dfrac{1}{w}\mathbf{i} - \dfrac{1}{w}\mathbf{j}$

$\mathbf{F}(w) = (3 - \ln w)\mathbf{i} + (2 + \ln w)^2\mathbf{j}$

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_1^{e^3} \left[(3 - \ln w)\left(\frac{1}{w}\right) - (2 + \ln w)^2\left(\frac{1}{w}\right)\right] dw = \left[-\frac{(3 - \ln w)^2}{2} - \frac{(2 + \ln w)^3}{3}\right]_1^{e^3} = -\frac{69}{2}$$

5. $\mathbf{F}(x, y) = e^x\sin y\mathbf{i} + e^x\cos y\mathbf{j}$

$\dfrac{\partial N}{\partial x} = e^x\cos y \qquad \dfrac{\partial M}{\partial y} = e^x\cos y$

Since $\dfrac{\partial N}{\partial x} = \dfrac{\partial M}{\partial y}$, **F** is conservative.

6. $\mathbf{F}(x, y) = 15x^2y^2\mathbf{i} + 10x^3y\mathbf{j}$

$\dfrac{\partial N}{\partial x} = 30x^2y \qquad \dfrac{\partial M}{\partial y} = 30x^2y$

Since $\dfrac{\partial N}{\partial x} = \dfrac{\partial M}{\partial y}$, **F** is conservative.

7. $\mathbf{F}(x, y) = \dfrac{1}{y}\mathbf{i} + \dfrac{x}{y^2}\mathbf{j}$

$\dfrac{\partial N}{\partial x} = \dfrac{1}{y^2} \qquad \dfrac{\partial M}{\partial y} = -\dfrac{1}{y^2}$

Since $\dfrac{\partial N}{\partial x} \ne \dfrac{\partial M}{\partial y}$, **F** is not conservative.

8. $\mathbf{F}(x, y, z) = y\ln z\mathbf{i} - x\ln z\mathbf{j} + \dfrac{xy}{z}\mathbf{k}$

curl F $\ne$ **0** so **F** is not conservative.

$\left(\dfrac{\partial P}{\partial y} = \dfrac{x}{z} \ne -\dfrac{x}{z} = \dfrac{\partial N}{\partial z}\right)$

9. $\mathbf{F}(x, y, z) = y^2z\mathbf{i} + 2xyz\mathbf{j} + xy^2\mathbf{k}$

$\mathbf{curl\ F} = \mathbf{0} \implies \mathbf{F}$ is conservative.

10. $\mathbf{F}(x, y, z) = \sin(yz)\mathbf{i} + xz\cos(yz)\mathbf{j} + xy\sin(yz)\mathbf{k}$

$\mathbf{curl\ F} \neq \mathbf{0}$, so $\mathbf{F}$ is not conservative.

11. $\mathbf{F}(x, y) = 2xy\mathbf{i} + x^2\mathbf{j}$

(a) $\mathbf{r}_1(t) = t\mathbf{i} + t^2\mathbf{j},\ 0 \le t \le 1$

$\mathbf{r}_1'(t) = \mathbf{i} + 2t\mathbf{j}$

$\mathbf{F}(t) = 2t^3\mathbf{i} + t^2\mathbf{j}$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 4t^3\,dt = 1$$

(b) $\mathbf{r}_2(t) = t\mathbf{i} + t^3\mathbf{j},\ 0 \le t \le 1$

$\mathbf{r}_2'(t) = \mathbf{i} + 3t^2\mathbf{j}$

$\mathbf{F}(t) = 2t^4\mathbf{i} + t^2\mathbf{j}$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 5t^4\,dt = 1$$

12. $\mathbf{F}(x, y) = ye^{xy}\mathbf{i} + xe^{xy}\mathbf{j}$

(a) $\mathbf{r}_1(t) = t\mathbf{i} - (t - 3)\mathbf{j},\ 0 \le t \le 3$

$\mathbf{r}_1'(t) = \mathbf{i} - \mathbf{j}$

$\mathbf{F}(t) = -(t - 3)e^{3t-t^2}\mathbf{i} + te^{3t-t^2}\mathbf{j}$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^3 [-(t - 3)e^{3t-t^2} - te^{3t-t^2}]\,dt$$

$$= \int_0^3 e^{3t-t^2}(3 - 2t)\,dt$$

$$= \Big[e^{3t-t^2}\Big]_0^3 = e^0 - e^0 = 0$$

(b) $\mathbf{F}(x, y)$ is conservative since

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = xye^{xy} + e^{xy}.$$

The potential function is $f(x, y) = e^{xy} + k$.

13. $\mathbf{F}(x, y) = y\mathbf{i} - x\mathbf{j}$

(a) $\mathbf{r}_1(t) = t\mathbf{i} + t\mathbf{j},\quad 0 \le t \le 1$

$\mathbf{r}_1'(t) = \mathbf{i} + \mathbf{j}$

$\mathbf{F}(t) = t\mathbf{i} - t\mathbf{j}$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = 0$$

(b) $\mathbf{r}_2(t) = t\mathbf{i} + t^2\mathbf{j},\quad 0 \le t \le 1$

$\mathbf{r}_2'(t) = \mathbf{i} + 2t\mathbf{j}$

$\mathbf{F}(t) = t^2\mathbf{i} - t\mathbf{j}$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 -t^2\,dt = -\frac{1}{3}$$

(c) $\mathbf{r}_3(t) = t\mathbf{i} + t^3\mathbf{j},\quad 0 \le t \le 1$

$\mathbf{r}_3'(t) = \mathbf{i} + 3t^2\mathbf{j}$

$\mathbf{F}(t) = t^3\mathbf{i} - t\mathbf{j}$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 -2t^3\,dt = -\frac{1}{2}$$

14. $\mathbf{F}(x, y) = xy^2\mathbf{i} + 2x^2y\mathbf{j}$

(a) $\mathbf{r}_1(t) = t\mathbf{i} + \frac{1}{t}\mathbf{j},\quad 1 \le t \le 3$

$\mathbf{r}_1'(t) = \mathbf{i} - \frac{1}{t^2}\mathbf{j}$

$\mathbf{F}(t) = \frac{1}{t}\mathbf{i} + 2t\mathbf{j}$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_1^3 -\frac{1}{t}\,dt$$

$$= \Big[-\ln|t|\Big]_1^3 = -\ln 3$$

(b) $\mathbf{r}_2(t) = (t + 1)\mathbf{i} - \frac{1}{3}(t - 3)\mathbf{j},\quad 0 \le t \le 2$

$\mathbf{r}_2'(t) = \mathbf{i} - \frac{1}{3}\mathbf{j}$

$\mathbf{F}(t) = \frac{1}{9}(t + 1)(t - 3)^2\mathbf{i} - \frac{2}{3}(t + 1)^2(t - 3)\mathbf{j}$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^2 \left[\frac{1}{9}(t + 1)(t - 3)^2 + \frac{2}{9}(t + 1)^2(t - 3)\right] dt$$

$$= \frac{1}{9}\int_0^2 (3t^3 - 7t^2 - 7t + 3)\,dt$$

$$= \frac{1}{9}\left[\frac{3t^4}{4} - \frac{7t^3}{3} - \frac{7t^2}{2} + 3t\right]_0^2 = -\frac{44}{27}$$

15. $\int_C y^2\,dx + 2xy\,dy$

Since $\partial M/\partial y = \partial N/\partial x = 2y$, $\mathbf{F}(x, y) = y^2\mathbf{i} + 2xy\mathbf{j}$ is conservative. The potential function is $f(x, y) = xy^2 + k$. Therefore, we can use the Fundamental Theorem of Line Integrals.

(a) $\displaystyle\int_C y^2\,dx + 2xy\,dy = \Big[x^2y\Big]_{(0,0)}^{(4,4)} = 64$

(b) $\displaystyle\int_C y^2\,dx + 2xy\,dy = \Big[x^2y\Big]_{(-1,0)}^{(1,0)} = 0$

(c) and (d) Since C is a closed curve, $\displaystyle\int_C y^2\,dx + 2xy\,dy = 0$.

16. $\int_C (2x - 3y + 1)\,dx - (3x + y - 5)\,dy$

Since $\partial M/\partial y = \partial N/\partial x = -3$, $\mathbf{F}(x, y) = (2x - 3y + 1)\mathbf{i} - (3x + y - 5)\mathbf{j}$ is conservative. The potential function is $f(x, y) = x^2 - 3xy - (y^2/2) + x + 5y + k$.

(a) and (d) Since C is a closed curve, $\displaystyle\int_C (2x - 3y + 1)\,dx - (3x + y - 5)\,dy = 0$.

(b) $\displaystyle\int_C (2x - 3y + 1)\,dx - (3x + y - 5)\,dy = \left[x^2 - 3xy - \frac{y^2}{2} + x + 5y\right]_{(0,-1)}^{(0,1)} = 10$

(c) $\displaystyle\int_C (2x - 3y + 1)\,dx - (3x + y - 5)\,dy = \left[x^2 - 3xy - \frac{y^2}{2} + x + 5y\right]_{(0,1)}^{(2,e^2)} = \frac{1}{2}(3 - 2e^2 - e^4)$

17. $\int_C 2xy\,dx + (x^2 + y^2)\,dy$

Since $\partial M/\partial y = \partial N/\partial x = 2x$,

$\mathbf{F}(x, y) = 2xy\mathbf{i} + (x^2 + y^2)\mathbf{j}$ is conservative.

The potential function is $f(x, y) = x^2y + \dfrac{y^3}{3} + k$.

(a) $\displaystyle\int_C 2xy\,dx + (x^2 + y^2)\,dy = \left[x^2y + \frac{y^3}{3}\right]_{(5,0)}^{(0,4)} = \frac{64}{3}$

(b) $\displaystyle\int_C 2xy\,dx + (x^2 + y^2)\,dy = \left[x^2y + \frac{y^3}{3}\right]_{(2,0)}^{(0,4)} = \frac{64}{3}$

18. $\int_C (x^2 + y^2)\,dx + 2xy\,dy$

Since $\partial M/\partial y = \partial N/\partial x = 2y$,

$\mathbf{F}(x, y) = (x^2 + y^2)\mathbf{i} + 2xy\mathbf{j}$

is conservative. The potential function is

$f(x, y) = (x^3/3) + xy^2 + k$.

(a) $\displaystyle\int_C (x^2 + y^2)\,dx + 2xy\,dy = \left[\frac{x^3}{3} + xy^2\right]_{(0,0)}^{(8,4)} = \frac{896}{3}$

(b) $\displaystyle\int_C (x^2 + y^2)\,dx + 2xy\,dy = \left[\frac{x^3}{3} + xy^2\right]_{(2,0)}^{(0,2)} = -\frac{8}{3}$

19. $\mathbf{F}(x, y, z) = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$

Since $\mathbf{curl\,F} = \mathbf{0}$, $\mathbf{F}(x, y, z)$ is conservative. The potential function is $f(x, y, z) = xyz + k$.

(a) $\mathbf{r}_1(t) = t\mathbf{i} + 2\mathbf{j} + t\mathbf{k}$, $0 \le t \le 4$

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \Big[xyz\Big]_{(0,2,0)}^{(4,2,4)} = 32$$

(b) $\mathbf{r}_2(t) = t^2\mathbf{i} + t\mathbf{j} + t^2\mathbf{k}$, $0 \le t \le 2$

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \Big[xyz\Big]_{(0,0,0)}^{(4,2,4)} = 32$$

20. $\mathbf{F}(x, y, z) = \mathbf{i} + z\mathbf{j} + y\mathbf{k}$

Since $\mathbf{curl\,F} = \mathbf{0}$, $\mathbf{F}(x, y, z)$ is conservative. The potential function is $f(x, y, z) = x + yz + k$.

(a) $\mathbf{r}_1(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t^2\mathbf{k}$, $0 \le t \le \pi$

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \Big[x + yz\Big]_{(1,0,0)}^{(-1,0,\pi^2)} = -2$$

(b) $\mathbf{r}_2(t) = (1 - 2t)\mathbf{i} + \pi^2 t\mathbf{k}$, $0 \le t \le 1$

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \Big[x + yz\Big]_{(1,0,0)}^{(-1,0,\pi^2)} = -2$$

21. $\mathbf{F}(x, y, z) = (2y + x)\mathbf{i} + (x^2 - z)\mathbf{j} + (2y - 4z)\mathbf{k}$

$\mathbf{F}(x, y, z)$ is not conservative.

(a) $\mathbf{r}_1(t) = t\mathbf{i} + t^2\mathbf{j} + \mathbf{k},\ 0 \le t \le 1$

$\mathbf{r}_1'(t) = \mathbf{i} + 2t\mathbf{j}$

$\mathbf{F}(t) = (2t^2 + t)\mathbf{i} + (t^2 - 1)\mathbf{j} + (2t^2 - 4)\mathbf{k}$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (2t^3 + 2t^2 - t)\,dt = \frac{2}{3}$$

(b) $\mathbf{r}_2(t) = t\mathbf{i} + t\mathbf{j} + (2t - 1)^2\mathbf{k},\ 0 \le t \le 1$

$\mathbf{r}_2'(t) = \mathbf{i} + \mathbf{j} + 4(2t - 1)\mathbf{k}$

$\mathbf{F}(t) = 3t\mathbf{i} + [t^2 - (2t - 1)^2]\mathbf{j} + [2t - 4(2t - 1)^2]\mathbf{k}$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 [3t + t^2 - (2t - 1)^2 + 8t(2t - 1) - 16(2t - 1)^3]\,dt$$

$$= \int_0^1 [17t^2 - 5t - (2t - 1)^2 - 16(2t - 1)^3]\,dt = \left[\frac{17t^3}{3} - \frac{5t^2}{2} - \frac{(2t - 1)^3}{6} - 2(2t - 1)^4\right]_0^1 = \frac{17}{6}$$

22. $\mathbf{F}(x, y, z) = -y\mathbf{i} + x\mathbf{j} + 3xz^2\mathbf{k}$

$\mathbf{F}(x, y, z)$ is not conservative.

(a) $\mathbf{r}_1(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k},\ 0 \le t \le \pi$

$\mathbf{r}_1'(t) = -\sin t\mathbf{i} + \cos t\mathbf{j} + \mathbf{k}$

$\mathbf{F}(t) = -\sin t\mathbf{i} + \cos t\mathbf{j} + 3t^2 \cos t\mathbf{k}$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^\pi [\sin^2 t + \cos^2 t + 3t^2 \cos t]\,dt = \int_0^\pi [1 + 3t^2 \cos t]\,dt$$

$$= \Big[t\Big]_0^\pi + 3\Big[t^2 \sin t\Big]_0^\pi - 6\int_0^\pi t \sin t\,dt = \Big[t + 3t^2 \sin t - 6(\sin t - t\cos t)\Big]_0^\pi = -5\pi$$

(b) $\mathbf{r}_2(t) = (1 - 2t)\mathbf{i} + \pi t\mathbf{k},\ 0 \le t \le 1$

$\mathbf{r}_2'(t) = -2\mathbf{i} + \pi\mathbf{k}$

$\mathbf{F}(t) = (1 - 2t)\mathbf{j} + 3\pi^2 t^2(1 - 2t)\mathbf{k}$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 3\pi^3 t^2(1 - 2t)\,dt = 3\pi^3 \int_0^1 (t^2 - 2t^3)\,dt = 3\pi^3\left[\frac{t^3}{3} - \frac{t^4}{2}\right]_0^1 = -\frac{\pi^3}{2}$$

23. $\mathbf{F}(x, y, z) = e^z(y\mathbf{i} + x\mathbf{j} + xy\mathbf{k})$

$\mathbf{F}(x, y, z)$ is conservative. The potential function is $f(x, y, z) = xye^z + k$.

(a) $\mathbf{r}_1(t) = 4\cos t\mathbf{i} + 4\sin t\mathbf{j} + 3\mathbf{k},\ 0 \le t \le \pi$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \Big[xye^z\Big]_{(4,0,3)}^{(-4,0,3)} = 0$$

(b) $\mathbf{r}_2(t) = (4 - 8t)\mathbf{i} + 3\mathbf{k},\ 0 \le t \le 1$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \Big[xye^z\Big]_{(4,0,3)}^{(-4,0,3)} = 0$$

24. $\mathbf{F}(x, y, z) = y \sin z\mathbf{i} + x \sin z\mathbf{j} + xy \cos x\mathbf{k}$

(a) $\mathbf{r}_1(t) = t^2\mathbf{i} + t^2\mathbf{j},\ 0 \le t \le 2$

$\mathbf{r}_1'(t) = 2t\mathbf{i} + 2t\mathbf{j}$

$\mathbf{F}(t) = t^4 \cos t^2\mathbf{k}$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^2 0\,dt = 0$$

(b) $\mathbf{r}_2(t) = 4t\mathbf{i} + 4t\mathbf{j},\ 0 \le t \le 1$

$\mathbf{r}_2'(t) = 4\mathbf{i} + 4\mathbf{j}$

$\mathbf{F}(t) = 16t^2 \cos(4t)\mathbf{k}$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 0\,dt = 0$$

25. $\displaystyle\int_C (y\mathbf{i} + x\mathbf{j}) \cdot d\mathbf{r} = \Big[xy\Big]_{(0,0)}^{(3,8)} = 24$

26. $\displaystyle\int_C [2(x+y)\mathbf{i} + 2(x+y)\mathbf{j}] \cdot d\mathbf{r} = \Big[(x+y)^2\Big]_{(-2,2)}^{(4,3)} = 49$

27. $\displaystyle\int_C \cos x \sin y \, dx + \sin x \cos y \, dy = \Big[\sin x \sin y\Big]_{(0,-\pi)}^{(3\pi/2,\pi/2)} = -1$

28. $\displaystyle\int_C \frac{y\,dx - x\,dy}{x^2+y^2} = \left[\arctan\left(\frac{x}{y}\right)\right]_{(1,1)}^{(2\sqrt{3},2)} = \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$

29. $\displaystyle\int_C e^x \sin y \, dx + e^x \cos y \, dy = \Big[e^x \sin y\Big]_{(0,0)}^{(2\pi,0)} = 0$

30. $\displaystyle\int_C \frac{2x}{(x^2+y^2)^2}\,dx + \frac{2y}{(x^2+y^2)^2}\,dy = \left[-\frac{1}{x^2+y^2}\right]_{(7,5)}^{(1,5)} = -\frac{1}{26} + \frac{1}{74} = \frac{-12}{481}$

31. $\displaystyle\int_C (y+2z)\,dx + (x-3z)\,dy + (2x-3y)\,dz$

$\mathbf{F}(x, y, z)$ is conservative and the potential function is $f(x, y, z) = xy - 3yz + 2xz$.

(a) $\Big[xy - 3yz + 2xz\Big]_{(0,0,0)}^{(1,1,1)} = 0 - 0 = 0$

(b) $\Big[xy - 3yz + 2xz\Big]_{(0,0,0)}^{(0,0,1)} + \Big[xy - 3yz + 2xz\Big]_{(0,0,1)}^{(1,1,1)} = 0 + 0 = 0$

(c) $\Big[xy - 3yz + 2xz\Big]_{(0,0,0)}^{(1,0,0)} + \Big[xy - 3yz + 2xz\Big]_{(1,0,0)}^{(1,1,0)} + \Big[xy - 3yz + 2xz\Big]_{(1,1,0)}^{(1,1,1)} = 0 + 1 + (-1) = 0$

32. $\displaystyle\int_C zy\,dx + xz\,dy + xy\,dz$

Note: Since $\mathbf{F}(x, y, z) = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$ is conservative and the potential function is $f(x, y, z) = xyz + k$, the integral is independent of path as illustrated below.

(a) $\Big[xyz\Big]_{(0,0,0)}^{(1,1,1)} = 1$

(b) $\Big[xyz\Big]_{(0,0,0)}^{(0,0,1)} + \Big[xyz\Big]_{(0,0,1)}^{(1,1,1)} = 0 + 1 = 1$

(c) $\Big[xyz\Big]_{(0,0,0)}^{(1,0,0)} + \Big[xyz\Big]_{(1,0,0)}^{(1,1,0)} + \Big[xyz\Big]_{(1,1,0)}^{(1,1,1)} = 0 + 0 + 1 = 1$

33. $\displaystyle\int_C -\sin x\,dx + z\,dy + y\,dz = \Big[\cos x + yz\Big]_{(0,0,0)}^{(\pi/2,3,4)} = 12 - 1 = 11$

34. $\displaystyle\int_C 6x\,dx - 4z\,dy - (4y - 20z)\,dz = \Big[3x^2 - 4yz + 10z^2\Big]_{(0,0,0)}^{(4,3,1)} = 46$

35. $\mathbf{F}(x, y) = 9x^2y^2\mathbf{i} + (6x^3y - 1)\mathbf{j}$ is conservative.

Work $= \Big[3x^3y^2 - y\Big]_{(0,0)}^{(5,9)} = 30{,}366$

36. $\mathbf{F}(x, y) = \dfrac{2x}{y}\mathbf{i} - \dfrac{x^2}{y^2}\mathbf{j}$ is conservative.

Work $= \left[\dfrac{x^2}{y}\right]_{(-3,2)}^{(1,4)} = \dfrac{1}{4} - \dfrac{9}{2} = -\dfrac{17}{4}$

37. $\mathbf{r}(t) = 2\cos 2\pi t\mathbf{i} + 2\sin 2\pi t\mathbf{j}$

$\mathbf{r}'(t) = -4\pi \sin 2\pi t\mathbf{i} + 4\pi\cos 2\pi t\mathbf{j}$

$\mathbf{a}(t) = -8\pi^2\cos 2\pi t\mathbf{i} - 8\pi^2\sin 2\pi t\mathbf{j}$

$\mathbf{F}(t) = m\mathbf{a}(t) = \dfrac{1}{32}\mathbf{a}(t) = -\dfrac{\pi^2}{4}(\cos 2\pi t\mathbf{i} + \sin 2\pi t\mathbf{j})$

$$W = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_C -\frac{\pi^2}{4}(\cos 2\pi t\mathbf{i} + \sin 2\pi t\mathbf{j}) \cdot 4\pi(-\sin 2\pi t\mathbf{i} + \cos 2\pi t\mathbf{j})\,dt = -\pi^3\int_C 0\,dt = 0$$

38. $\mathbf{F}(x, y, z) = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$

Since $\mathbf{F}(x, y, z)$ is conservative, the work done in moving a particle along any path from P to Q is

$$f(x, y, z) = \Big[a_1x + a_2y + a_3z\Big]_{P=(p_1, p_2, p_3)}^{Q=(q_1, q_2, q_3)}$$

$$= a_1(q_1 - p_1) + a_2(q_2 - p_2) + a_3(q_3 - p_3) = \mathbf{F}\cdot\overrightarrow{PQ}.$$

39. $\mathbf{F} = -150\mathbf{j}$

(a) $\mathbf{r}(t) = t\mathbf{i} + (50 - t)\mathbf{j},\ 0 \le t \le 50$

$d\mathbf{r} = (\mathbf{i} - \mathbf{j})\,dt$

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{50} 150\,dt = 7500 \text{ ft}\cdot\text{lbs}$$

(b) $\mathbf{r}(t) = t\mathbf{i} + \frac{1}{50}(50 - t)^2\mathbf{j}$

$d\mathbf{r} = \left(\mathbf{i} - \frac{1}{25}(50 - t)\mathbf{j}\right)dt$

$$\int_C \mathbf{F}\cdot d\mathbf{r} = 6\int_0^{50}(50 - t)\,dt = 7500 \text{ ft}\cdot\text{lbs}$$

40. No. The force field is conservative.

41. See Theorem 15.5.

42. A line integral is independent of path if $\int_C \mathbf{F}\cdot d\mathbf{r}$ does not depend on the curve joining P and Q. See Theorem 15.6.

43. (a) The direct path along the line segment joining $(-4, 0)$ to $(3, 4)$ requires less work than the path going from $(-4, 0)$ to $(-4, 4)$ and then to $(3, 4)$.

(b) The closed curve given by the line segments joining $(-4, 0)$, $(-4, 4)$, $(3, 4)$, and $(-4, 0)$ satisfies $\int_C \mathbf{F}\cdot d\mathbf{r} \neq 0$.

44. No, the amount of fuel is not independent of the flight path. The vector field is not conservative.

45. Conservative. $\int_C \mathbf{F}\cdot d\mathbf{r}$ is independent of path.

46. Not conservative. The value of $\int_C \mathbf{F}\cdot d\mathbf{r}$ from $(-1, 0)$ to $(1, 0)$ is positive if the path is above the x-axis, and negative if the path is below the x-axis.

47. False, it would be true if $\mathbf{F}$ were conservative.

48. True

49. True

50. False, the requirement is $\partial M/\partial y = \partial N/\partial x$.

51. Let

$$\mathbf{F} = M\mathbf{i} + N\mathbf{j} = \frac{\partial f}{\partial y}\mathbf{i} - \frac{\partial f}{\partial x}\mathbf{j}.$$

Then $\dfrac{\partial M}{\partial y} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial y}\right) = \dfrac{\partial^2 f}{\partial y^2}$ and $\dfrac{\partial N}{\partial x} = \dfrac{\partial}{\partial x}\left(-\dfrac{\partial f}{\partial x}\right) = -\dfrac{\partial^2 f}{\partial x^2}$. Since

$\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} = 0$ we have $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$.

Thus, $\mathbf{F}$ is conservative. Therefore, by Theorem 15.7, we have

$$\int_C \left(\frac{\partial f}{\partial y}\,dx - \frac{\partial f}{\partial x}\,dy\right) = \int_C (M\,dx + N\,dy) = \int_C \mathbf{F}\cdot d\mathbf{r} = 0$$

for every closed curve in the plane.

52. Since the sum of the potential and kinetic energies remains constant from point to point, if the kinetic energy is decreasing at a rate of 10 units per minute, then the potential energy is increasing at a rate of 10 units per minute.

53. $\mathbf{F}(x, y) = \dfrac{y}{x^2 + y^2}\mathbf{i} - \dfrac{x}{x^2 + y^2}\mathbf{j}$

(a) $M = \dfrac{y}{x^2 + y^2}$

$$\frac{\partial M}{\partial y} = \frac{(x^2 + y^2)(1) - y(2y)}{(x^2 + y^2)^2} = \frac{x^2 - y^2}{(x^2 + y^2)^2}$$

$$N = -\frac{x}{x^2 + y^2}$$

$$\frac{\partial N}{\partial x} = \frac{(x^2 + y^2)(-1) + x(2x)}{(x^2 + y^2)^2} = \frac{x^2 - y^2}{(x^2 + y^2)^2}$$

Thus, $\dfrac{\partial N}{\partial x} = \dfrac{\partial M}{\partial y}$.

(b) $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j},\ 0 \le t \le \pi$

$\mathbf{F} = \sin t\mathbf{i} - \cos t\mathbf{j}$

$d\mathbf{r} = (-\sin t\mathbf{i} + \cos t\mathbf{j})\,dt$

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{\pi} (-\sin^2 t - \cos^2 t)\,dt = \Big[-t\Big]_0^{\pi} = -\pi$$

(c) $\mathbf{r}(t) = \cos t\mathbf{i} - \sin t\mathbf{j},\ 0 \le t \le \pi$

$\mathbf{F} = -\sin t\mathbf{i} - \cos t\mathbf{j}$

$d\mathbf{r} = (-\sin t\mathbf{i} - \cos t\mathbf{j})\,dt$

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{\pi} (\sin^2 t + \cos^2 t)\,dt$$

$$= \Big[t\Big]_0^{\pi} = \pi$$

(d) $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j},\ 0 \le t \le 2\pi$

$\mathbf{F} = \sin t\mathbf{i} - \cos t\mathbf{j}$

$d\mathbf{r} = (-\sin t\mathbf{i} + \cos t\mathbf{j})\,dt$

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} (-\sin^2 t - \cos^2 t)\,dt$$

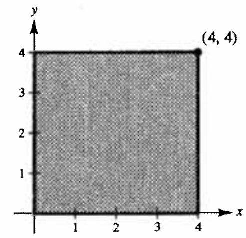
$$= \Big[-t\Big]_0^{2\pi} = -2\pi$$

This does not contradict Theorem 15.7 since $\mathbf{F}$ is not continuous at (0, 0) in R enclosed by curve C.

(e) $$\nabla\left(\arctan\frac{x}{y}\right) = \frac{1/y}{1 + (x/y)^2}\mathbf{i} + \frac{-x/y^2}{1 + (x/y)^2}\mathbf{j}$$

$$= \frac{y}{x^2 + y^2}\mathbf{i} - \frac{x}{x^2 + y^2}\mathbf{j} = \mathbf{F}$$

Section 15.4 Green's Theorem



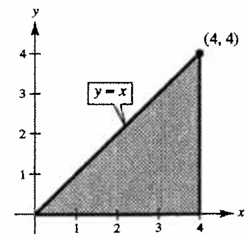
1. $\mathbf{r}(t) = \begin{cases} t\mathbf{i}, & 0 \le t \le 4 \\ 4\mathbf{i} + (t-4)\mathbf{j}, & 4 \le t \le 8 \\ (12-t)\mathbf{i} + 4\mathbf{j}, & 8 \le t \le 12 \\ (16-t)\mathbf{j}, & 12 \le t \le 16 \end{cases}$

$$\int_C y^2\,dx + x^2\,dy = \int_0^4 [0\,dt + t^2(0)] + \int_4^8 [(t-4)^2(0) + 16\,dt]$$

$$+ \int_8^{12} [16(-dt) + (12-t)^2(0)] + \int_{12}^{16} [(16-t)^2(0) + 0(-dt)]$$

$$= 0 + 64 - 64 + 0 = 0$$

By Green's Theorem, $\displaystyle\int_R\int \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \int_0^4\int_0^4 (2x - 2y)\,dy\,dx = \int_0^4 (8x - 16)\,dx = 0.$



2. $\mathbf{r}(t) = \begin{cases} t\mathbf{i}, & 0 \le t \le 4 \\ 4\mathbf{i} + (t-4)\mathbf{j}, & 4 \le t \le 8 \\ (12-t)\mathbf{i} + (12-t)\mathbf{j}, & 8 \le t \le 12 \end{cases}$

$$\int_C y^2\,dx + x^2\,dy = \int_0^4 [0\,dt + t^2(0)] + \int_4^8 [(t-4)^2(0) + 16\,dt]$$

$$+ \int_8^{12} [(12-t)^2(-dt) + (12-t)^2(-dt)] = 0 + 64 - \frac{128}{3} = \frac{64}{3}$$

By Green's Theorem, $\displaystyle\int_R\int \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \int_0^4\int_0^x (2x - 2y)\,dy\,dx = \int_0^4 x^2\,dx = \frac{64}{3}.$

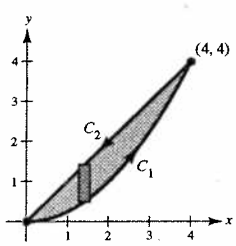
3. $\mathbf{r}(t) = \begin{cases} t\mathbf{i} + t^2/4\mathbf{j}, & 0 \le t \le 4 \\ (8-t)\mathbf{i} + (8-t)\mathbf{j}, & 4 \le t \le 8 \end{cases}$

$$\int_C y^2\,dx + x^2\,dy = \int_0^4 \left[\frac{t^4}{16}(dt) + t^2\left(\frac{t}{2}\,dt\right)\right] + \int_4^8 [(8-t)^2(-dt) + (8-t)^2(-dt)]$$

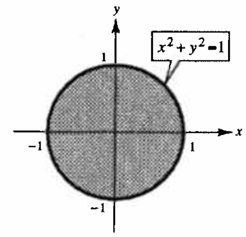
$$= \int_0^4 \left[\frac{t^4}{16} + \frac{t^3}{2}\right] dt + \int_4^8 -2(8-t)^2\,dt = \frac{224}{5} - \frac{128}{3} = \frac{32}{15}$$

By Green's Theorem,

$$\int_R\int \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \int_0^4\int_{x^2/4}^x (2x - 2y)\,dy\,dx = \int_0^4 \left(x^2 - \frac{x^3}{2} + \frac{x^4}{16}\right) dx = \frac{32}{15}.$$



4. $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j},\ 0 \le t \le 2\pi$



$$\int_C y^2\,dx + x^2\,dy = \int_0^{2\pi} [\sin^2 t(-\sin t\,dt) + \cos^2 t(\cos t\,dt)]$$

$$= \int_0^{2\pi} (\cos^3 t - \sin^3 t)\,dt$$

$$= \int_0^{2\pi} [\cos t(1 - \sin^2 t) - \sin t(1 - \cos^2 t)]\,dt$$

$$= \left[\sin t - \frac{\sin^3 t}{3} + \cos t - \frac{\cos^3 t}{3}\right]_0^{2\pi} = 0$$

By Green's Theorem,

$$\int_R\int \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (2x - 2y)\,dy\,dx$$

$$= \int_0^{2\pi}\int_0^1 (2r\cos\theta - 2r\sin\theta)r\,dr\,d\theta = \frac{2}{3}\int_0^{2\pi} (\cos\theta - \sin\theta)\,d\theta = \frac{2}{3}(0) = 0.$$

5. C: $x^2 + y^2 = 4$

Let $x = 2\cos t$ and $y = 2\sin t,\ 0 \le t \le 2\pi.$

$$\int_C xe^y\,dx + e^x\,dy = \int_0^{2\pi} [2\cos t e^{2\sin t}(-2\sin t) + e^{2\cos t}(2\cos t)]\,dt \approx 19.99$$

$$\int_R\int \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (e^x - xe^y)\,dy\,dx = \int_{-2}^{2} \left[2\sqrt{4-x^2}\,e^x - xe^{\sqrt{4-x^2}} + xe^{-\sqrt{4-x^2}}\right] dx \approx 19.99$$

6. C: boundary of the region lying between the graphs of $y = x$ and $y = x^3$

$$\int_C xe^y\,dx + e^x\,dy = \int_0^1 (xe^{x^3} + 3x^2e^x)\,dx + \int_1^0 (xe^x + e^x)\,dx \approx 2.936 - 2.718 \approx 0.22$$

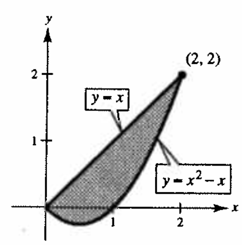
$$\int_R\int \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \int_0^1\int_{x^3}^{x} (e^x - xe^y)\,dy\,dx = \int_0^1 (xe^{x^3} - x^3e^x)\,dx \approx 0.22$$

In Exercises 7-10, $\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y} = 1$.

7. $$\int_C (y - x)\,dx + (2x - y)\,dy = \int_0^2\int_{x^2-x}^{x} dy\,dx$$

$$= \int_0^2 (2x - x^2)\,dx$$

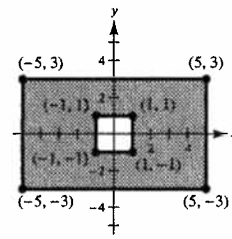
$$= \frac{4}{3}$$



8. Since C is an ellipse with $a = 2$ and $b = 1$, then R is an ellipse of area $\pi ab = 2\pi$. Thus, Green's Theorem yields

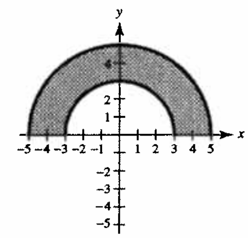
$$\int_C (y - x)\,dx + (2x - y)\,dy = \int_R\int 1\,dA = \text{Area of ellipse} = 2\pi.$$

9. From the accompanying figure, we see that R is the shaded region. Thus, Green's Theorem yields



$$\int_C (y - x)\,dx + (2x - y)\,dy = \iint_R 1\,dA$$
$$= \text{Area of } R$$
$$= 6(10) - 2(2)$$
$$= 56.$$

10. R is the shaded region of the accompanying figure.



$$\int_C (y - x)\,dx + (2x - y)\,dy = \iint_R 1\,dA$$
$$= \text{Area of shaded region}$$
$$= \frac{1}{2}\pi[25 - 9] = 8\pi$$

11. Since the curves $y = 0$ and $y = 4 - x^2$ intersect at $(-2, 0)$ and $(2, 0)$, Green's Theorem yields

$$\int_C 2xy\,dx + (x + y)\,dy = \iint_R (1 - 2x)\,dA = \int_{-2}^{2}\int_0^{4-x^2} (1 - 2x)\,dy\,dx$$
$$= \int_{-2}^{2}\Big[y - 2xy\Big]_0^{4-x^2}\,dx$$
$$= \int_{-2}^{2} (4 - 8x - x^2 + 2x^3)\,dx$$
$$= \left[4x - 4x^2 - \frac{x^3}{3} + \frac{x^4}{2}\right]_{-2}^{2}$$
$$= -\frac{8}{3} - \frac{8}{3} + 16 = \frac{32}{3}.$$

12. The given curves intersect at $(0, 0)$ and $(9, 3)$. Thus, Green's Theorem yields

$$\int_C y^2\,dx + xy\,dy = \iint_R (y - 2y)\,dA$$
$$= \int_0^9\int_0^{\sqrt{x}} -y\,dy\,dx = \int_0^9 \left[\frac{-y^2}{2}\right]_0^{\sqrt{x}} dx = \int_0^9 \frac{-x}{2}\,dx = \left[\frac{-x^2}{4}\right]_0^9 = -\frac{81}{4}.$$

13. Since R is the interior of the circle $x^2 + y^2 = a^2$, Green's Theorem yields

$$\int_C (x^2 - y^2)\,dx + 2xy\,dy = \iint_R (2y + 2y)\,dA$$
$$= \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} 4y\,dy\,dx = 4\int_{-a}^{a} 0\,dx = 0.$$

14. In this case, let $y = r\sin\theta$, $x = r\cos\theta$. Then $dA = r\,dr\,d\theta$ and Green's Theorem yields

$$\begin{aligned}\int_C (x^2 - y^2)\,dx + 2xy\,dy &= \int\!\!\int_R 4y\,dA = 4\int_0^{2\pi}\int_0^{1+\cos\theta} r\sin\theta r\,dr\,d\theta \\ &= 4\int_0^{2\pi}\int_0^{1+\cos\theta} r^2\sin\theta\,dr\,d\theta \\ &= \frac{4}{3}\int_0^{2\pi}\sin\theta(1+\cos\theta)^3\,d\theta \\ &= \left[-\frac{(1+\cos\theta)^4}{3}\right]_0^{2\pi} = 0.\end{aligned}$$

15. Since $\dfrac{\partial M}{\partial y} = \dfrac{2x}{x^2+y^2} = \dfrac{\partial N}{\partial x}$,

we have path independence and

$$\int\!\!\int_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = 0.$$

16. Since $\dfrac{\partial M}{\partial y} = -2e^x\sin 2y = \dfrac{\partial N}{\partial x}$ we have

$$\int\!\!\int_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = 0.$$

17. By Green's Theorem,

$$\begin{aligned}\int_C \sin x\cos y\,dx + (xy + \cos x\sin y)\,dy &= \int\!\!\int_R [(y - \sin x\sin y) - (-\sin x\sin y)]\,dA \\ &= \int_0^1\int_x^{\sqrt{x}} y\,dy\,dx = \frac{1}{2}\int_0^1 (x - x^2)\,dx = \frac{1}{2}\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{12}.\end{aligned}$$

18. By Green's Theorem,

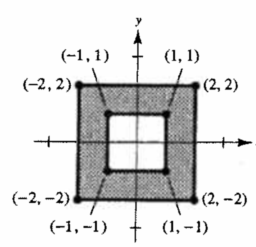
$$\int_C (e^{-x^2/2} - y)\,dx + (e^{-y^2/2} + x)\,dy = \int\!\!\int_R 2\,dA = 2(\text{Area of } R) = 2[\pi(6)^2 - \pi(2)(3)] = 60\pi.$$

19. By Green's Theorem,

$$\begin{aligned}\int_C xy\,dx + (x+y)\,dy &= \int\!\!\int_R (1 - x)\,dA \\ &= \int_0^{2\pi}\int_1^3 (1 - r\cos\theta)r\,dr\,d\theta = \int_0^{2\pi}\left(4 - \frac{26}{3}\cos\theta\right)d\theta = 8\pi.\end{aligned}$$

20. By Green's Theorem,

$$\begin{aligned}\int_C 3x^2e^y\,dx + e^y\,dy &= \int\!\!\int_R -3x^2e^y\,dA \\ &= \int_1^2\int_{-2}^2 -3x^2e^y\,dy\,dx + \int_{-1}^1\int_1^2 -3x^2e^y\,dy\,dx \\ &\quad + \int_{-2}^{-1}\int_{-2}^2 -3x^2e^y\,dy\,dx + \int_{-1}^1\int_{-2}^{-1} -3x^2e^y\,dy\,dx \\ &= -7(e^2 - e^{-2}) - 2(e^2 - e) - 7(e^2 - e^{-2}) - 2(e^{-1} - e^{-2}) \\ &= -16e^2 + 16e^{-2} + 2e - 2e^{-1}.\end{aligned}$$



21. $\mathbf{F}(x, y) = xy\mathbf{i} + (x + y)\mathbf{j}$

C: $x^2 + y^2 = 4$

$$\text{Work} = \int_C xy\,dx + (x+y)\,dy = \int\!\!\int_R (1 - x)\,dA = \int_0^{2\pi}\int_0^2 (1 - r\cos\theta)r\,dr\,d\theta = \int_0^{2\pi}\left(2 - \frac{8}{3}\cos\theta\right)d\theta = 4\pi$$

22. $\mathbf{F}(x, y) = (e^x - 3y)\mathbf{i} + (e^y + 6x)\mathbf{j}$

C: $r = 2\cos\theta$

Work $= \displaystyle\int_C (e^x - 3y)\,dx + (e^y + 6x)\,dy = \iint_R 9\,dA = 9\pi$ since $r = 2\cos\theta$ is a circle with a radius of one.

23. $\mathbf{F}(x, y) = (x^{3/2} - 3y)\mathbf{i} + (6x + 5\sqrt{y})\mathbf{j}$

C: boundary of the triangle with vertices $(0, 0)$, $(5, 0)$, $(0, 5)$

Work $= \displaystyle\int_C (x^{3/2} - 3y)\,dx + \left(6x + 5\sqrt{y}\right)dy = \iint_R 9\,dA = 9\left(\tfrac{1}{2}\right)(5)(5) = \tfrac{225}{2}$

24. $\mathbf{F}(x, y) = (3x^2 + y)\mathbf{i} + 4xy^2\mathbf{j}$

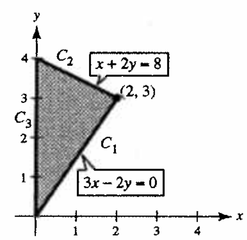
C: boundary of the region bounded by the graphs of $y = \sqrt{x}$, $y = 0$, $x = 9$

Work $= \displaystyle\int_C (3x^2 + y)\,dx + 4xy^2\,dy = \int_0^9\int_0^{\sqrt{x}} (4y^2 - 1)\,dy\,dx = \int_0^9 \left(\frac{4}{3}x^{3/2} - x^{1/2}\right)dx = \frac{558}{5}$

25. C: let $x = a\cos t$, $y = a\sin t$, $0 \le t \le 2\pi$. By Theorem 15.9, we have

$$A = \frac{1}{2}\int_C x\,dy - y\,dx = \frac{1}{2}\int_0^{2\pi} [a\cos t(a\cos t) - a\sin t(-a\sin t)]\,dt = \frac{1}{2}\int_0^{2\pi} a^2\,dt = \left[\frac{a^2}{2}t\right]_0^{2\pi} = \pi a^2.$$

26. From the figure we see that



C_1: $y = \dfrac{3}{2}x$, $dy = \dfrac{3}{2}dx$, $0 \le x \le 2$

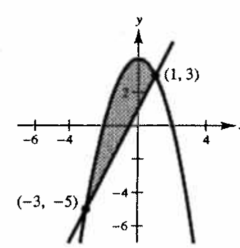
C_2: $y = -\dfrac{x}{2} + 4$, $dy = -\dfrac{1}{2}dx$

C_3: $x = 0$, $dx = 0$.

$$A = \frac{1}{2}\int_0^2 \left(\frac{3}{2}x - \frac{3}{2}x\right)dx + \frac{1}{2}\int_2^0 \left(-\frac{1}{2}x + \frac{x}{2} - 4\right)dx + \frac{1}{2}(0)$$

$$= \frac{1}{2}\int_2^0 (-4)\,dx = 2\int_0^2 dx = 4$$

27. From the accompanying figure we see that



C_1: $y = 2x + 1$, $dy = 2\,dx$

C_2: $y = 4 - x^2$, $dy = -2x\,dx$.

Thus, by Theorem 15.9, we have

$$A = \frac{1}{2}\int_{-3}^{1} [x(2) - (2x + 1)]\,dx + \frac{1}{2}\int_1^{-3} [x(-2x) - (4 - x^2)]\,dx$$

$$= \frac{1}{2}\int_{-3}^{1} (-1)\,dx + \frac{1}{2}\int_1^{-3} (-x^2 - 4)\,dx$$

$$= \frac{1}{2}\int_{-3}^{1} (-1)\,dx + \frac{1}{2}\int_{-3}^{1} (x^2 + 4)\,dx = \frac{1}{2}\int_{-3}^{1} (3 + x^2)\,dx = \frac{1}{2}\left[3x + \frac{x^3}{3}\right]_{-3}^{1} = \frac{32}{3}.$$

28. Since the loop of the folium is formed on the interval $0 \le t \le \infty$,

$$dx = \frac{3(1 - 2t^3)}{(t^3 + 1)^2}\,dt \text{ and } dy = \frac{3(2t - t^4)}{(t^3 + 1)^2}\,dt,$$

we have

$$A = \frac{1}{2}\int_0^{\infty}\left[\left(\frac{3t}{t^3+1}\right)\frac{3(2t - t^4)}{(t^3+1)^2} - \left(\frac{3t^2}{t^3+1}\right)\frac{3(1 - 2t^3)}{(t^3+1)^2}\right]dt$$

$$= \frac{9}{2}\int_0^{\infty}\frac{t^5 + t^2}{(t^3+1)^3}\,dt = \frac{9}{2}\int_0^{\infty}\frac{t^2(t^3+1)}{(t^3+1)^3}\,dt = \frac{3}{2}\int_0^{\infty}3t^2(t^3+1)^{-2}\,dt = \left[\frac{-3}{2(t^3+1)}\right]_0^{\infty} = \frac{3}{2}.$$

29. See Theorem 15.8, page 1089.

30. See Theorem 15.9: $A = \dfrac{1}{2}\displaystyle\int_C x\,dy - y\,dx.$

31. For the moment about the x-axis, $M_x = \displaystyle\iint_R y\,dA$. Let $N = 0$ and $M = -y^2/2$. By Green's Theorem,

$$M_x = \int_C -\frac{y^2}{2}\,dx = -\frac{1}{2}\int_C y^2\,dx \text{ and } \bar{y} = \frac{M_x}{2A} = -\frac{1}{2A}\int_C y^2\,dx.$$

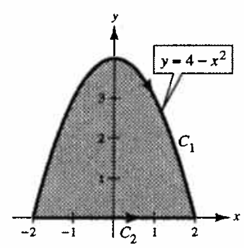
For the moment about the y-axis, $M_y = \displaystyle\iint_R x\,dA$. Let $N = x^2/2$ and $M = 0$. By Green's Theorem,

$$M_y = \int_C \frac{x^2}{2}\,dy = \frac{1}{2}\int_C x^2\,dy \text{ and } \bar{x} = \frac{M_y}{2A} = \frac{1}{2A}\int_C x^2\,dy.$$

32. By Theorem 15.9 and the fact that $x = r\cos\theta,\ y = r\sin\theta$, we have

$$A = \frac{1}{2}\int x\,dy - y\,dx = \frac{1}{2}\int (r\cos\theta)(r\cos\theta)\,d\theta - (r\sin\theta)(-r\sin\theta)\,d\theta = \frac{1}{2}\int_C r^2\,d\theta.$$

33. $A = \displaystyle\int_{-2}^{2}(4 - x^2)\,dx = \left[4x - \frac{x^3}{3}\right]_{-2}^{2} = \frac{32}{3}$



$$\bar{x} = \frac{1}{2A}\int_{C_1} x^2\,dy + \frac{1}{2A}\int_{C_2} x^2\,dy$$

For C_1, $dy = -2x\,dx$ and for C_2, $dy = 0$. Thus,

$$\bar{x} = \frac{1}{2(32/3)}\int_2^{-2} x^2(-2x\,dx) = \left[\frac{3}{64}\left(-\frac{x^4}{2}\right)\right]_2^{-2} = 0.$$

To calculate $\bar{y}$, note that $y = 0$ along C_2. Thus,

$$\bar{y} = \frac{-1}{2(32/3)}\int_2^{-2}(4 - x^2)^2\,dx = \frac{3}{64}\int_{-2}^{2}(16 - 8x^2 + x^4)\,dx = \frac{3}{64}\left[16x - \frac{8x^3}{3} + \frac{x^5}{5}\right]_{-2}^{2} = \frac{8}{5}.$$

$$(\bar{x}, \bar{y}) = \left(0, \frac{8}{5}\right)$$

34. Since A = area of semicircle $= \dfrac{\pi a^2}{2}$, we have $\dfrac{1}{2A} = \dfrac{1}{\pi a^2}$. Note that $y = 0$ and $dy = 0$ along the boundary $y = 0$. Let $x = a\cos t,\ y = a\sin t,\ 0 \le t \le \pi$, then

$$\bar{x} = \frac{1}{\pi a^2}\int_0^{\pi} a^2\cos^2 t(a\cos t)\,dt = \frac{a}{\pi}\int_0^{\pi}\cos^3 t\,dt = \frac{a}{\pi}\int_0^{\pi}(1 - \sin^2 t)\cos t\,dt = \frac{a}{\pi}\left[\sin t - \frac{\sin^3 t}{3}\right]_0^{\pi} = 0$$

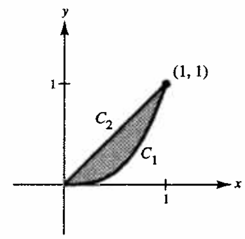
$$\bar{y} = \frac{-1}{\pi a^2}\int_0^{\pi} a^2\sin^2 t(-a\sin t\,dt) = \frac{a}{\pi}\int_0^{\pi}\sin^3 t\,dt = \frac{a}{\pi}\left[-\cos t + \frac{\cos^3 t}{3}\right]_0^{\pi} = \frac{4a}{3\pi}.$$

$$(\bar{x}, \bar{y}) = \left(0, \frac{4a}{3\pi}\right)$$

35. Since $A = \int_0^1 (x - x^3)\, dx = \left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1 = \frac{1}{4}$, we have $\frac{1}{2A} = 2$. On C_1 we have $y = x^3$, $dy = 3x^2\, dx$ and on C_2 we have $y = x$, $dy = dx$. Thus,

$$\begin{aligned}\bar{x} &= 2\int_C x^2\, dy = 2\int_{C_1} x^2(3x^2\, dx) + 2\int_{C_2} x^2\, dx\\ &= 6\int_0^1 x^4\, dx + 2\int_1^0 x^2\, dx = \frac{6}{5} - \frac{2}{3} = \frac{8}{15}\\ \bar{y} &= -2\int_C y^2\, dx\\ &= -2\int_0^1 x^6\, dx - 2\int_1^0 x^2\, dx = -\frac{2}{7} + \frac{2}{3} = \frac{8}{21}.\end{aligned}$$

$$(\bar{x}, \bar{y}) = \left(\frac{8}{15}, \frac{8}{21}\right)$$

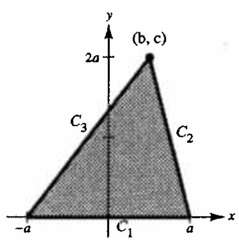


36. Since $A = \frac{1}{2}(2a)(c) = ac$, we have $\frac{1}{2A} = \frac{1}{2ac}$,

C_1: $y = 0$, $dy = 0$

C_2: $y = \frac{c}{b-a}(x - a)$, $dy = \frac{c}{b-a}\, dx$

C_3: $y = \frac{c}{b+a}(x + a)$, $dy = \frac{c}{b+a}\, dx$.



Thus,

$$\bar{x} = \frac{1}{2ac}\int_C x^2\, dy = \frac{1}{2ac}\left[\int_{-a}^{a} 0 + \int_a^b x^2 \frac{c}{b-a}\, dx + \int_b^{-a} x^2 \frac{c}{b+a}\, dx\right] = \frac{1}{2ac}\left[0 + \frac{2abc}{3}\right] = \frac{b}{3}$$

$$\begin{aligned}\bar{y} &= \frac{-1}{2ac}\int_C y^2\, dx = \frac{-1}{2ac}\left[0 + \int_a^b \left(\frac{c}{b-a}\right)^2 (x-a)^2\, dx + \int_b^{-a}\left(\frac{c}{b+a}\right)^2 (x+a)^2\, dx\right]\\ &= \frac{-1}{2ac}\left[\frac{c^2(b-a)}{3} - \frac{c^2(b+a)}{3}\right] = \frac{c}{3}.\end{aligned}$$

$$(\bar{x}, \bar{y}) = \left(\frac{b}{3}, \frac{c}{3}\right)$$

37.
$$\begin{aligned}A &= \frac{1}{2}\int_0^{2\pi} a^2(1 - \cos\theta)^2\, d\theta\\ &= \frac{a^2}{2}\int_0^{2\pi}\left(1 - 2\cos\theta + \frac{1}{2} + \frac{\cos 2\theta}{2}\right) d\theta = \frac{a^2}{2}\left[\frac{3\theta}{2} - 2\sin\theta + \frac{1}{4}\sin 2\theta\right]_0^{2\pi} = \frac{a^2}{2}(3\pi) = \frac{3\pi a^2}{2}\end{aligned}$$

38.
$$A = \frac{1}{2}\int_0^{\pi} a^2\cos^2 3\theta\, d\theta = \frac{a^2}{2}\int_0^{\pi}\frac{1 + \cos 6\theta}{2}\, d\theta = \frac{a^2}{4}\left[\theta + \frac{\sin 6\theta}{6}\right]_0^{\pi} = \frac{\pi a^2}{4}$$

Note: In this case R is enclosed by $r = a\cos 3\theta$ where $0 \le \theta \le \pi$.

39. In this case the inner loop has domain $\frac{2\pi}{3} \le \theta \le \frac{4\pi}{3}$. Thus,

$$\begin{aligned}A &= \frac{1}{2}\int_{2\pi/3}^{4\pi/3} (1 + 4\cos\theta + 4\cos^2\theta)\, d\theta\\ &= \frac{1}{2}\int_{2\pi/3}^{4\pi/3} (3 + 4\cos\theta + 2\cos 2\theta)\, d\theta = \frac{1}{2}\Big[3\theta + 4\sin\theta + \sin 2\theta\Big]_{2\pi/3}^{4\pi/3} = \pi - \frac{3\sqrt{3}}{2}.\end{aligned}$$

40. In this case, $0 \le \theta \le 2\pi$ and we let

$$u = \frac{\sin\theta}{1+\cos\theta},\ \cos\theta = \frac{1-u^2}{1+u^2},\ d\theta = \frac{2\,du}{1+u^2}.$$

Now $u \Rightarrow \infty$ as $\theta \Rightarrow \pi$ and we have

$$A = 2\left(\frac{1}{2}\right)\int_0^{\pi}\frac{9}{(2-\cos\theta)^2}\,d\theta = 9\int_0^{\infty}\frac{\dfrac{2du}{1+u^2}}{4-4\left(\dfrac{1-u^2}{1+u^2}\right)+\dfrac{(1-u^2)^2}{(1+u^2)^2}} = 18\int_0^{\infty}\frac{1+u^2}{(1+3u^2)^2}\,du$$

$$= 18\int_0^{\infty}\frac{1/3}{1+3u^2}\,du + 18\int_0^{\infty}\frac{2/3}{(1+3u^2)^2}\,du = \left[\frac{6}{\sqrt{3}}\arctan\sqrt{3}\,u\right]_0^{\infty} + \frac{12}{\sqrt{3}}\left(\frac{1}{2}\right)\left[\frac{u}{1+3u^2} + \int\frac{\sqrt{3}}{1+3u^2}\,du\right]_0^{\infty}$$

$$= \frac{6}{\sqrt{3}}\left(\frac{\pi}{2}\right) + \frac{6}{\sqrt{3}}\left[\frac{u}{1+3u^2}\right]_0^{\infty} + \left[\frac{6}{\sqrt{3}}\arctan\sqrt{3}u\right]_0^{\infty} = \frac{3\pi}{\sqrt{3}} + 0 + \frac{3\pi}{\sqrt{3}} = 2\sqrt{3}\pi.$$

41. $I = \displaystyle\int_C \frac{y\,dx - x\,dy}{x^2+y^2}$

(a) Let $\mathbf{F} = \dfrac{y}{x^2+y^2}\mathbf{i} - \dfrac{x}{x^2+y^2}\mathbf{j}.$

$\mathbf{F}$ is conservative since $\dfrac{\partial N}{\partial x} = \dfrac{\partial M}{\partial y} = \dfrac{x^2-y^2}{(x^2+y^2)^2}.$

$\mathbf{F}$ is defined and has continuous first partials everywhere except at the origin. If C is a circle (a closed path) that does not contain the origin, then

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C M\,dx + N\,dy = \iint_R\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dA = 0.$$

(b) Let $\mathbf{r} = a\cos t\,\mathbf{i} - a\sin t\,\mathbf{j}$, $0 \le t \le 2\pi$ be a circle C_1 oriented clockwise inside C (see figure). Introduce line segments C_2 and C_3 as illustrated in Example 6 of this section in the text. For the region inside C and outside C_1, Green's Theorem applies. Note that since C_2 and C_3 have opposite orientations, the line integrals over them cancel. Thus, $C_4 = C_1 + C_2 + C + C_3$ and

$$\int_{C_4}\mathbf{F}\cdot d\mathbf{r} = \int_{C_1}\mathbf{F}\cdot d\mathbf{r} + \int_C \mathbf{F}\cdot d\mathbf{r} = 0.$$

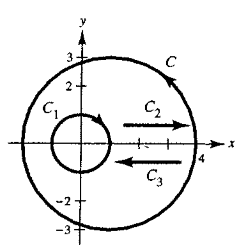
But,

$$\int_{C_1}\mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi}\left[\frac{(-a\sin t)(-a\sin t)}{a^2\cos^2 t + a^2\sin^2 t} + \frac{(-a\cos t)(-a\cos t)}{a^2\cos^2 t + a^2\sin^2 t}\right]dt$$

$$= \int_0^{2\pi}(\sin^2 t + \cos^2 t)\,dt = \Big[t\Big]_0^{2\pi} = 2\pi.$$

Finally, $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r} = -\int_{C_1}\mathbf{F}\cdot d\mathbf{r} = -2\pi.$

Note: If C were orientated clockwise, then the answer would have been 2π.



42. (a) Let C be the line segment joining (x_1, y_1) and (x_2, y_2).

$$y = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) + y_1$$

$$dy = \frac{y_2 - y_1}{x_2 - x_1}\,dx$$

$$\int_C -y\,dx + x\,dy = \int_{x_1}^{x_2}\left[-\frac{y_2 - y_1}{x_2 - x_1}(x - x_1) - y_1 + x\left(\frac{y_2 - y_1}{x_2 - x_1}\right)\right]dx = \int_{x_1}^{x_2}\left[x_1\left(\frac{y_2 - y_1}{x_2 - x_1}\right) - y_1\right]dx$$

$$= \left[\left[x_1\left(\frac{y_2 - y_1}{x_2 - x_1}\right) - y_1\right]x\right]_{x_1}^{x_2} = \left[x_1\left(\frac{y_2 - y_1}{x_2 - x_1}\right) - y_1\right](x_2 - x_1)$$

$$= x_1(y_2 - y_1) - y_1(x_2 - x_1) = x_1 y_2 - x_2 y_1$$

(b) Let C be the boundary of the region $A = \dfrac{1}{2}\displaystyle\int_C -y\,dx + x\,dy = \frac{1}{2}\int_R\int (1 - (-1))\,dA = \int_R\int dA.$

Therefore,

$$\int_R\int dA = \frac{1}{2}\left[\int_{C_1} -y\,dx + x\,dy + \int_{C_2} -y\,dx + x\,dy + \cdots + \int_{C_n} -y\,dx + x\,dy\right]$$

where C_1 is the line segment joining (x_1, y_1) and (x_2, y_2), C_2 is the line segment joining (x_2, y_2) and $(x_3, y_3), \ldots$, and C_n is the line segment joining (x_n, y_n) and (x_1, y_1). Thus,

$$\int_R\int dA = \frac{1}{2}[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + \cdots + (x_{n-1}y_n - x_ny_{n-1}) + (x_ny_1 - x_1y_n)].$$

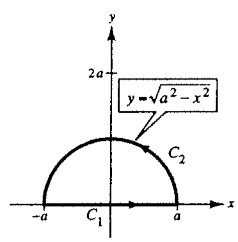
43. Pentagon: $(0, 0), (2, 0), (3, 2), (1, 4), (-1, 1)$

$A = \frac{1}{2}[(0 - 0) + (4 - 0) + (12 - 2) + (1 + 4) + (0 - 0)] = \frac{19}{2}$

44. Hexagon: $(0, 0), (2, 0), (3, 2), (2, 4), (0, 3), (-1, 1)$

$A = \frac{1}{2}[(0 - 0) + (4 - 0) + (12 - 4) + (6 - 0) + (0 + 3) + (0 - 0)] = \frac{21}{2}$

45. $\displaystyle\int_C y^n\,dx + x^n\,dy = \int_R\int\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dA$



For the line integral, use the two paths

C_1: $\mathbf{r}_1(x) = x\mathbf{i}, \ -a \le x \le a$

C_2: $\mathbf{r}_2(x) = x\mathbf{i} + \sqrt{a^2 - x^2}\,\mathbf{j}, \ \ x = a$ to $x = -a$

$$\int_{C_1} y^n\,dx + x^n\,dy = 0$$

$$\int_{C_2} y^n\,dx + x^n\,dy = \int_a^{-a}\left[(a^2 - x^2)^{n/2} + x^n\frac{-x}{\sqrt{a^2 - x^2}}\right]dx$$

$$\int_R\int\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dA = \int_{-a}^{a}\int_0^{\sqrt{a^2-x^2}}[nx^{n-1} - ny^{n-1}]\,dy\,dx$$

(a) For $n = 1, 3, 5, 7$, both integrals give 0.

(b) For n even, you obtain

$$n = 2: -\frac{4}{3}a^3 \qquad n = 4: -\frac{16}{15}a^5 \qquad n = 6: -\frac{32}{35}a^7 \qquad n = 8: -\frac{256}{315}a^9$$

(c) If n is odd then the integral equals 0.

46. Since $\int_C \mathbf{F} \cdot \mathbf{N}\, ds = \iint_R \operatorname{div} \mathbf{F}\, dA$, then

$$\int_C f D_{\mathbf{N}} g\, ds = \int_C f \nabla g \cdot \mathbf{N}\, ds$$

$$= \iint_R \operatorname{div}(f\nabla g)\, dA = \iint_R (f \operatorname{div}(\nabla g) + \nabla f \cdot \nabla g)\, dA = \iint_R (f\nabla^2 g + \nabla f \cdot \nabla g)\, dA.$$

47. $\int_C (f D_{\mathbf{N}} g - g D_{\mathbf{N}} f)\, ds = \int_C f D_{\mathbf{N}} g\, ds - \int_C g D_{\mathbf{N}} f\, ds$

$$= \iint_R (f\nabla^2 g + \nabla f \cdot \nabla g)\, dA - \iint_R (g\nabla^2 f + \nabla g \cdot \nabla f)\, dA = \iint_R (f\nabla^2 g - g\nabla^2 f)\, dA$$

48. $\int_C f(x)\, dx + g(y)\, dy = \iint_R \left[\frac{\partial}{\partial x} g(y) - \frac{\partial}{\partial y} f(x)\right] dA = \iint_R (0 - 0)\, dA = 0$

49. $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y} \implies \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0$$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C M\, dx + N\, dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \iint_R (0)\, dA = 0$$

Section 15.5 Parametric Surfaces

1. $\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + uv\mathbf{k}$

$z = xy$

Matches (c)

2. $\mathbf{r}(u, v) = u \cos v\,\mathbf{i} + u \sin v\,\mathbf{j} + u\mathbf{k}$

$x^2 + y^2 = z^2$

Matches (d)

3. $\mathbf{r}(u, v) = 2 \cos v \cos u\,\mathbf{i} + 2 \cos v \sin u\,\mathbf{j} + 2 \sin v\,\mathbf{k}$

$x^2 + y^2 + z^2 = 4$

Matches (b)

4. $\mathbf{r}(u, v) = 4 \cos u\,\mathbf{i} + 4 \sin u\,\mathbf{j} + v\mathbf{k}$

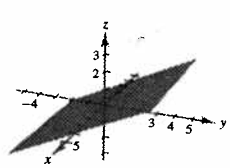
$x^2 + y^2 = 16$

Matches (a)

5. $\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + \frac{v}{2}\mathbf{k}$

$y - 2z = 0$

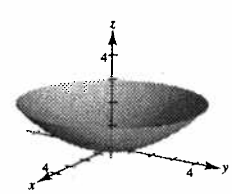
Plane



6. $\mathbf{r}(u, v) = 2u \cos v\,\mathbf{i} + 2u \sin v\,\mathbf{j} + \frac{1}{2}u^2\mathbf{k}$

$z = \frac{1}{2}u^2,\ x^2 + y^2 = 4u^2 \implies z = \frac{1}{8}(x^2 + y^2)$

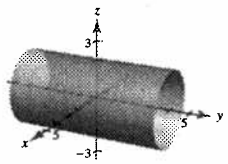
Paraboloid



7. $\mathbf{r}(u, v) = 2\cos u\mathbf{i} + v\mathbf{j} + 2\sin u\mathbf{k}$

$x^2 + z^2 = 4$

Cylinder



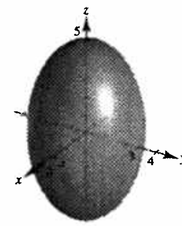
8. $\mathbf{r}(u, v) = 3\cos v\cos u\mathbf{i} + 3\cos v\sin u\mathbf{j} + 5\sin v\mathbf{k}$

$x^2 + y^2 = 9\cos^2 v\cos^2 u + 9\cos^2 v\sin^2 u = 9\cos^2 v$

$$\frac{x^2 + y^2}{9} + \frac{z^2}{25} = \cos^2 v + \sin^2 v = 1$$

$$\frac{x^2}{9} + \frac{y^2}{9} + \frac{z^2}{25} = 1$$

Ellipsoid

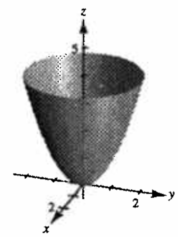


For Exercises 9-12,

$\mathbf{r}(u, v) = u\cos v\mathbf{i} + u\sin v\mathbf{j} + u^2\mathbf{k},\ 0 \le u \le 2,\ 0 \le v \le 2\pi.$

Eliminating the parameter yields

$z = x^2 + y^2,\ 0 \le z \le 4.$



9. $\mathbf{s}(u, v) = u\cos v\mathbf{i} + u\sin v\mathbf{j} - u^2\mathbf{k},\ 0 \le u \le 2,\ 0 \le v \le 2\pi$

$z = -(x^2 + y^2)$

The paraboloid is reflected (inverted) through the xy-plane.

10. $\mathbf{s}(u, v) = u\cos v\mathbf{i} + u^2\mathbf{j} + u\sin v\mathbf{k},\ 0 \le u \le 2,\ 0 \le v \le 2\pi$

$y = x^2 + z^2$

The paraboloid opens along the y-axis instead of the z-axis.

11. $\mathbf{s}(u, v) = u\cos v\mathbf{i} + u\sin v\mathbf{j} + u^2\mathbf{k},\ 0 \le u \le 3,\ 0 \le v \le 2\pi$

The height of the paraboloid is increased from 4 to 9.

12. $\mathbf{s}(u, v) = 4u\cos v\mathbf{i} + 4u\sin v\mathbf{j} + u^2\mathbf{k},\ 0 \le u \le 2,\ 0 \le v \le 2\pi$

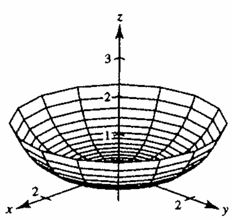
$$z = \frac{x^2 + y^2}{16}$$

The paraboloid is "wider." The top is now the circle $x^2 + y^2 = 64$. It was $x^2 + y^2 = 4$.

13. $\mathbf{r}(u, v) = 2u\cos v\mathbf{i} + 2u\sin v\mathbf{j} + u^4\mathbf{k}$,

$0 \le u \le 1,\ 0 \le v \le 2\pi$

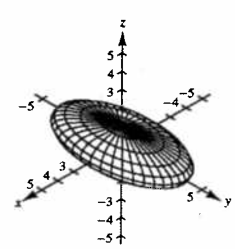
$$z = \frac{(x^2 + y^2)^2}{16}$$



14. $\mathbf{r}(u, v) = 2\cos v\cos u\mathbf{i} + 4\cos v\sin u\mathbf{j} + \sin v\mathbf{k}$,

$0 \le u \le 2\pi,\ 0 \le v \le 2\pi$

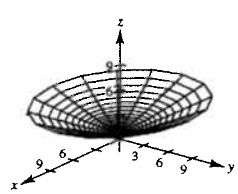
$$\frac{x^2}{4} + \frac{y^2}{16} + \frac{z^2}{1} = 1$$



15. $\mathbf{r}(u, v) = 2\sinh u\cos v\mathbf{i} + \sinh u\sin v\mathbf{j} + \cosh u\mathbf{k}$,

$0 \le u \le 2,\ 0 \le v \le 2\pi$

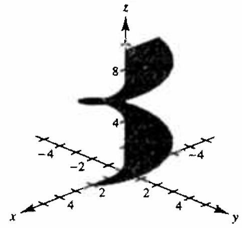
$$\frac{z^2}{1} - \frac{x^2}{4} - \frac{y^2}{1} = 1$$



16. $\mathbf{r}(u, v) = 2u \cos v\mathbf{i} + 2u \sin v\mathbf{j} + v\mathbf{k},$

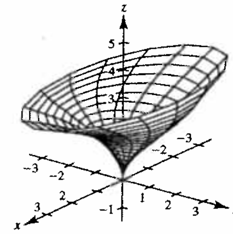
$0 \le u \le 1,\ 0 \le v \le 3\pi$

$\tan z = \dfrac{y}{x}$



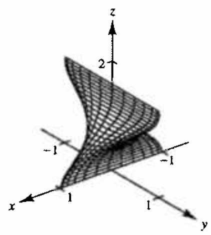
17. $\mathbf{r}(u, v) = (u - \sin u) \cos v\mathbf{i} + (1 - \cos u) \sin v\mathbf{j} + u\mathbf{k},$

$0 \le u \le \pi,\ 0 \le v \le 2\pi$



18. $\mathbf{r}(u, v) = \cos^3 u \cos v\mathbf{i} + \sin^3 u \sin v\mathbf{j} + u\mathbf{k},$

$0 \le u \le \dfrac{\pi}{2},\ 0 \le v \le 2\pi$



19. $z = y$

$\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + v\mathbf{k}$

20. $z = 6 - x - y$

$\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + (6 - u - v)\mathbf{k}$

21. $x^2 + y^2 = 16$

$\mathbf{r}(u, v) = 4 \cos u\mathbf{i} + 4 \sin u\mathbf{j} + v\mathbf{k}$

22. $4x^2 + y^2 = 16$

$\mathbf{r}(u, v) = 2 \cos u\mathbf{i} + 4 \sin u\mathbf{j} + v\mathbf{k}$

23. $z = x^2$

$\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + u^2\mathbf{k}$

24. $\dfrac{x^2}{9} + \dfrac{y^2}{4} + \dfrac{z^2}{1} = 1$

$\mathbf{r}(u, v) = 3 \cos v \cos u\mathbf{i} + 2 \cos v \sin u\mathbf{j} + \sin v\mathbf{k}$

25. $z = 4$ inside $x^2 + y^2 = 9.$

$\mathbf{r}(u, v) = v \cos u\mathbf{i} + v \sin u\mathbf{j} + 4\mathbf{k},\ 0 \le v \le 3$

26. $z = x^2 + y^2$ inside $x^2 + y^2 = 9.$

$\mathbf{r}(u, v) = v \cos u\mathbf{i} + v \sin u\mathbf{j} + v^2\mathbf{k},\ 0 \le v \le 3$

27. Function: $y = \dfrac{x}{2},\ 0 \le x \le 6$

Axis of revolution: x-axis

$x = u,\ y = \dfrac{u}{2} \cos v,\ z = \dfrac{u}{2} \sin v$

$0 \le u \le 6,\ 0 \le v \le 2\pi$

28. Function: $y = x^{3/2},\ 0 \le x \le 4$

Axis of revolution: x-axis

$x = u,\ y = u^{3/2} \cos v,\ z = u^{3/2} \sin v$

$0 \le u \le 4,\ 0 \le v \le 2\pi$

29. Function: $x = \sin z,\ 0 \le z \le \pi$

Axis of revolution: z-axis

$x = \sin u \cos v,\ y = \sin u \sin v,\ z = u$

$0 \le u \le \pi,\ 0 \le v \le 2\pi$

30. Function: $z = 4 - y^2,\ 0 \le y \le 2$

Axis of revolution: y-axis

$x = (4 - u^2) \cos v,\ y = u,\ z = (4 - u^2) \sin v$

$0 \le u \le 2,\ 0 \le v \le 2\pi$

31. $\mathbf{r}(u, v) = (u + v)\mathbf{i} + (u - v)\mathbf{j} + v\mathbf{k},\ (1, -1, 1)$

$\mathbf{r}_u(u, v) = \mathbf{i} + \mathbf{j},\ \mathbf{r}_v(u, v) = \mathbf{i} - \mathbf{j} + \mathbf{k}$

At $(1, -1, 1)$, $u = 0$ and $v = 1$.

$\mathbf{r}_u(0, 1) = \mathbf{i} + \mathbf{j},\ \mathbf{r}_v(0, 1) = \mathbf{i} - \mathbf{j} + \mathbf{k}$

$$\mathbf{N} = \mathbf{r}_u(0, 1) \times \mathbf{r}_v(0, 1) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 1 & -1 & 1 \end{vmatrix} = \mathbf{i} - \mathbf{j} - 2\mathbf{k}$$

Tangent plane: $(x - 1) - (y + 1) - 2(z - 1) = 0$

$$x - y - 2z = 0$$

(The original plane!)

32. $\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + \sqrt{uv}\,\mathbf{k},\ (1, 1, 1)$

$$\mathbf{r}_u(u, v) = \mathbf{i} + \frac{v}{2\sqrt{uv}}\mathbf{k},\ \mathbf{r}_v(u, v) = \mathbf{j} + \frac{u}{2\sqrt{uv}}\mathbf{k}$$

At $(1, 1, 1)$, $u = 1$ and $v = 1$.

$$\mathbf{r}_u(1, 1) = \mathbf{i} + \frac{1}{2}\mathbf{k},\ \mathbf{r}_v(1, 1) = \mathbf{j} + \frac{1}{2}\mathbf{k}$$

$$\mathbf{N} = \mathbf{r}_u(1, 1) \times \mathbf{r}_v(1, 1) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \end{vmatrix} = -\tfrac{1}{2}\mathbf{i} - \tfrac{1}{2}\mathbf{j} + \mathbf{k}$$

Direction numbers: $1, 1, -2$

Tangent plane: $(x - 1) + (y - 1) - 2(z - 1) = 0$

$$x + y - 2z = 0$$

33. $\mathbf{r}(u, v) = 2u\cos v\mathbf{i} + 3u\sin v\mathbf{j} + u^2\mathbf{k},\ (0, 6, 4)$

$\mathbf{r}_u(u, v) = 2\cos v\mathbf{i} + 3\sin v\mathbf{j} + 2u\mathbf{k}$

$\mathbf{r}_v(u, v) = -2u\sin v\mathbf{i} + 3u\cos v\mathbf{j}$

At $(0, 6, 4)$, $u = 2$ and $v = \pi/2$.

$$\mathbf{r}_u\left(2, \frac{\pi}{2}\right) = 3\mathbf{j} + 4\mathbf{k},\ \mathbf{r}_v\left(2, \frac{\pi}{2}\right) = -4\mathbf{i}$$

$$\mathbf{N} = \mathbf{r}_u\left(2, \frac{\pi}{2}\right) \times \mathbf{r}_v\left(2, \frac{\pi}{2}\right)$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 3 & 4 \\ -4 & 0 & 0 \end{vmatrix} = -16\mathbf{j} + 12\mathbf{k}$$

Direction numbers: $0, 4, -3$

Tangent plane: $4(y - 6) - 3(z - 4) = 0$

$$4y - 3z = 12$$

34. $\mathbf{r}(u, v) = 2u\cosh v\mathbf{i} + 2u\sinh v\mathbf{j} + \frac{1}{2}u^2\mathbf{k},$

$\mathbf{r}_u(u, v) = 2\cosh v\mathbf{i} + 2\sinh v\mathbf{j} + u\mathbf{k}$

$\mathbf{r}_v(u, v) = 2u\sinh v\mathbf{i} + 2u\cosh v\mathbf{j}$

At $(-4, 0, 2)$, $u = -2$ and $v = 0$.

$\mathbf{r}_u(-2, 0) = 2\mathbf{i} - 2\mathbf{k},\ \mathbf{r}_v(-2, 0) = -4\mathbf{j}$

$\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v = -8\mathbf{i} - 8\mathbf{k}$

Direction numbers: $1, 0, 1$

Tangent plane: $(x + 4) + (z - 2) = 0$

$$x + z = -2$$

35. $\mathbf{r}(u, v) = 2u\mathbf{i} - \frac{v}{2}\mathbf{j} + \frac{v}{2}\mathbf{k},\ 0 \le u \le 2,\ 0 \le v \le 1$

$$\mathbf{r}_u(u, v) = 2\mathbf{i},\ \mathbf{r}_v(u, v) = -\frac{1}{2}\mathbf{j} + \frac{1}{2}\mathbf{k}$$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = -\mathbf{j} - \mathbf{k}$$

$\|\mathbf{r}_u \times \mathbf{r}_v\| = \sqrt{2}$

$$A = \int_0^1 \int_0^2 \sqrt{2}\, du\, dv = 2\sqrt{2}$$

36. $\mathbf{r}(u, v) = 4u\cos v\mathbf{i} + 4u\sin v\mathbf{j} + u^2\mathbf{k},\ 0 \le u \le 2,\ 0 \le v \le 2\pi$

$\mathbf{r}_u(u, v) = 4\cos v\mathbf{i} + 4\sin v\mathbf{j} + 2u\mathbf{k}$

$\mathbf{r}_v(u, v) = -4u\sin v\mathbf{i} + 4u\cos v\mathbf{j}$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4\cos v & 4\sin v & 2u \\ -4u\sin v & 4u\cos v & 0 \end{vmatrix} = -8u^2\cos v\mathbf{i} - 8u^2\sin v\mathbf{j} + 16u\mathbf{k}$$

$\|\mathbf{r}_u \times \mathbf{r}_v\| = \sqrt{64u^4 + 256u^2} = 8u\sqrt{u^2 + 4}$

$$A = \int_0^{2\pi}\int_0^2 8u\sqrt{u^2 + 4}\,du\,dv = \int_0^{2\pi}\left(\frac{128\sqrt{2}}{3} - \frac{64}{3}\right)dv = \frac{128\pi}{3}\left(2\sqrt{2} - 1\right)$$

37. $\mathbf{r}(u, v) = a\cos u\mathbf{i} + a\sin u\mathbf{j} + v\mathbf{k},\ 0 \le u \le 2\pi,\ 0 \le v \le b$

$\mathbf{r}_u(u, v) = -a\sin u\mathbf{i} + a\cos u\mathbf{j}$

$\mathbf{r}_v(u, v) = \mathbf{k}$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a\sin u & a\cos u & 0 \\ 0 & 0 & 1 \end{vmatrix} = a\cos u\mathbf{i} + a\sin u\mathbf{j}$$

$\|\mathbf{r}_u \times \mathbf{r}_v\| = a$

$$A = \int_0^b\int_0^{2\pi} a\,du\,dv = 2\pi ab$$

38. $\mathbf{r}(u, v) = a\sin u\cos v\mathbf{i} + a\sin u\sin v\mathbf{j} + a\cos u\mathbf{k},\ 0 \le u \le \pi,\ 0 \le v \le 2\pi$

$\mathbf{r}_u(u, v) = a\cos u\cos v\mathbf{i} + a\cos u\sin v\mathbf{j} - a\sin u\mathbf{k}$

$\mathbf{r}_v(u, v) = -a\sin u\sin v\mathbf{i} + a\sin u\cos v\mathbf{j}$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a\cos u\cos v & a\cos u\sin v & -a\sin u \\ -a\sin u\sin v & a\sin u\cos v & 0 \end{vmatrix} = a^2\sin^2 u\cos v\mathbf{i} + a^2\sin^2 u\sin v\mathbf{j} + a^2\sin u\cos u\mathbf{k}$$

$\|\mathbf{r}_u \times \mathbf{r}_v\| = a^2\sin u$

$$A = \int_0^{2\pi}\int_0^{\pi} a^2\sin u\,du\,dv = 4\pi a^2$$

39. $\mathbf{r}(u, v) = au\cos v\mathbf{i} + au\sin v\mathbf{j} + u\mathbf{k},\ 0 \le u \le b,\ 0 \le v \le 2\pi$

$\mathbf{r}_u(u, v) = a\cos v\mathbf{i} + a\sin v\mathbf{j} + \mathbf{k}$

$\mathbf{r}_v(u, v) = -au\sin v\mathbf{i} + au\cos v\mathbf{j}$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a\cos v & a\sin v & 1 \\ -au\sin v & au\cos v & 0 \end{vmatrix} = -au\cos v\mathbf{i} - au\sin v\mathbf{j} + a^2u\mathbf{k}$$

$\|\mathbf{r}_u \times \mathbf{r}_v\| = au\sqrt{1 + a^2}$

$$A = \int_0^{2\pi}\int_0^b a\sqrt{1 + a^2}\,u\,du\,dv = \pi ab^2\sqrt{1 + a^2}$$

40. $\mathbf{r}(u, v) = (a + b\cos v)\cos u\mathbf{i} + (a + b\cos v)\sin u\mathbf{j} + b\sin v\mathbf{k},\ a > b,\ 0 \le u \le 2\pi,\ 0 \le v \le 2\pi$

$\mathbf{r}_u(u, v) = -(a + b\cos v)\sin u\mathbf{i} + (a + b\cos v)\cos u\mathbf{j}$

$\mathbf{r}_v(u, v) = -b\sin v\cos u\mathbf{i} - b\sin v\sin u\mathbf{j} + b\cos v\mathbf{k}$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -(a + b\cos v)\sin u & (a + b\cos v)\cos u & 0 \\ -b\sin v\cos u & -b\sin v\sin u & b\cos v \end{vmatrix}$$

$$= b\cos u\cos v(a + b\cos v)\mathbf{i} + b\sin u\cos v(a + b\cos v)\mathbf{j} + b\sin v(a + b\cos v)\mathbf{k}$$

$\|\mathbf{r}_u \times \mathbf{r}_v\| = b(a + b\cos v)$

$$A = \int_0^{2\pi}\int_0^{2\pi} b(a + b\cos v)\,du\,dv = 4\pi^2 ab$$

41. $\mathbf{r}(u, v) = \sqrt{u}\cos v\mathbf{i} + \sqrt{u}\sin v\mathbf{j} + u\mathbf{k},\ 0 \le u \le 4,\ 0 \le v \le 2\pi$

$\mathbf{r}_u(u, v) = \dfrac{\cos v}{2\sqrt{u}}\mathbf{i} + \dfrac{\sin v}{2\sqrt{u}}\mathbf{j} + \mathbf{k}$

$\mathbf{r}_v(u, v) = -\sqrt{u}\sin v\mathbf{i} + \sqrt{u}\cos v\mathbf{j}$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\cos v}{2\sqrt{u}} & \dfrac{\sin v}{2\sqrt{u}} & 1 \\ -\sqrt{u}\sin v & \sqrt{u}\cos v & 0 \end{vmatrix} = -\sqrt{u}\cos v\mathbf{i} - \sqrt{u}\sin v\mathbf{j} + \frac{1}{2}\mathbf{k}$$

$\|\mathbf{r}_u \times \mathbf{r}_v\| = \sqrt{u + \dfrac{1}{4}}$

$$A = \int_0^{2\pi}\int_0^{4} \sqrt{u + \frac{1}{4}}\,du\,dv = \frac{\pi}{6}\left(17\sqrt{17} - 1\right) \approx 36.177$$

42. $\mathbf{r}(u, v) = \sin u\cos v\mathbf{i} + u\mathbf{j} + \sin u\sin v\mathbf{k},\ 0 \le u \le \pi,\ 0 \le v \le 2\pi$

$\mathbf{r}_u(u, v) = \cos u\cos v\mathbf{i} + \mathbf{j} + \cos u\sin v\mathbf{k}$

$\mathbf{r}_v(u, v) = -\sin u\sin v\mathbf{i} + \sin u\cos v\mathbf{k}$

$\mathbf{r}_u \times \mathbf{r}_v = \sin u\cos v\mathbf{i} - \cos u\sin u\mathbf{j} + \sin u\sin v\mathbf{k}$

$\|\mathbf{r}_u \times \mathbf{r}_v\| = \sin u\sqrt{1 + \cos^2 u}$

$$A = \int_0^{2\pi}\int_0^{\pi} \sin u\sqrt{1 + \cos^2 u}\,du\,dv = \pi\left[2\sqrt{2} + \ln\left|\frac{\sqrt{2} + 1}{\sqrt{2} - 1}\right|\right]$$

43. See the definition, page 1098.

44. See the definition, page 1102.

45. (a) From $(-10, 10, 0)$

(b) From $(10, 10, 10)$

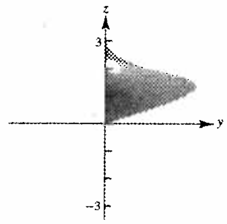
(c) From $(0, 10, 0)$

(d) From $(10, 0, 0)$

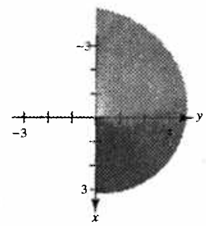
46. Graph of $\mathbf{r}(u, v) = u \cos v\mathbf{i} + u \sin v\mathbf{j} + v\mathbf{k}$

$0 \le u \le \pi,\ 0 \le v \le \pi$ from

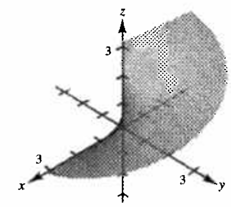
(a) $(10, 0, 0)$



(b) $(0, 0, 10)$

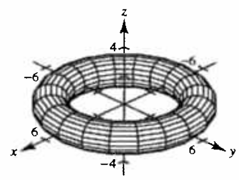


(c) $(10, 10, 10)$



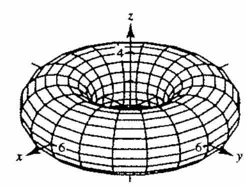
47. (a) $\mathbf{r}(u, v) = (4 + \cos v) \cos u\mathbf{i} + (4 + \cos v) \sin u\mathbf{j} + \sin v\mathbf{k},$

$0 \le u \le 2\pi,\ 0 \le v \le 2\pi$



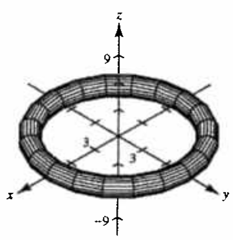
(b) $\mathbf{r}(u, v) = (4 + 2\cos v) \cos u\mathbf{i} + (4 + 2\cos v) \sin u\mathbf{j} + 2\sin v\mathbf{k},$

$0 \le u \le 2\pi, 0 \le v \le 2\pi$



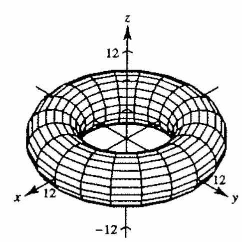
(c) $\mathbf{r}(u, v) = (8 + \cos v) \cos u\mathbf{i} + (8 + \cos v) \sin u\mathbf{j} + \sin v\mathbf{k},$

$0 \le u \le 2\pi,\ 0 \le v \le 2\pi$



(d) $\mathbf{r}(u, v) = (8 + 3\cos v) \cos u\mathbf{i} + (8 + 3\cos v) \sin u\mathbf{j} + 3\sin v\mathbf{k},$

$0 \le u \le 2\pi, 0 \le v \le 2\pi$



The radius of the generating circle that is revolved about the z-axis is b, and its center is a units from the axis of revolution.

48. $\mathbf{r}(u, v) = 2u \cos v\mathbf{i} + 2u \sin v\mathbf{j} + v\mathbf{k},\ 0 \le u \le 1, 0 \le v \le 3\pi$

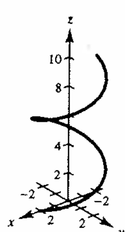
(a) If $u = 1$:

$\mathbf{r}(1, v) = 2\cos v\mathbf{i} + 2\sin v\mathbf{j} + v\mathbf{k}$

$x^2 + y^2 = 4$

$0 \le z \le 3\pi$

Helix



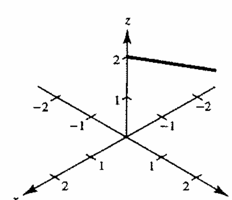
(b) If $v = \dfrac{2\pi}{3}$:

$$\mathbf{r}\left(u, \frac{2\pi}{3}\right) = -u\mathbf{i} + \sqrt{3}u\mathbf{j} + \frac{2\pi}{3}\mathbf{k}$$

$y = -\sqrt{3}x$

$z = \dfrac{2\pi}{3}$

Line



(c) If one parameter is held constant, the result is a **curve** in 3-space.

49. $\mathbf{r}(u, v) = 20 \sin u \cos v\mathbf{i} + 20 \sin u \sin v\mathbf{j} + 20 \cos u\mathbf{k},\ 0 \le u \le \pi/3,\ 0 \le v \le 2\pi$

$\mathbf{r}_u = 20 \cos u \cos v\mathbf{i} + 20 \cos u \sin v\mathbf{j} - 20 \sin u\mathbf{k}$

$\mathbf{r}_v = -20 \sin u \sin v\mathbf{i} + 20 \sin u \cos v\mathbf{j}$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 20\cos u \cos v & 20 \cos u \sin v & -20 \sin u \\ -20 \sin u \sin v & 20 \sin u \cos v & 0 \end{vmatrix}$$

$$= 400 \sin^2 u \cos v\mathbf{i} + 400 \sin^2 u \sin v\mathbf{j} + 400(\cos u \sin u \cos^2 v + \cos u \sin u \sin^2 v)\mathbf{k}$$

$$= 400[\sin^2 u \cos v\mathbf{i} + \sin^2 u \sin v\mathbf{j} + \cos u \sin u\mathbf{k}]$$

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = 400\sqrt{\sin^4 u \cos^2 v + \sin^4 u \sin^2 v + \cos^2 u \sin^2 u}$$

$$= 400\sqrt{\sin^4 u + \cos^2 u \sin^2 u}$$

$$= 400\sqrt{\sin^2 u} = 400 \sin u$$

$$S = \iint_S dS = \int_0^{2\pi}\int_0^{\pi/3} 400 \sin u\, du\, dv = \int_0^{2\pi}\Big[-400 \cos u\Big]_0^{\pi/3} dv = \int_0^{2\pi} 200\, dv = 400\pi \text{ m}^2$$

50. $x^2 + y^2 - z^2 = 1$

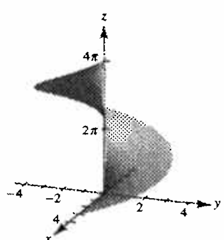
Let $x = u \cos v$, $y = u \sin v$, and $z = \sqrt{u^2 - 1}$. Then,

$$\mathbf{r}_u(u, v) = \cos v\mathbf{i} + \sin v\mathbf{j} + \frac{u}{\sqrt{u^2 - 1}}\mathbf{k}$$

$$\mathbf{r}_v(u, v) = -u \sin v\mathbf{i} + u \cos v\mathbf{j}.$$

At $(1, 0, 0)$, $u = 1$ and $v = 0$. $\mathbf{r}_u(1, 0)$ is undefined and $\mathbf{r}_v(1, 0) = \mathbf{j}$. The tangent plane at $(1, 0, 0)$ is $x = 1$.

51. $\mathbf{r}(u, v) = u \cos v\mathbf{i} + u \sin v\mathbf{j} + 2v\mathbf{k},\ 0 \le u \le 3,\ 0 \le v \le 2\pi$



$\mathbf{r}_u(u, v) = \cos v\mathbf{i} + \sin v\mathbf{j}$

$\mathbf{r}_v(u, v) = -u \sin v\mathbf{i} + u \cos v\mathbf{j} + 2\mathbf{k}$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos v & \sin v & 0 \\ -u \sin v & u \cos v & 2 \end{vmatrix} = 2 \sin v\mathbf{i} - 2 \cos v\mathbf{j} + u\mathbf{k}$$

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = \sqrt{4 + u^2}$$

$$A = \int_0^{2\pi}\int_0^3 \sqrt{4 + u^2}\, du\, dv = \pi\left[3\sqrt{13} + 4 \ln\left(\frac{3 + \sqrt{13}}{2}\right)\right]$$

52. $\mathbf{r}(u, v) = u\mathbf{i} + f(u) \cos v\mathbf{j} + f(u) \sin v\mathbf{k},\ a \le u \le b,\ 0 \le v \le 2\pi$

$\mathbf{r}_u(u, v) = \mathbf{i} + f'(u) \cos v\mathbf{j} + f'(u) \sin v\mathbf{k}$

$\mathbf{r}_v(u, v) = -f(u) \sin v\mathbf{j} + f(u) \cos v\mathbf{k}$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & f'(u) \cos v & f'(u) \sin v \\ 0 & -f(u) \sin v & f(u) \cos v \end{vmatrix}$$

$$= f(u)f'(u)\mathbf{i} - f(u) \cos v\mathbf{j} - f(u) \sin v\mathbf{k}$$

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = f(u)\sqrt{1 + [f'(u)]^2}$$

$$A = \int_0^{2\pi}\int_a^b f(u)\sqrt{1 + [f'(u)]^2}\, du\, dv$$

$$= 2\pi \int_a^b f(x)\sqrt{1 + [f'(x)]^2}\, dx \quad (\text{since } u = x)$$

53. Answers will vary.

54. Answers will vary.

Section 15.6 Surface Integrals

1. S: $z = 4 - x,\ 0 \le x \le 4,\ 0 \le y \le 4,\ \dfrac{\partial z}{\partial x} = -1,\ \dfrac{\partial z}{\partial y} = 0$

$$\iint_S (x - 2y + z)\, dS = \int_0^4 \int_0^4 (x - 2y + 4 - x)\sqrt{1 + (-1)^2 + (0)^2}\, dy\, dx$$
$$= \sqrt{2}\int_0^4 \int_0^4 (4 - 2y)\, dy\, dx = 0$$

2. S: $z = 15 - 2x + 3y,\ 0 \le x \le 2,\ 0 \le y \le 4,\ \dfrac{\partial z}{\partial x} = -2,\ \dfrac{\partial z}{\partial y} = 3,\ dS = \sqrt{1 + 4 + 9}\, dy\, dx = \sqrt{14}\, dy\, dx$

$$\iint_S (x - 2y + z)\, dS = \int_0^2 \int_0^4 (x - 2y + 15 - 2x + 3y)\sqrt{14}\, dy\, dx$$
$$= \sqrt{14}\int_0^2 \int_0^4 (15 - x + y)\, dy\, dx = 128\sqrt{14}$$

3. S: $z = 10,\ x^2 + y^2 \le 1,\ \dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial y} = 0$

$$\iint_S (x - 2y + z)\, dS = \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (x - 2y + 10)\sqrt{1 + (0)^2 + (0)^2}\, dy\, dx$$
$$= \int_0^{2\pi} \int_0^1 (r\cos\theta - 2r\sin\theta + 10)r\, dr\, d\theta$$
$$= \int_0^{2\pi} \left(\frac{1}{3}\cos\theta - \frac{2}{3}\sin\theta + 5\right) d\theta$$
$$= \left[\frac{1}{3}\sin\theta + \frac{2}{3}\cos\theta + 5\theta\right]_0^{2\pi} = 10\pi$$

4. S: $z = \dfrac{2}{3}x^{3/2},\ 0 \le x \le 1,\ 0 \le y \le x,\ \dfrac{\partial z}{\partial x} = x^{1/2},\ \dfrac{\partial z}{\partial y} = 0$

$$\iint_S (x - 2y + z)\, dS = \int_0^1 \int_0^x \left(x - 2y + \frac{2}{3}x^{3/2}\right)\sqrt{1 + (x^{1/2})^2 + (0)^2}\, dy\, dx$$
$$= \int_0^1 \int_0^x \left(x - 2y + \frac{2}{3}x^{3/2}\right)\sqrt{1 + x}\, dy\, dx$$
$$= \frac{2}{3}\int_0^1 x^{5/2}\sqrt{x + 1}\, dx$$
$$= \frac{2}{3}\left[\frac{1}{4}x^{5/2}(1 + x)^{3/2}\right]_0^1 - \frac{5}{12}\int_0^1 x^{3/2}\sqrt{1 + x}\, dx$$
$$= \left[\frac{1}{6}x^{5/2}(1 + x)^{3/2}\right]_0^1 - \frac{5}{12}\left(\frac{1}{3}\right)\left[x^{3/2}(1 + x)^{3/2}\right]_0^1 + \frac{5}{24}\int_0^1 x^{1/2}\sqrt{1 + x}\, dx$$

—CONTINUED—

4. —CONTINUED—

$$= \frac{\sqrt{2}}{3} - \frac{5\sqrt{2}}{18} + \frac{5}{24}\int_0^1 \sqrt{x + x^2}\,dx$$

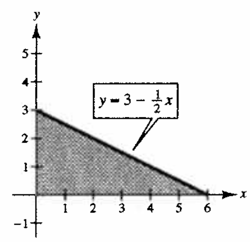
$$= \frac{\sqrt{2}}{18} + \frac{5}{24}\int_0^1 \sqrt{\left(x + \frac{1}{2}\right)^2 - \frac{1}{4}}\,dx$$

$$= \frac{\sqrt{2}}{18} + \frac{5}{24}\left(\frac{1}{2}\right)\left[\left(x + \frac{1}{2}\right)\sqrt{x^2 + x} - \frac{1}{4}\ln\left|\left(x + \frac{1}{2}\right) + \sqrt{x^2 + x}\right|\right]_0^1$$

$$= \frac{\sqrt{2}}{18} + \frac{5}{48}\left[\frac{3}{2}\sqrt{2} - \frac{1}{4}\ln\left|\frac{3}{2} + \sqrt{2}\right| + \frac{1}{4}\ln\left|\frac{1}{2}\right|\right]$$

$$= \frac{\sqrt{2}}{18} + \frac{15\sqrt{2}}{96} + \frac{5}{192}\ln\left|\frac{1}{3 + 2\sqrt{2}}\right| = \frac{61\sqrt{2}}{288} - \frac{5}{192}\ln|3 + 2\sqrt{2}| \approx 0.2536$$

5. S: $z = 6 - x - 2y$, (first octant) $\dfrac{\partial z}{\partial x} = -1$, $\dfrac{\partial z}{\partial y} = -2$



$$\iint_S xy\,dS = \int_0^6 \int_0^{3-(x/2)} xy\sqrt{1 + (-1)^2 + (-2)^2}\,dy\,dx$$

$$= \sqrt{6}\int_0^6 \left[\frac{xy^2}{2}\right]_0^{3-(x/2)} dx$$

$$= \frac{\sqrt{6}}{2}\int_0^6 x\left(9 - 3x + \frac{1}{4}x^2\right) dx$$

$$= \frac{\sqrt{6}}{2}\left[\frac{9x^2}{2} - x^3 + \frac{x^4}{16}\right]_0^6 = \frac{27\sqrt{6}}{2}$$

6. S: $z = h$, $0 \le x \le 2$, $0 \le y \le \sqrt{4 - x^2}$, $\dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial y} = 0$

$$\iint_S dx\,dS = \int_0^2 \int_0^{\sqrt{4-x^2}} xy\,dy\,dx = \frac{1}{2}\int_0^2 x(4 - x^2)\,dx = \frac{1}{2}\left[2x^2 - \frac{x^4}{4}\right]_0^2 = 2$$

7. S: $z = 9 - x^2$, $0 \le x \le 2$, $0 \le y \le x$,

$$\frac{\partial z}{\partial x} = -2x, \quad \frac{\partial z}{\partial y} = 0$$

$$\iint_S xy\,dS = \int_0^2 \int_y^2 xy\sqrt{1 + 4x^2}\,dx\,dy = \frac{391\sqrt{17} + 1}{240}$$

8. S: $z = \dfrac{1}{2}xy$, $0 \le x \le 4, 0 \le y \le 4$, $\dfrac{\partial z}{\partial x} = \dfrac{1}{2}y$, $\dfrac{\partial z}{\partial y} = \dfrac{1}{2}x$

$$\iint_S xy\,dS = \int_0^4 \int_0^4 xy\sqrt{1 + \frac{y^2}{4} + \frac{x^2}{4}}\,dy\,dx = \frac{3904}{15} - \frac{160\sqrt{5}}{3}$$

9. S: $z = 10 - x^2 - y^2$, $0 \le x \le 2$, $0 \le y \le 2$

$$\iint_S (x^2 - 2xy)\,dS = \int_0^2 \int_0^2 (x^2 - 2xy)\sqrt{1 + 4x^2 + 4y^2}\,dy\,dx \approx -11.47$$

10. S: $z = \cos x$, $0 \le x \le \frac{\pi}{2}$, $0 \le y \le \frac{x}{2}$

$$\iint_S (x^2 - 2xy)\, dS = \int_0^{\pi/2}\int_0^{x/2} (x^2 - 2xy)\sqrt{1 + \sin^2 x}\, dy\, dx = \int_0^{\pi/2} \frac{x^3}{4}\sqrt{1 + \sin^2 x}\, dx \approx 0.52$$

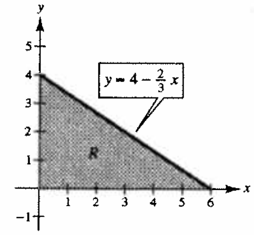
11. S: $2x + 3y + 6z = 12$ (first octant) $\Rightarrow z = 2 - \frac{1}{3}x - \frac{1}{2}y$

$\rho(x, y, z) = x^2 + y^2$

$$m = \iint_R (x^2 + y^2)\sqrt{1 + \left(-\frac{1}{3}\right)^2 + \left(-\frac{1}{2}\right)^2}\, dA$$

$$= \frac{7}{6}\int_0^6\int_0^{4-(2x/3)} (x^2 + y^2)\, dy\, dx$$

$$= \frac{7}{6}\int_0^6\left[x^2\left(4 - \frac{2}{3}x\right) + \frac{1}{3}\left(4 - \frac{2}{3}x\right)^3\right] dx = \frac{7}{6}\left[\frac{4}{3}x^3 - \frac{1}{6}x^4 - \frac{1}{8}\left(4 - \frac{2}{3}x\right)^4\right]_0^6 = \frac{364}{3}$$



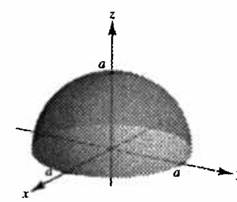
12. S: $z = \sqrt{a^2 - x^2 - y^2}$

$\rho(x, y, z) = kz$

$$m = \iint_S kz\, dS = \iint_R k\sqrt{a^2 - x^2 - y^2}\sqrt{1 + \left(\frac{-x}{\sqrt{a^2 - x^2 - y^2}}\right)^2 + \left(\frac{-y}{\sqrt{a^2 - x^2 - y^2}}\right)^2}\, dA$$

$$= \iint_R k\sqrt{a^2 - x^2 - y^2}\left(\frac{a}{\sqrt{a^2 - x^2 - y^2}}\right) dA$$

$$= \iint_R ka\, dA = ka\iint_R dA = ka(2\pi a^2) = 2ka^3\pi$$



13. S: $\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + \frac{v}{2}\mathbf{k}$, $0 \le u \le 1$, $0 \le v \le 2$

$\mathbf{r}_u = \mathbf{i}$, $\mathbf{r}_v = \mathbf{j} + \frac{1}{2}\mathbf{k}$

$\mathbf{r}_u \times \mathbf{r}_v = \frac{-1}{2}\mathbf{j} + \mathbf{k}$

$\|\mathbf{r}_u \times \mathbf{r}_v\| = \left\|-\frac{1}{2}\mathbf{j} + \mathbf{k}\right\| = \frac{\sqrt{5}}{2}$

$$\iint_S (y + 5)\, dS = \int_0^2\int_0^1 (v + 5)\frac{\sqrt{5}}{2}\, du\, dv = 6\sqrt{5}$$

14. S: $\mathbf{r}(u, v) = 2\cos u\mathbf{i} + 2\sin u\mathbf{j} + v\mathbf{k}$, $0 \le u \le \frac{\pi}{2}$, $0 \le v \le 2$

$\|\mathbf{r}_u \times \mathbf{r}_v\| = \|2\cos u\mathbf{i} + 2\sin u\mathbf{j}\| = 2$

$$\iint_S (x + y)\, dS = \int_0^2\int_0^{\pi/2} (2\cos u + 2\sin u)2\, du\, dv = 16$$

15. S: $\mathbf{r}(u, v) = 2\cos u\mathbf{i} + 2\sin u\mathbf{j} + v\mathbf{k}$, $0 \le u \le \frac{\pi}{2}$, $0 \le v \le 2$

$\mathbf{r}_u = -2\sin u\mathbf{i} + 2\cos u\mathbf{j}$

$\mathbf{r}_v = \mathbf{k}$

$\mathbf{r}_u \times \mathbf{r}_v = 2\cos u\mathbf{i} + 2\sin u\mathbf{j}$

$\|\mathbf{r}_u \times \mathbf{r}_v\| = \|2\cos u\mathbf{i} + 2\sin u\mathbf{j}\| = 2$

$$\iint_S xy\, dS = \int_0^2\int_0^{\pi/2} 8\cos u \sin u\, du\, dv = 8$$

16. S: $\mathbf{r}(u, v) = 4u\cos v\mathbf{i} + 4u\sin v\mathbf{j} + 3u\mathbf{k},\ 0 \le u \le 4,\ 0 \le v \le \pi$

$\|\mathbf{r}_u \times \mathbf{r}_v\| = \|-12u\cos v\mathbf{i} - 12u\sin v\mathbf{j} + 16u\mathbf{k}\| = 20u$

$$\iint_S (x + y)\, dS = \int_0^{\pi}\int_0^4 (4u\cos v + 4u\sin v)20u\, du\, dv = \frac{10{,}240}{3}$$

17. $f(x, y, z) = x^2 + y^2 + z^2$

S: $z = x + 2,\ x^2 + y^2 \le 1$

$$\iint_S f(x, y, z)\, dS = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} [x^2 + y^2 + (x + 2)^2]\sqrt{1 + (1)^2 + (0)^2}\, dy\, dx$$

$$= \sqrt{2}\int_0^{2\pi}\int_0^1 [r^2 + (r\cos\theta + 2)^2]r\, dr\, d\theta$$

$$= \sqrt{2}\int_0^{2\pi}\int_0^1 [r^2 + r^2\cos^2\theta + 4r\cos\theta + 4]r\, dr\, d\theta$$

$$= \sqrt{2}\int_0^{2\pi}\left[\frac{r^4}{4} + \frac{r^4}{4}\cos^2\theta + \frac{4r^3}{3}\cos\theta + 2r^2\right]_0^1 d\theta$$

$$= \sqrt{2}\int_0^{2\pi}\left[\frac{9}{4} + \left(\frac{1}{4}\right)\frac{1 + \cos 2\theta}{2} + \frac{4}{3}\cos\theta\right] d\theta$$

$$= \sqrt{2}\left[\frac{9}{4}\theta + \frac{1}{8}\left(\theta + \frac{1}{2}\sin 2\theta\right) + \frac{4}{3}\sin\theta\right]_0^{2\pi} = \sqrt{2}\left[\frac{18\pi}{4} + \frac{\pi}{4}\right] = \frac{19\sqrt{2}\pi}{4}$$

18. $f(x, y, z) = \dfrac{xy}{z}$

S: $z = x^2 + y^2,\ 4 \le x^2 + y^2 \le 16$

$$\iint_S f(x, y, z)\, dS = \iint_S \frac{xy}{x^2 + y^2}\sqrt{1 + 4x^2 + 4y^2}\, dy\, dx = \int_0^{2\pi}\int_2^4 \frac{r^2\sin\theta\cos\theta}{r^2}\sqrt{1 + 4r^2}\, r\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_2^4 r\sqrt{1 + 4r^2}\sin\theta\cos\theta\, dr\, d\theta = \int_0^{2\pi}\left[\frac{1}{12}(1 + 4r^2)^{3/2}\right]_2^4 \sin\theta\cos\theta\, d\theta$$

$$= \left[\frac{65\sqrt{65} - 17\sqrt{17}}{12}\left(\frac{\sin^2\theta}{2}\right)\right]_0^{2\pi} = 0$$

19. $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$

S: $z = \sqrt{x^2 + y^2},\ x^2 + y^2 \le 4$

$$\iint_S f(x, y, z)\, dS = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{x^2 + y^2 + \left(\sqrt{x^2 + y^2}\right)^2}\sqrt{1 + \left(\frac{x}{\sqrt{x^2 + y^2}}\right)^2 + \left(\frac{y}{\sqrt{x^2 + y^2}}\right)^2}\, dy\, dx$$

$$= \sqrt{2}\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{x^2 + y^2}\sqrt{\frac{x^2 + y^2 + x^2 + y^2}{x^2 + y^2}}\, dy\, dx$$

$$= 2\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{x^2 + y^2}\, dy\, dx$$

$$= 2\int_0^{2\pi}\int_0^2 r^2\, dr\, d\theta = 2\int_0^{2\pi}\left[\frac{r^3}{3}\right]_0^2 d\theta = \left[\frac{16}{3}\theta\right]_0^{2\pi} = \frac{32\pi}{3}$$

20. $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$

S: $z = \sqrt{x^2 + y^2}$, $(x - 1)^2 + y^2 \le 1$

$$\iint_S f(x, y, z)\, dS = \iint_S \sqrt{x^2 + y^2 + \left(\sqrt{x^2 + y^2}\right)^2}\sqrt{1 + \left(\frac{x}{\sqrt{x^2 + y^2}}\right)^2 + \left(\frac{y}{\sqrt{x^2 + y^2}}\right)^2}\, dy\, dx$$

$$= \iint_S \sqrt{2(x^2 + y^2)}\sqrt{\frac{2(x^2 + y^2)}{x^2 + y^2}}\, dy\, dx$$

$$= 2\iint_S \sqrt{x^2 + y^2}\, dy\, dx = 2\int_0^{\pi}\int_0^{2\cos\theta} r^2\, dr\, d\theta$$

$$= \frac{16}{3}\int_0^{\pi} \cos^3\theta\, d\theta = \frac{16}{3}\int_0^{\pi}(1 - \sin^2\theta)\cos\theta\, d\theta$$

$$= \left[\frac{16}{3}\left(\sin\theta - \frac{\sin^3\theta}{3}\right)\right]_0^{\pi} = 0$$

21. $f(x, y, z) = x^2 + y^2 + z^2$

S: $x^2 + y^2 = 9$, $0 \le x \le 3$, $0 \le y \le 3, 0 \le z \le 9$

Project the solid onto the yz-plane; $x = \sqrt{9 - y^2}$, $0 \le y \le 3$, $0 \le z \le 9$.

$$\iint_S f(x, y, z)\, dS = \int_0^3\int_0^9 [(9 - y^2) + y^2 + z^2]\sqrt{1 + \left(\frac{-y}{\sqrt{9 - y^2}}\right)^2 + (0)^2}\, dz\, dy$$

$$= \int_0^3\int_0^9 (9 + z^2)\frac{3}{\sqrt{9 - y^2}}\, dz\, dy = \int_0^3\left[\frac{3}{\sqrt{9 - y^2}}\left(9z + \frac{z^3}{3}\right)\right]_0^9 dy$$

$$= 324\int_0^3 \frac{3}{\sqrt{9 - y^2}}\, dy = \left[972 \arcsin\left(\frac{y}{3}\right)\right]_0^3 = 972\left(\frac{\pi}{2} - 0\right) = 486\pi$$

22. $f(x, y, z) = x^2 + y^2 + z^2$

S: $x^2 + y^2 = 9$, $0 \le x \le 3$, $0 \le z \le x$

Project the solid onto the xz-plane; $y = \sqrt{9 - x^2}$.

$$\iint_S f(x, y, z)\, dS = \int_0^3\int_0^x [x^2 + (9 - x^2) + z^2]\sqrt{1 + \left(\frac{-x}{\sqrt{9 - x^2}}\right)^2 + (0)^2}\, dz\, dx$$

$$= \int_0^3\int_0^x (9 + z^2)\frac{3}{\sqrt{9 - x^2}}\, dz\, dx = \int_0^3\left[\frac{3}{\sqrt{9 - x^2}}\left(9z + \frac{z^3}{3}\right)\right]_0^x dx$$

$$= \int_0^3 \frac{3}{\sqrt{9 - x^2}}\left(9x + \frac{x^3}{3}\right) dx = \int_0^3 27x(9 - x^2)^{-1/2}\, dx + \int_0^3 x^3(9 - x^2)^{-1/2}\, dx$$

Let $u = x^2$, $dv = x(9 - x^2)^{-1/2}\, dx$, then $du = 2x\, dx$, $v = -\sqrt{9 - x^2}$.

$$= \left[-27\sqrt{9 - x^2}\right]_0^3 + \left[\left[-x^2\sqrt{9 - x^2}\right]_0^3 + \int_0^3 2x\sqrt{9 - x^2}\, dx\right]$$

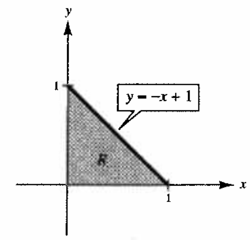
$$= \left[81 - \frac{2}{3}(9 - x^2)^{3/2}\right]_0^3 = 81 + 18 = 99$$

23. $\mathbf{F}(x, y, z) = 3z\mathbf{i} - 4\mathbf{j} + y\mathbf{k}$

S: $x + y + z = 1$ (first octant)

$G(x, y, z) = x + y + z - 1$

$\nabla G(x, y, z) = \mathbf{i} + \mathbf{j} + \mathbf{k}$



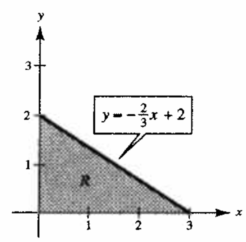
$$\iint_S \mathbf{F} \cdot \mathbf{N}\, dS = \iint_R \mathbf{F} \cdot \nabla G\, dA = \int_0^1 \int_0^{1-x} (3z - 4 + y)\, dy\, dx$$
$$= \int_0^1 \int_0^{1-x} [3(1 - x - y) - 4 + y]\, dy\, dx$$
$$= \int_0^1 \int_0^{1-x} (-1 - 3x - 2y)\, dy\, dx$$
$$= \int_0^1 \Big[-y - 3xy - y^2\Big]_0^{1-x} dx$$
$$= -\int_0^1 [(1 - x) + 3x(1 - x) + (1 - x)^2]\, dx$$
$$= -\int_0^1 (2 - 2x^2)\, dx = -\frac{4}{3}$$

24. $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j}$

S: $2x + 3y + z = 6$ (first octant)

$G(x, y, z) = 2x + 3y + z - 6$

$\nabla G(x, y, z) = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$



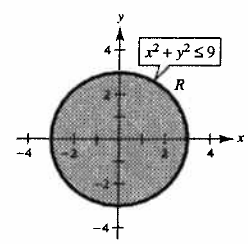
$$\iint_S \mathbf{F} \cdot \mathbf{N}\, dS = \iint_R \mathbf{F} \cdot \nabla G\, dA = \int_0^3 \int_0^{-(2x/3)+2} (2x + 3y)\, dy\, dx$$
$$= \int_0^3 \left[-\frac{4}{3}x^2 + 4x + \frac{3}{2}\left(-\frac{2}{3}x + 2\right)^2\right] dx$$
$$= \left[-\frac{4}{9}x^3 + 2x^2 - \frac{3}{4}\left(-\frac{2}{3}x + 2\right)^3\right]_0^3 = 12$$

25. $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

S: $z = 9 - x^2 - y^2,\ 0 \le z$

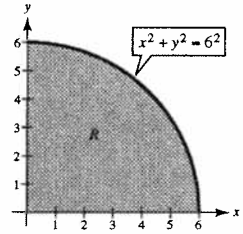
$G(x, y, z) = x^2 + y^2 + z - 9$

$\nabla G(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k}$



$$\iint_S \mathbf{F} \cdot \mathbf{N}\, dS = \iint_R \mathbf{F} \cdot \nabla G\, dA = \iint_R (2x^2 + 2y^2 + z)\, dA$$
$$= \iint_R [2x^2 + 2y^2 + (9 - x^2 - y^2)]\, dA$$
$$= \iint_R (x^2 + y^2 + 9)\, dA$$
$$= \int_0^{2\pi} \int_0^3 (r^2 + 9)r\, dr\, d\theta$$
$$= \int_0^{2\pi} \left[\frac{r^4}{4} + \frac{9r^2}{2}\right]_0^3 d\theta = \frac{243\pi}{2}$$

26. $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$



S: $x^2 + y^2 + z^2 = 36$ (first octant)

$z = \sqrt{36 - x^2 - y^2}$

$G(x, y, z) = z - \sqrt{36 - x^2 - y^2}$

$$\nabla G(x, y, z) = \frac{x}{\sqrt{36 - x^2 - y^2}}\mathbf{i} + \frac{y}{\sqrt{36 - x^2 - y^2}}\mathbf{j} + \mathbf{k}$$

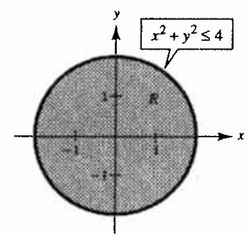
$$\mathbf{F} \cdot \nabla G = \frac{x^2}{\sqrt{36 - x^2 - y^2}} + \frac{y^2}{\sqrt{36 - x^2 - y^2}} + z = \frac{36}{\sqrt{36 - x^2 - y^2}}$$

$$\int_S\int \mathbf{F} \cdot \mathbf{N}\, dS = \int_R\int \mathbf{F} \cdot \nabla G\, dA = \int_R\int \frac{36}{\sqrt{36 - x^2 - y^2}}\, dA$$

$$= \int_0^{\pi/2}\int_0^6 \frac{36}{\sqrt{36 - r^2}}\, r\, dr\, d\theta \quad \text{(improper)}$$

$$= 108\pi$$

27. $\mathbf{F}(x, y, z) = 4\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$



S: $z = x^2 + y^2$, $x^2 + y^2 \le 4$

$G(x, y, z) = -x^2 - y^2 + z$

$\nabla G(x, y, z) = -2x\mathbf{i} - 2y\mathbf{j} + \mathbf{k}$

$$\int_S\int \mathbf{F} \cdot \mathbf{N}\, dS = \int_R\int \mathbf{F} \cdot \nabla G\, dA = \int_R\int (-8x + 6y + 5)\, dA$$

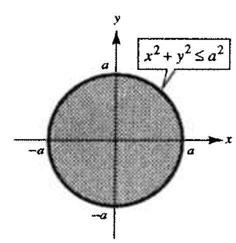
$$= \int_0^{2\pi}\int_0^2 [-8r\cos\theta + 6r\sin\theta + 5]\, r\, dr\, d\theta$$

$$= \int_0^{2\pi}\left[-\frac{8}{3}r^3\cos\theta + 2r^3\sin\theta + \frac{5}{2}r^2\right]_0^2 d\theta$$

$$= \int_0^{2\pi}\left[-\frac{64}{3}\cos\theta + 16\sin\theta + 10\right] d\theta$$

$$= \left[-\frac{64}{3}\sin\theta - 16\cos\theta + 10\theta\right]_0^{2\pi} = 20\pi$$

28. $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} - 2z\mathbf{k}$



S: $z = \sqrt{a^2 - x^2 - y^2}$

$G(x, y, z) = z - \sqrt{a^2 - x^2 - y^2}$

$$\nabla G(x, y, z) = \frac{x}{\sqrt{a^2 - x^2 - y^2}}\mathbf{i} + \frac{y}{\sqrt{a^2 - x^2 - y^2}}\mathbf{j} + \mathbf{k}$$

$$\mathbf{F} \cdot \nabla G = \frac{x^2}{\sqrt{a^2 - x^2 - y^2}} + \frac{y^2}{\sqrt{a^2 - x^2 - y^2}} - 2\sqrt{a^2 - x^2 - y^2} = \frac{3x^2 + 3y^2 - 2a^2}{\sqrt{a^2 - x^2 - y^2}}$$

$$\int_S\int \mathbf{F} \cdot \mathbf{N}\, dS = \int_R\int \mathbf{F} \cdot \nabla G\, dA = \int_R\int \frac{3x^2 + 3y^2 - 2a^2}{\sqrt{a^2 - x^2 - y^2}}\, dA$$

$$= \int_0^{2\pi}\int_0^a \frac{3r^2 - 2a^2}{\sqrt{a^2 - r^2}}\, r\, dr\, d\theta$$

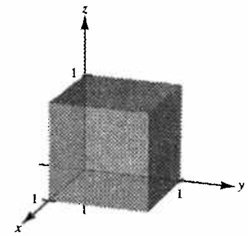
$$= 3\int_0^{2\pi}\int_0^a \frac{r^3}{\sqrt{a^2 - r^2}}\, dr\, d\theta - 2a^2\int_0^{2\pi}\int_0^a \frac{r}{\sqrt{a^2 - r^2}}\, dr\, d\theta$$

$$= 3\left[\int_0^{2\pi}\left[-r^2\sqrt{a^2 - r^2} - \frac{2}{3}(a^2 - r^2)^{3/2}\right]_0^a d\theta\right] - 2a^2\int_0^{2\pi}\left[-\sqrt{a^2 - r^2}\right]_0^a d\theta$$

$$= 3\int_0^{2\pi}\frac{2}{3}a^3\, d\theta - 2a^2\int_0^{2\pi} a\, d\theta = 0$$

29. $\mathbf{F}(x, y, z) = 4xy\mathbf{i} + z^2\mathbf{j} + yz\mathbf{k}$

S: unit cube bounded by $x = 0,\ x = 1,\ y = 0,\ y = 1,\ z = 0,\ z = 1$



S_1: The top of the cube

$\mathbf{N} = \mathbf{k},\ z = 1$

$$\int_{S_1}\!\!\int \mathbf{F} \cdot \mathbf{N}\, dS = \int_0^1\!\!\int_0^1 y(1)\, dy\, dx = \frac{1}{2}$$

S_2: The bottom of the cube

$\mathbf{N} = -\mathbf{k},\ z = 0$

$$\int_{S_2}\!\!\int \mathbf{F} \cdot \mathbf{N}\, dS = \int_0^1\!\!\int_0^1 -y(0)\, dy\, dx = 0$$

S_3: The front of the cube

$\mathbf{N} = \mathbf{i},\ x = 1$

$$\int_{S_3}\!\!\int \mathbf{F} \cdot \mathbf{N}\, dS = \int_0^1\!\!\int_0^1 4(1)y\, dy\, dz = 2$$

S_4: The back of the cube

$\mathbf{N} = -\mathbf{i},\ x = 0$

$$\int_{S_4}\!\!\int \mathbf{F} \cdot \mathbf{N}\, dS = \int_0^1\!\!\int_0^1 -4(0)y\, dy\, dx = 0$$

S_5: The right side of the cube

$\mathbf{N} = \mathbf{j},\ y = 1$

$$\int_{S_5}\!\!\int \mathbf{F} \cdot \mathbf{N}\, dS = \int_0^1\!\!\int_0^1 z^2\, dz\, dx = \frac{1}{3}$$

S_6: The left side of the cube

$\mathbf{N} = -\mathbf{j},\ y = 0$

$$\int_{S_6}\!\!\int \mathbf{F} \cdot \mathbf{N}\, dS = \int_0^1\!\!\int_0^1 -z^2\, dz\, dx = -\frac{1}{3}$$

Therefore,

$$\int_S\!\!\int \mathbf{F} \cdot \mathbf{N}\, dS = \frac{1}{2} + 0 + 2 + 0 + \frac{1}{3} - \frac{1}{3} = \frac{5}{2}.$$

30. $\mathbf{F}(x, y, z) = (x + y)\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

S: $z = 1 - x^2 - y^2,\ z = 0$

$G(x, y, z) = z + x^2 + y^2 - 1$

$\nabla G(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k}$

$\mathbf{F} \cdot \nabla G = 2x(x + y) + 2y(y) + (1 - x^2 - y^2) = x^2 + 2xy + y^2 + 1$

$$\int_S\!\!\int \mathbf{F} \cdot \mathbf{N}\, dS = \int_R\!\!\int \mathbf{F} \cdot \nabla G\, dA = \int_R\!\!\int (x^2 + 2xy + y^2 + 1)\, dA$$

$$= \int_0^{2\pi}\!\!\int_0^1 (r^2 + 2r^2 \cos\theta \sin\theta + 1)r\, dr\, d\theta$$

$$= \int_0^{2\pi} \left(\frac{3}{4} + \frac{1}{2}\sin\theta\cos\theta\right) d\theta = \left[\frac{3}{4}\theta + \frac{\sin^2\theta}{4}\right]_0^{2\pi} = \frac{3\pi}{2}$$

The flux across the bottom $z = 0$ is zero.

31. The surface integral of f over a surface S, where S is given by $z = g(x, y)$, is defined as

$$\int_S\!\!\int f(x, y, z)\, dS = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(x_i, y_i, z_i)\Delta S_i. \text{ (page 1108)}$$

See Theorem 15.10, page 1108.

32. A surface is orientable if a unit normal vector N can be defined at every nonboundary point of S in such a way that the normal vectors vary continuously over the surface S.

33. See the definition, page 1114.

See Theorem 15.11, page 1114.

34. Orientable

35. $\mathbf{E} = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$

S: $z = \sqrt{1 - x^2 - y^2}$

$$\iint_S \mathbf{E} \cdot \mathbf{N}\, dS = \iint_R \mathbf{E} \cdot (-g_x(x, y)\mathbf{i} - g_y(x, y)\mathbf{j} + \mathbf{k})\, dA$$

$$= \iint_R (yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}) \cdot \left(\frac{x}{\sqrt{1 - x^2 - y^2}}\mathbf{i} + \frac{y}{\sqrt{1 - x^2 - y^2}}\mathbf{j} + \mathbf{k}\right) dA$$

$$= \iint_R \left(\frac{2xyz}{\sqrt{1 - x^2 - y^2}} + xy\right) dA = \iint_R 3xy\, dA = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 3xy\, dy\, dx = 0$$

36. $\mathbf{E} = x\mathbf{i} + y\mathbf{j} + 2z\mathbf{k}$

S: $z = \sqrt{1 - x^2 - y^2} = g(x, y)$

$$\iint_S \mathbf{E} \cdot \mathbf{N}\, dS = \iint_R \mathbf{E} \cdot (-g_x(x, y)\mathbf{i} - g_y(x, y)\mathbf{j} + \mathbf{k})\, dA$$

$$= \iint_R (x\mathbf{i} + y\mathbf{j} + 2z\mathbf{k}) \cdot \left(\frac{x}{\sqrt{1 - x^2 - y^2}}\mathbf{i} + \frac{y}{\sqrt{1 - x^2 - y^2}}\mathbf{j} + \mathbf{k}\right) dA$$

$$= \iint_R \left(\frac{x^2}{\sqrt{1 - x^2 - y^2}} + \frac{y^2}{\sqrt{1 - x^2 - y^2}} + 2z\right) dA$$

$$= \iint_R \frac{x^2 + y^2 + 2(1 - x^2 - y^2)}{\sqrt{1 - x^2 - y^2}}\, dA$$

$$= \iint_R \frac{2 - x^2 - y^2}{\sqrt{1 - x^2 - y^2}}\, dA$$

$$= \int_0^{2\pi} \int_0^1 \frac{2 - r^2}{\sqrt{1 - r^2}}\, r\, dr\, d\theta$$

$$= \frac{8\pi}{3}$$

37. $z = \sqrt{x^2 + y^2},\ 0 \le z \le a$

$$m = \iint_S k\, dS = k\iint_R \sqrt{1 + \left(\frac{x}{\sqrt{x^2 + y^2}}\right)^2 + \left(\frac{y}{\sqrt{x^2 + y^2}}\right)^2}\, dA = k\iint_R \sqrt{2}\, dA = \sqrt{2}\, k\pi a^2$$

$$I_z = \iint_S k(x^2 + y^2)\, dS = \iint_R k(x^2 + y^2)\sqrt{2}\, dA$$

$$= \sqrt{2}k \int_0^{2\pi} \int_0^a r^3\, dr\, d\theta = \frac{\sqrt{2}ka^4}{4}(2\pi)$$

$$= \frac{\sqrt{2}k\pi a^4}{2} = \frac{a^2}{2}\left(\sqrt{2}k\pi a^2\right) = \frac{a^2 m}{2}$$

38. $x^2 + y^2 + z^2 = a^2$

$z = \pm\sqrt{a^2 - x^2 - y^2}$

$$m = 2\iint_S k\,dS = 2k\iint_R \sqrt{1 + \left(\frac{-x}{\sqrt{a^2 - x^2 - y^2}}\right)^2 + \left(\frac{-y}{\sqrt{a^2 - x^2 - y^2}}\right)^2}\,dA$$

$$= 2k\iint_R \frac{a}{\sqrt{a^2 - x^2 - y^2}}\,dA = 2ka\int_0^{2\pi}\int_0^a \frac{r}{\sqrt{a^2 - r^2}}\,dr\,d\theta$$

$$= 2ka\left[-\sqrt{a^2 - r^2}\right]_0^a (2\pi) = 4\pi k a^2$$

$$I_z = 2\iint_S k(x^2 + y^2)\,dS$$

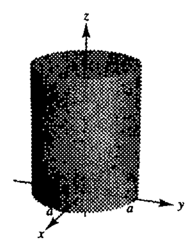
$$= 2k\iint_R (x^2 + y^2)\frac{a}{\sqrt{a^2 - x^2 - y^2}}\,dA = 2ka\int_0^{2\pi}\int_0^a \frac{r^3}{\sqrt{a^2 - r^2}}\,dr\,d\theta \quad \text{(use integration by parts)}$$

$$= 2ka\left[-r^2\sqrt{a^2 - r^2} - \frac{2}{3}(a^2 - r^2)^{3/2}\right]_0^a (2\pi)$$

$$= 2ka\left(\frac{2}{3}a^3\right)(2\pi) = \frac{2}{3}a^2(4\pi k a^2) = \frac{2}{3}a^2 m$$

Let $u = r^2,\ dv = r(a^2 - r^2)^{-1/2}\,dr,\ du = 2r\,dr,\ v = -\sqrt{a^2 - r^2}.$

39. $x^2 + y^2 = a^2,\ 0 \le z \le h$



$\rho(x, y, z) = 1$

$y = \pm\sqrt{a^2 - x^2}$

Project the solid onto the xz-plane.

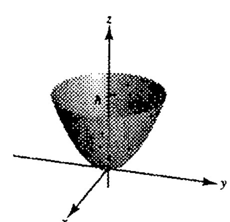
$$I_z = 4\iint_S (x^2 + y^2)(1)\,dS$$

$$= 4\int_0^h\int_0^a [x^2 + (a^2 - x^2)]\sqrt{1 + \left(\frac{-x}{\sqrt{a^2 - x^2}}\right)^2 + (0)^2}\,dx\,dz$$

$$= 4a^3\int_0^h\int_0^a \frac{1}{\sqrt{a^2 - x^2}}\,dx\,dz$$

$$= 4a^3\int_0^h\left[\arcsin\frac{x}{a}\right]_0^a dz = 4a^3\left(\frac{\pi}{2}\right)(h) = 2\pi a^3 h$$

40. $z = x^2 + y^2,\ 0 \le z \le h$



Project the solid onto the xy-plane.

$$I_z = \iint_S (x^2 + y^2)(1)\,dS$$

$$= \int_{-\sqrt{h}}^{\sqrt{h}}\int_{-\sqrt{h-x^2}}^{\sqrt{h-x^2}} (x^2 + y^2)\sqrt{1 + 4x^2 + 4y^2}\,dy\,dx$$

$$= \int_0^{2\pi}\int_0^{\sqrt{h}} r^2\sqrt{1 + 4r^2}\,r\,dr\,d\theta$$

$$= 2\pi\left[\frac{h}{12}(1 + 4h)^{3/2} - \frac{1}{120}(1 + 4h)^{5/2}\right] + \frac{2\pi}{120}$$

$$= \frac{(1 + 4h)^{3/2}\pi}{60}[10h - (1 + 4h)] + \frac{\pi}{60} = \frac{\pi}{60}[(1 + 4h)^{3/2}(6h - 1) + 1]$$

41. S: $z = 16 - x^2 - y^2$, $z \geq 0$

$\mathbf{F}(x, y, z) = 0.5z\mathbf{k}$

$$\int_S\int \rho\mathbf{F} \cdot \mathbf{N}\, dS = \int_R\int \rho\mathbf{F} \cdot (-g_x(x, y)\mathbf{i} - g_y(x, y)\mathbf{j} + \mathbf{k})\, dA = \int_R\int 0.5\rho z\mathbf{k} \cdot (2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k})\, dA$$

$$= \int_R\int 0.5\rho z\, dA = \int_R\int 0.5\rho(16 - x^2 - y^2)\, dA$$

$$= 0.5\rho\int_0^{2\pi}\int_0^4 (16 - r^2)r\, dr\, d\theta = 0.5\rho\int_0^{2\pi} 64\, d\theta = 64\pi\rho$$

42. S: $z = \sqrt{16 - x^2 - y^2}$

$\mathbf{F}(x, y, z) = 0.5z\mathbf{k}$

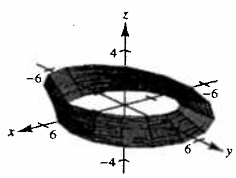
$$\int_S\int \rho\mathbf{F} \cdot \mathbf{N}\, dS = \int_R\int \rho\mathbf{F} \cdot (-g_x(x, y)\mathbf{i} - g_y(x, y)\mathbf{j} + \mathbf{k})\, dA$$

$$= \int_R\int 0.5\rho z\mathbf{k} \cdot \left[\frac{x}{\sqrt{16 - x^2 - y^2}}\mathbf{i} + \frac{y}{\sqrt{16 - x^2 - y^2}}\mathbf{j} + \mathbf{k}\right] dA$$

$$= \int_R\int 0.5\, \rho z\, dA = \int_R\int 0.5\rho\sqrt{16 - x^2 - y^2}\, dA$$

$$= 0.5\rho\int_0^{2\pi}\int_0^4 \sqrt{16 - r^2}r\, dr\, d\theta = 0.5\rho\int_0^{2\pi} \frac{64}{3}\, d\theta = \frac{64\pi\rho}{3}$$

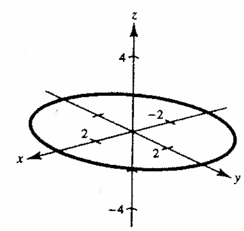
43. (a)



(b) If a normal vector at a point P on the surface is moved around the Möbius strip once, it will point in the opposite direction.

(c) $\mathbf{r}(u, 0) = 4\cos(2u)\mathbf{i} + 4\sin(2u)\mathbf{j}$

This is a circle.



(d) (construction)

(e) You obtain a strip with a double twist and twice as long as the original Möbius strip.

Section 15.7 Divergence Theorem

1. Surface Integral: There are six surfaces to the cube, each with $dS = \sqrt{1}\,dA$.

$$z = 0, \quad \mathbf{N} = -\mathbf{k}, \quad \mathbf{F} \cdot \mathbf{N} = -z^2, \quad \int_{S_1}\!\!\int 0\,dA = 0$$

$$z = a, \quad \mathbf{N} = \mathbf{k}, \quad \mathbf{F} \cdot \mathbf{N} = z^2, \quad \int_{S_2}\!\!\int a^2\,dA = \int_0^a\!\!\int_0^a a^2\,dx\,dy = a^4$$

$$x = 0, \quad \mathbf{N} = -\mathbf{i}, \quad \mathbf{F} \cdot \mathbf{N} = -2x, \quad \int_{S_3}\!\!\int 0\,dA = 0$$

$$x = a, \quad \mathbf{N} = \mathbf{i}, \quad \mathbf{F} \cdot \mathbf{N} = 2x, \quad \int_{S_4}\!\!\int 2a\,dy\,dz = \int_0^a\!\!\int_0^a 2a\,dy\,dz = 2a^3$$

$$y = 0, \quad \mathbf{N} = -\mathbf{j}, \quad \mathbf{F} \cdot \mathbf{N} = 2y, \quad \int_{S_5}\!\!\int 0\,dA = 0$$

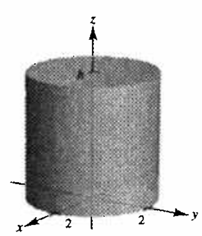
$$y = a, \quad \mathbf{N} = \mathbf{j}, \quad \mathbf{F} \cdot \mathbf{N} = -2y, \quad \int_{S_6}\!\!\int -2a\,dA = \int_0^a\!\!\int_0^a -2a\,dz\,dx = -2a^3$$

Therefore, $\displaystyle\int_S\!\!\int \mathbf{F} \cdot \mathbf{N}\,dS = a^4 + 2a^3 - 2a^3 = a^4$.

Divergence Theorem: Since div $\mathbf{F} = 2z$, the Divergence Theorem yields

$$\iiint_Q \text{div}\,\mathbf{F}\,dV = \int_0^a\!\!\int_0^a\!\!\int_0^a 2z\,dz\,dy\,dx = \int_0^a\!\!\int_0^a a^2\,dy\,dx = a^4.$$

2. Surface Integral: There are three surfaces to the cylinder.



Bottom: $z = 0$, $\mathbf{N} = -\mathbf{k}$, $\mathbf{F} \cdot \mathbf{N} = -z^2$

$$\int_{S_1}\!\!\int 0\,dS = 0$$

Top: $z = h$, $\mathbf{N} = \mathbf{k}$, $\mathbf{F} \cdot \mathbf{N} = z^2$

$$\int_{S_2}\!\!\int h^2\,dS = h^2\,(\text{Area of circle}) = 4\pi h^2$$

Side: $\mathbf{r}(u, v) = 2\cos u\mathbf{i} + 2\sin u\mathbf{j} + v\mathbf{k}$, $0 \le u \le 2\pi$, $0 \le v \le h$

$$\mathbf{r}_u = -2\sin u\mathbf{i} + 2\cos u\mathbf{j}, \quad r_v = \mathbf{k}$$

$$\mathbf{r}_u \times \mathbf{r}_v = 2\cos u\mathbf{i} + 2\sin u\mathbf{j}$$

$$\mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v) = 8\cos^2 u - 8\sin^2 u$$

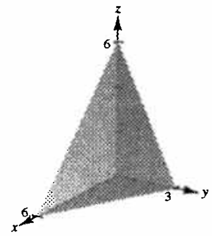
$$\int_{S_3}\!\!\int \mathbf{F} \cdot \mathbf{N}\,dS = \int_0^h\!\!\int_0^{2\pi} (8\cos^2 u - 8\sin^2 u)\,du\,dv = 0$$

Therefore, $\displaystyle\int_S\!\!\int \mathbf{F} \cdot \mathbf{N}\,dS = 0 + 4\pi h^2 + 0 = 4\pi h^2$.

Divergence Theorem: div $\mathbf{F} = 2 - 2 + 2z = 2z$

$$\iiint_Q 2z\,dV = \int_0^{2\pi}\!\!\int_0^2\!\!\int_0^h 2zr\,dz\,dr\,d\theta = 4\pi h^2.$$

3. Surface Integral: There are four surfaces to this solid.



$z = 0,\ \mathbf{N} = -\mathbf{k},\ \mathbf{F} \cdot \mathbf{N} = -z$

$$\int_{S_1}\int 0\, dS = 0$$

$y = 0,\ \mathbf{N} = -\mathbf{j},\ \mathbf{F} \cdot \mathbf{N} = 2y - z,\ dS = dA = dx\, dz$

$$\int_{S_2}\int -z\, dS = \int_0^6 \int_0^{6-z} -z\, dx\, dz = \int_0^6 (z^2 - 6z)\, dz = -36$$

$x = 0,\ \mathbf{N} = -\mathbf{i},\ \mathbf{F} \cdot \mathbf{N} = y - 2x,\ dS = dA = dz\, dy$

$$\int_{S_3}\int y\, dS = \int_0^3 \int_0^{6-2y} y\, dz\, dy = \int_0^3 (6y - 2y^2)\, dy = 9$$

$x + 2y + z = 6,\ \mathbf{N} = \dfrac{\mathbf{i} + 2\mathbf{j} + \mathbf{k}}{\sqrt{6}},\ \mathbf{F} \cdot \mathbf{N} = \dfrac{2x - 5y + 3z}{\sqrt{6}},\ dS = \sqrt{6}\, dA$

$$\int_{S_4}\int (2x - 5y + 3z)\, dz\, dy = \int_0^3 \int_0^{6-2y} (18 - x - 11y)\, dx\, dy = \int_0^3 (90 - 90y + 20y^2)\, dy = 45$$

Therefore, $\displaystyle\int_S\int \mathbf{F} \cdot \mathbf{N}\, dS = 0 - 36 + 9 + 45 = 18.$

Divergence Theorem: Since div $\mathbf{F} = 1$, we have

$$\iiint_Q dV = (\text{Volume of solid}) = \frac{1}{3}(\text{Area of base}) \times (\text{Height}) = \frac{1}{3}(9)(6) = 18.$$

4. $\mathbf{F}(x, y, z) = xy\mathbf{i} + z\mathbf{j} + (x + y)\mathbf{k}$

S: surface bounded by the planes $y = 4,\ z = 4 - x$ and the coordinate planes

Surface Integral: There are five surfaces to this solid.

$z = 0,\ \mathbf{N} = -\mathbf{k},\ \mathbf{F} \cdot \mathbf{N} = -(x + y)$

$$\int_{S_1}\int -(x + y)\, dS = \int_0^4 \int_0^4 -(x + y)\, dy\, dx = -\int_0^4 (4x + 8)\, dx = -64$$

$y = 0,\ \mathbf{N} = -\mathbf{j},\ \mathbf{F} \cdot \mathbf{N} = -z$

$$\int_{S_2}\int -z\, dS = \int_0^4 \int_0^{4-x} -z\, dz\, dx = -\int_0^4 \frac{(4 - x)^2}{2}\, dx = -\frac{32}{3}$$

$y = 4,\ \mathbf{N} = \mathbf{j},\ \mathbf{F} \cdot \mathbf{N} = z$

$$\int_{S_3}\int z\, dS = \int_0^4 \int_0^{4-x} z\, dz\, dx = \int_0^4 \frac{(4 - x)^2}{2}\, dx = \frac{32}{3}$$

$x = 0,\ \mathbf{N} = -\mathbf{i},\ \mathbf{F} \cdot \mathbf{N} = -xy$

$$\int_{S_4}\int -xy\, dS = \int_0^4 \int_0^4 0\, dS = 0$$

$x + z = 4,\ \mathbf{N} = \dfrac{\mathbf{i} + \mathbf{k}}{\sqrt{2}},\ \mathbf{F} \cdot \mathbf{N} = \dfrac{1}{\sqrt{2}}[xy + x + y],\ dS = \sqrt{2}\, dA$

$$\int_{S_5}\int \frac{1}{\sqrt{2}}[xy + x + y]\sqrt{2}\, dA = \int_0^4 \int_0^4 (xy + x + y)\, dy\, dx = 128$$

Therefore, $\displaystyle\int_S\int \mathbf{F} \cdot \mathbf{N}\, dS = -64 - \tfrac{32}{3} + \tfrac{32}{3} + 0 + 128 = 64.$

Divergence Theorem: Since div $\mathbf{F} = y$, we have

$$\iiint_Q \text{div}\, \mathbf{F}\, dV = \int_0^4 \int_0^4 \int_0^{4-x} y\, dz\, dy\, dx = 64.$$

5. Since div $\mathbf{F} = 2x + 2y + 2z$, we have

$$\iiint_Q \operatorname{div} \mathbf{F}\, dV = \int_0^a \int_0^a \int_0^a (2x + 2y + 2z)\, dz\, dy\, dx$$

$$= \int_0^a \int_0^a (2ax + 2ay + a^2)\, dy\, dx = \int_0^a (2a^2x + 2a^3)\, dx = \Big[a^2x^2 + 2a^3x\Big]_0^a = 3a^4.$$

6. Since div $\mathbf{F} = 2xz^2 - 2 + 3xy$ we have

$$\iiint_Q \operatorname{div} \mathbf{F}\, dV = \int_0^a \int_0^a \int_0^a (2xz^2 - 2 + 3xy)\, dz\, dy\, dx = \int_0^a \int_0^a \left(\frac{2}{3}xa^3 - 2a + 3xya\right) dy\, dx$$

$$= \int_0^a \left(\frac{2}{3}xa^4 - 2a^2 + \frac{3}{2}xa^3\right) dx$$

$$= \frac{1}{3}a^6 - 2a^3 + \frac{3}{4}a^5.$$

7. Since div $\mathbf{F} = 2x - 2x + 2xyz = 2xyz$

$$\iiint_Q \operatorname{div} \mathbf{F}\, dV = \iiint_Q 2xyz\, dV = \int_0^a \int_0^{2\pi} \int_0^{\pi/2} 2(\rho \sin\phi \cos\theta)(\rho \sin\phi \sin\theta)(\rho\cos\phi)\rho^2 \sin\phi\, d\phi\, d\theta\, d\rho$$

$$= \int_0^a \int_0^{2\pi} \int_0^{\pi/2} 2\rho^5(\sin\theta\cos\theta)(\sin^3\phi\cos\phi)\, d\phi\, d\theta\, d\rho$$

$$= \int_0^a \int_0^{2\pi} \frac{1}{2}\rho^5 \sin\theta\cos\theta\, d\theta\, d\rho = \int_0^a \left[\left(\frac{\rho^5}{2}\right)\frac{\sin^2\theta}{2}\right]_0^{2\pi} d\rho = 0.$$

8. Since div $\mathbf{F} = y + z - y = z$, we have

$$\iiint_Q \operatorname{div} \mathbf{F}\, dV = \int_{-a}^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} z\, dz\, dy\, dx = \int_0^{2\pi} \int_0^a \int_0^{\sqrt{a^2-r^2}} zr\, dz\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^a \left[\frac{a^2r}{2} - \frac{r^3}{2}\right] dr\, d\theta = \int_0^{2\pi} \left[\frac{a^2r^2}{4} - \frac{r^4}{8}\right]_0^a d\theta = \int_0^{2\pi} \frac{a^4}{8}\, d\theta = \frac{\pi a^4}{4}.$$

9. Since div $\mathbf{F} = 3$, we have

$$\iiint_Q 3\, dV = 3(\text{Volume of sphere}) = 3\left[\frac{4}{3}\pi(2)^3\right] = 32\pi.$$

10. Since div $\mathbf{F} = xz$, we have

$$\iiint_Q xz\, dV = \int_0^4 \int_{-3}^3 \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} xz\, dx\, dy\, dz = \int_0^4 \int_{-3}^3 \frac{z}{2}(0)\, dy\, dz = 0.$$

11. Since div $\mathbf{F} = 1 + 2y - 1 = 2y$, we have

$$\iiint_Q 2y\, dV = \int_0^4 \int_{-3}^3 \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} 2y\, dx\, dy\, dz = \int_0^4 \int_{-3}^3 4y\sqrt{9-y^2}\, dy\, dz = \int_0^4 \left[-\frac{4}{3}(9-y^2)^{3/2}\right]_{-3}^3 dz = 0.$$

12. Since div $\mathbf{F} = y^2 + x^2 + e^z$, we have

$$\iiint_Q (x^2 + y^2 + e^z)\,dV = \int_0^{16}\int_{-\sqrt{256-x^2}}^{\sqrt{256-x^2}}\int_{(1/2)\sqrt{x^2+y^2}}^{8} (x^2 + y^2 + e^z)\,dz\,dy\,dx$$

$$= \int_0^{2\pi}\int_0^{16}\int_{r/2}^{8} (r^2 + e^z)r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^{16}\left(8r^3 + re^8 - \frac{1}{2}r^4 - re^{r/2}\right)dr\,d\theta$$

$$= \int_0^{2\pi}\left(\frac{131{,}052}{5} + 100e^8\right)d\theta = \frac{262{,}104}{5}\pi + 200e^8\pi.$$

13. Since div $\mathbf{F} = 3x^2 + x^2 + 0 = 4x^2$, we have

$$\iiint_Q 4x^2\,dV = \int_0^6\int_0^4\int_0^{4-y} 4x^2\,dz\,dy\,dx = \int_0^6\int_0^4 4x^2(4 - y)\,dy\,dx = \int_0^6 32x^2\,dx = 2304.$$

14. Since div $\mathbf{F} = e^z + e^z + e^z = 3e^z$, we have

$$\iiint_Q 3e^z\,dV = \int_0^6\int_0^4\int_0^{4-y} 3e^z\,dz\,dy\,dx = \int_0^6\int_0^4 3[e^{4-y} - 1]\,dy\,dx = \int_0^6 3(e^4 - 5)\,dx = 18(e^4 - 5).$$

15. $\mathbf{F}(x, y, z) = xy\mathbf{i} + 4y\mathbf{j} + xz\mathbf{k}$

div $\mathbf{F} = y + 4 + x$

$$\iint_S \mathbf{F}\cdot\mathbf{N}\,dS = \iiint_Q \text{div}\,\mathbf{F}\,dV = \iiint_Q (y + x + 4)\,dV$$

$$= \int_0^3\int_0^{\pi}\int_0^{2\pi} (\rho\sin\phi\sin\theta + \rho\sin\phi\cos\theta + 4)\rho^2\sin\phi\,d\theta\,d\phi\,d\rho$$

$$= \int_0^3\int_0^{\pi}\int_0^{2\pi} [\rho^3\sin^2\phi\sin\theta + \rho^3\sin^2\phi\cos\theta + 4\rho^2\sin\phi]\,d\theta\,d\phi\,d\rho$$

$$= \int_0^3\int_0^{\pi}\left[-\rho^3\sin^2\phi\cos\theta + \rho^3\sin^2\phi\sin\theta + 4\rho^2\sin\phi\cdot\theta\right]_0^{2\pi}d\phi\,d\rho$$

$$= \int_0^3\int_0^{\pi} 8\pi\rho^2\sin\phi\,d\phi\,d\rho$$

$$= \int_0^3\left[-8\pi\rho^2\cos\phi\right]_0^{\pi}d\rho$$

$$= \int_0^3 16\pi\rho^2\,d\rho = \left[\frac{16\pi\rho^3}{3}\right]_0^3 = 144\pi$$

16. div $\mathbf{F} = 2$

$$\iint_S \mathbf{F}\cdot\mathbf{N}\,dS = \iiint_Q \text{div}\,\mathbf{F}\,dV = \iiint_Q 2\,dV.$$

The surface S is the upper half of a hemisphere of radius 2. Since the volume is $\frac{1}{2}\left(\frac{4}{3}\pi(2^3)\right) = 16\pi/3$, you have

$$\iint_S \mathbf{F}\cdot\mathbf{N}\,dS = 2(\text{Volume}) = \frac{32\pi}{3}.$$

17. Using the Divergence Theorem, we have

$$\int_S\int \mathbf{curl\ F} \cdot \mathbf{N}\, dS = \iiint_Q \text{div}\,(\mathbf{curl\ F})\, dV$$

$$\mathbf{curl\ F}(x, y, z) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 4xy + z^2 & 2x^2 + 6yz & 2xz \end{vmatrix} = -6y\mathbf{i} - (2z - 2z)\mathbf{j} + (4x - 4x)\mathbf{k} = -6y\mathbf{i}$$

$$\text{div}\,(\mathbf{curl\ F}) = 0.$$

Therefore, $\displaystyle\iiint_Q \text{div}\,(\mathbf{curl\ F})\, dV = 0.$

18. Using the Divergence Theorem, we have

$$\int_S\int \mathbf{curl\ F} \cdot \mathbf{N}\, dS = \iiint_Q \text{div}\,(\mathbf{curl\ F})\, dV$$

$$\mathbf{curl\ F}(x, y, z) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xy\cos z & yz\sin x & xyz \end{vmatrix} = (xz - y\sin x)\mathbf{i} - (yz + xy\sin z)\mathbf{j} + (yz\cos x - x\cos z)\mathbf{k}.$$

Now, div $\mathbf{curl\ F}(x, y, z) = (z - y\cos x) - (z + x\sin z) + (y\cos x + x\sin z) = 0$. Therefore,

$$\int_S\int \mathbf{curl\ F} \cdot \mathbf{N}\, dS = \iiint_Q \text{div}\,(\mathbf{curl\ F})\, dV = 0.$$

19. See Theorem 15.12.

20. If div $\mathbf{F}(x, y, z) > 0$, then source.

If div $\mathbf{F}(x, y, z) < 0$, then sink.

If div $\mathbf{F}(x, y, z) = 0$, then incompressible.

21. Using the triple integral to find volume, we need $\mathbf{F}$ so that

$$\text{div}\,\mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z} = 1.$$

Hence, we could have $\mathbf{F} = x\mathbf{i}$, $\mathbf{F} = y\mathbf{j}$, or $\mathbf{F} = z\mathbf{k}$.

For $dA = dy\,dz$ consider $\mathbf{F} = x\mathbf{i}$, $x = f(y, z)$, then $\mathbf{N} = \dfrac{\mathbf{i} + f_y\mathbf{j} + f_z\mathbf{k}}{\sqrt{1 + f_y^2 + f_z^2}}$ and $dS = \sqrt{1 + f_y^2 + f_z^2}\, dy\, dz$.

For $dA = dz\,dx$ consider $\mathbf{F} = y\mathbf{j}$, $y = f(x, z)$, then $\mathbf{N} = \dfrac{f_x\mathbf{i} + \mathbf{j} + f_z\mathbf{k}}{\sqrt{1 + f_x^2 + f_z^2}}$ and $dS = \sqrt{1 + f_x^2 + f_z^2}\, dz\, dx$.

For $dA = dx\,dy$ consider $\mathbf{F} = z\mathbf{k}$, $z = f(x, y)$, then $\mathbf{N} = \dfrac{f_x\mathbf{i} + f_y\mathbf{j} + \mathbf{k}}{\sqrt{1 + f_x^2 + f_y^2}}$ and $dS = \sqrt{1 + f_x^2 + f_y^2}\, dx\, dy$.

Correspondingly, we then have $V = \displaystyle\int_S\int \mathbf{F} \cdot \mathbf{N}\, dS = \int_S\int x\, dy\, dz = \int_S\int y\, dz\, dx = \int_S\int z\, dx\, dy.$

22. $\displaystyle v = \int_0^a\int_0^a x\, dy\, dz = \int_0^a\int_0^a a\, dy\, dz = \int_0^a a^2\, dz = a^3$

Similarly, $\displaystyle\int_0^a\int_0^a y\, dz\, dx = \int_0^a\int_0^a z\, dx\, dy = a^3.$

23. Using the Divergence Theorem, we have $\displaystyle\int_S\!\!\int \mathbf{curl\,F}\cdot\mathbf{N}\,dS = \iiint_Q \text{div}\,(\mathbf{curl\,F})\,dV$. Let

$$\mathbf{F}(x, y, z) = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$$

$$\mathbf{curl\,F} = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\mathbf{i} - \left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z}\right)\mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k}$$

$$\text{div}\,(\mathbf{curl\,F}) = \frac{\partial^2 P}{\partial x\partial y} - \frac{\partial^2 N}{\partial x\partial z} - \frac{\partial^2 P}{\partial y\partial x} + \frac{\partial^2 M}{\partial y\partial z} + \frac{\partial^2 N}{\partial z\partial x} - \frac{\partial^2 M}{\partial z\partial y} = 0.$$

Therefore, $\displaystyle\int_S\!\!\int \mathbf{curl\,F}\cdot\mathbf{N}\,dS = \iiint_Q 0\,dV = 0.$

24. If $\mathbf{F}(x, y, z) = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, then div $\mathbf{F} = 0$. Therefore,

$$\int_S\!\!\int \mathbf{F}\cdot\mathbf{N}\,dS = \iiint_Q \text{div}\,\mathbf{F}\,dV = \iiint_Q 0\,dV = 0.$$

25. If $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, then div $\mathbf{F} = 3$.

$$\int_S\!\!\int \mathbf{F}\cdot\mathbf{N}\,dS = \iiint_Q \text{div}\,\mathbf{F}\,dV = \iiint_Q 3\,dV = 3V.$$

26. If $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, then div $\mathbf{F} = 3$.

$$\frac{1}{\|\mathbf{F}\|}\int_S\!\!\int \mathbf{F}\cdot\mathbf{N}\,dS = \frac{1}{\|\mathbf{F}\|}\iiint_Q \text{div}\,\mathbf{F}\,dV = \frac{1}{\|\mathbf{F}\|}\iiint_Q 3\,dV = \frac{3}{\|\mathbf{F}\|}\iiint_Q dV$$

27.

$$\begin{aligned}\int_S\!\!\int fD_{\mathbf{N}}g\,dS &= \int_S\!\!\int f\nabla g\cdot\mathbf{N}\,dS\\ &= \iiint_Q \text{div}\,(f\nabla g)\,dV = \iiint_Q (f\,\text{div}\,\nabla g + \nabla f\cdot\nabla g)\,dV = \iiint_Q (f\nabla^2 g + \nabla f\cdot\nabla g)\,dV\end{aligned}$$

28.

$$\begin{aligned}\int_S\!\!\int (fD_{\mathbf{N}}g - gD_{\mathbf{N}}f)\,dS &= \int_S\!\!\int fD_{\mathbf{N}}g\,dS - \int_S\!\!\int gD_{\mathbf{N}}f\,dS\\ &= \iiint_Q (f\nabla^2 g + \nabla f\cdot\nabla g)\,dV - \iiint_Q (g\nabla^2 f + \nabla g\cdot\nabla f)\,dV = \iiint_Q (f\nabla^2 g - g\nabla^2 f)\,dV\end{aligned}$$

Section 15.8 Stokes's Theorem

1. $\mathbf{F}(x, y, z) = (2y - z)\mathbf{i} + xyz\mathbf{j} + e^z\mathbf{k}$

$$\mathbf{curl\,F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ 2y - z & xyz & e^z \end{vmatrix} = -xy\mathbf{i} - \mathbf{j} + (yz - 2)\mathbf{k}$$

2. $\mathbf{F}(x, y, z) = x^2\mathbf{i} + y^2\mathbf{j} + x^2\mathbf{k}$

$$\mathbf{curl\,F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ x^2 & y^2 & x^2 \end{vmatrix} = -2x\mathbf{j}$$

3. $\mathbf{F}(x, y, z) = 2z\mathbf{i} - 4x^2\mathbf{j} + \arctan x\mathbf{k}$

$$\mathbf{curl\,F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ 2z & -4x^2 & \arctan x \end{vmatrix} = \left(2 - \frac{1}{1 + x^2}\right)\mathbf{j} - 8x\mathbf{k}$$

4. $\mathbf{F}(x, y, z) = x\sin y\mathbf{i} - y\cos x\mathbf{j} + yz^2\mathbf{k}$

$$\begin{aligned}\mathbf{curl\,F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ x\sin y & -y\cos x & yz^2 \end{vmatrix}\\ &= z^2\mathbf{i} + (y\sin x - x\cos y)\mathbf{k}\end{aligned}$$

5. $\mathbf{F}(x, y, z) = e^{x^2+y^2}\mathbf{i} + e^{y^2+z^2}\mathbf{j} + xyz\mathbf{k}$

$$\textbf{curl F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ e^{x^2+y^2} & e^{y^2+z^2} & xyz \end{vmatrix}$$

$$= (xz - 2ze^{y^2+z^2})\mathbf{i} - yz\mathbf{j} - 2ye^{x^2+y^2}\mathbf{k}$$

$$= z(x - 2e^{y^2+z^2})\mathbf{i} - yz\mathbf{j} - 2ye^{x^2+y^2}\mathbf{k}$$

6. $\mathbf{F}(x, y, z) = \arcsin y\mathbf{i} + \sqrt{1 - x^2}\,\mathbf{j} + y^2\mathbf{k}$

$$\textbf{curl F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \arcsin y & \sqrt{1 - x^2} & y^2 \end{vmatrix}$$

$$= 2y\mathbf{i} + \left[\frac{-x}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-y^2}}\right]\mathbf{k}$$

$$= 2y\mathbf{i} - \left[\frac{x}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-y^2}}\right]\mathbf{k}$$

7. In this case, $M = -y + z,\ N = x - z,\ P = x - y$ and C is the circle $x^2 + y^2 = 1,\ z = 0,\ dz = 0$.

Line Integral: $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C (-y + z)\,dx + (x - z)\,dy + (x - y)\,dz = \int_C -y\,dx + x\,dy$

Letting $x = \cos t,\ y = \sin t,$ we have $dx = -\sin t\,dt,\ dy = \cos t\,dt$ and

$$\int_C -y\,dx + x\,dy = \int_0^{2\pi} (\sin^2 t + \cos^2 t)\,dt = 2\pi.$$

Double Integral: Consider $F(x, y, z) = x^2 + y^2 + z^2 - 1$.

Then

$$\mathbf{N} = \frac{\nabla F}{\|\nabla F\|} = \frac{2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}}{2\sqrt{x^2 + y^2 + z^2}} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

Since

$$z^2 = 1 - x^2 - y^2,\ z_x = \frac{-2x}{2z} = \frac{-x}{z},\ \text{and } z_y = \frac{-y}{z},\ dS = \sqrt{1 + \frac{x^2}{z^2} + \frac{y^2}{z^2}}\,dA = \frac{1}{z}\,dA.$$

Now, since **curl F** $= 2\mathbf{k}$, we have

$$\int_S\int (\textbf{curl F}) \cdot \mathbf{N}\,dS = \int_R\int 2z\left(\frac{1}{z}\right)dA = \int_R\int 2\,dA = 2(\text{Area of circle of radius } 1) = 2\pi.$$

8. In this case C is the circle $x^2 + y^2 = 4,\ z = 0,\ dz = 0$.

Line Integral: $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C -y\,dx + x\,dy$

Let $x = 2\cos t,\ y = 2\sin t,$ then $dx = -2\sin t\,dt,\ dy = 2\cos t\,dt,$ and $\displaystyle\int_C -y\,dx + x\,dy = \int_0^{2\pi} 4\,dt = 8\pi.$

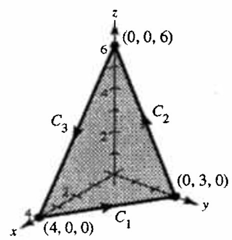
Double Integral: $F(x, y, z) = z + x^2 + y^2 - 4,\ \mathbf{N} = \dfrac{\nabla F}{\|\nabla F\|} = \dfrac{2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k}}{\sqrt{1 + 4x^2 + 4y^2}},\ dS = \sqrt{1 + 4x^2 + 4y^2}\,dA$

curl F $= 2\mathbf{k}$, therefore

$$\int\int (\textbf{curl F}) \cdot \mathbf{N}dS = \int_R\int 2\,dA = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 2\,dy\,dx = 2\int_{-2}^{2} 2\sqrt{4 - x^2}\,dx$$

$$= 4\int_{-2}^{2} \sqrt{4 - x^2}\,dx = 2\left[x\sqrt{4 - x^2} + 4\arcsin\frac{x}{2}\right]_{-2}^{2} = 8\pi.$$

9. Line Integral: From the accompanying figure we see that for



C_1: $z = 0, \ dz = 0$

C_2: $x = 0, \ dx = 0$

C_3: $y = 0, \ dy = 0.$

Hence, $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C xyz\,dx + y\,dy + z\,dz$

$$= \int_{C_1} y\,dy + \int_{C_2} y\,dy + z\,dz + \int_{C_3} z\,dz$$

$$= \int_0^3 y\,dy + \int_3^0 y\,dy + \int_0^6 z\,dz + \int_6^0 z\,dz = 0.$$

Double Integral: $\mathbf{curl\ F} = xy\mathbf{j} - xz\mathbf{k}$

Considering $F(x, y, z) = 3x + 4y + 2z - 12$, then

$$\mathbf{N} = \frac{\nabla F}{\|\nabla F\|} = \frac{3\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}}{\sqrt{29}} \text{ and } dS = \sqrt{29}\,dA.$$

Thus,

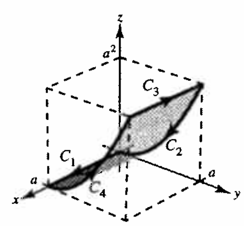
$$\int_S\int (\mathbf{curl\ F}) \cdot \mathbf{N}\,dS = \int_R\int (4xy - 2xz)\,dA$$

$$= \int_0^4\int_0^{(-3x+12)/4} \left[4xy - 2x\left(6 - 2y - \frac{3}{2}x\right)\right] dy\,dx$$

$$= \int_0^4\int_0^{(12-3x)/4} (8xy + 3x^2 - 12x)\,dy\,dx$$

$$= \int_0^4 0\,dx = 0.$$

10. Line Integral: From the figure we see that



C_1: $y = 0, \ z = 0, \ dy = dz = 0$

C_2: $z = y^2, \ x = 0, \ dx = 0, \ dz = 2y\,dy$

C_3: $y = a, \ z = a^2, \ dy = dz = 0$

C_4: $z = y^2, \ x = a, \ dx = 0, \ dz = 2y\,dy.$

Hence,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C z^2\,dx + x^2\,dy + y^2\,dz$$

$$= \int_{C_1} 0\,dx + \int_{C_2} 2y^3\,dy + \int_{C_3} a^4\,dx + \int_{C_4} [a^2\,dy + 2y^3\,dy]$$

$$= \int_a^0 2y^3\,dy + \int_a^0 a^4\,dx + \int_0^a (a^2 2y^3)dy = \Big[a^4x\Big]_a^0 + \Big[a^2y\Big]_0^a = -a^5 + a^3 = a^3(1 - a).$$

Double Integral: Since S is given by $-y^2 + z = 0$, we have

$$\mathbf{N} = \frac{2y\mathbf{j} - \mathbf{k}}{\sqrt{1 + 4y^2}} \text{ and } dS = \sqrt{1 + 4y^2}\,dA.$$

Furthermore, $\mathbf{curl\ F} = 2y\mathbf{i} + 2z\mathbf{j} + 2x\mathbf{k}$. Therefore,

$$\int_S\int (\mathbf{curl\ F}) \cdot \mathbf{N}\,dS = \int_R\int (4yz - 2x)\,dA = \int_0^a\int_0^a (-4yz + 2x)\,dA = \int_0^a\int_0^a (-4y^3 + 2^x)\,dydx$$

$$= \int_0^a (-a^4 + 2ax)\,dx = \Big[-a^4x + ax^2\Big]_0^a = -a^5 + a^3 = a^3(1 - a^2).$$

11. Let $A = (0, 0, 0)$, $B = (1, 1, 1)$ and $C = (0, 2, 0)$. Then $\mathbf{U} = \overrightarrow{AB} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{V} = \overrightarrow{AC} = 2\mathbf{j}$. Thus,

$$\mathbf{N} = \frac{\mathbf{U} \times \mathbf{V}}{\|\mathbf{U} \times \mathbf{V}\|} = \frac{-2\mathbf{i} + 2\mathbf{k}}{2\sqrt{2}} = \frac{-\mathbf{i} + \mathbf{k}}{\sqrt{2}}.$$

Surface S has direction numbers $-1, 0, 1$, with equation $z - x = 0$ and $dS = \sqrt{2}\, dA$. Since $\mathbf{curl\, F} = -3\mathbf{i} - \mathbf{j} - 2\mathbf{k}$, we have

$$\int_S\int (\mathbf{curl\, F}) \cdot \mathbf{N}\, dS = \int_R\int \frac{1}{\sqrt{2}}(\sqrt{2})\, dA = \int_R\int dA = (\text{Area of triangle with } h = 1, b = 2) = 1.$$

12. Let $A = (0, 0, 0)$, $B = (1, 1, 1)$, and $C = (0, 0, 2)$. Then $\mathbf{U} = \overrightarrow{AB} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, and $\mathbf{V} = \overrightarrow{AC} = 2\mathbf{k}$, and

$$\mathbf{N} = \frac{\mathbf{U} \times \mathbf{V}}{\|\mathbf{U} \times \mathbf{V}\|} = \frac{2\mathbf{i} - 2\mathbf{j}}{2\sqrt{2}} = \frac{\mathbf{i} - \mathbf{j}}{\sqrt{2}}.$$

Hence, $F(x, y, z) = x - y$ and $dS = \sqrt{2}\, dA$. Since $\mathbf{curl\, F} = \dfrac{2x}{x^2 + y^2}\mathbf{k}$, we have $\displaystyle\int_S\int (\mathbf{curl\, F}) \cdot \mathbf{N}\, dS = \int_R\int 0\, dS = 0.$

13. $\mathbf{F}(x, y, z) = z^2\mathbf{i} + x^2\mathbf{j} + y^2\mathbf{k}$, S: $z = 4 - x^2 - y^2$, $0 \le z$

$$\mathbf{curl\, F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ z^2 & x^2 & y^2 \end{vmatrix} = 2y\mathbf{i} + 2z\mathbf{j} + 2x\mathbf{k}$$

$G(x, y, z) = x^2 + y^2 + z - 4$

$\nabla G(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k}$

$$\begin{aligned}\int_S\int (\mathbf{curl\, F}) \cdot \mathbf{N}\, dS = \int_R\int (4xy + 4yz + 2x)\, dA &= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} [4xy + 4y(4 - x^2 - y^2) + 2x]\, dy\, dx \\ &= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} [4xy + 16y - 4x^2y - 4y^3 + 2x]\, dy\, dx \\ &= \int_{-2}^{2} 4x\sqrt{4 - x^2}\, dx = 0\end{aligned}$$

14. $\mathbf{F}(x, y, z) = 4xz\mathbf{i} + y\mathbf{j} + 4xy\mathbf{k}$, S: $9 - x^2 - y^2$, $z \le 0$

$\mathbf{curl\, F} = 4x\mathbf{i} + (4x - 4y)\mathbf{j}$

$G(x, y, z) = x^2 + y^2 + z - 9$

$\nabla G(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k}$

$$\begin{aligned}\int_S\int (\mathbf{curl\, F}) \cdot \mathbf{N}\, dS = \int_R\int [8x^2 + 2y(4x - 4y)]\, dA &= \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} [8x^2 + 8xy - 8y^2]\, dy\, dx \\ &= \int_{-3}^{3}\left(16x^2\sqrt{9 - x^2} - \frac{16}{3}(9 - x^2)^{3/2}\right) dx = 0\end{aligned}$$

15. $\mathbf{F}(x, y, z) = z^2\mathbf{i} + y\mathbf{j} + xz\mathbf{k}$, S: $z = \sqrt{4 - x^2 - y^2}$

$$\mathbf{curl\, F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ z^2 & y & xz \end{vmatrix} = z\mathbf{j}$$

$G(x, y, z) = z - \sqrt{4 - x^2 - y^2}$

$$\nabla G(x, y, z) = \frac{x}{\sqrt{4 - x^2 - y^2}}\mathbf{i} + \frac{y}{\sqrt{4 - x^2 - y^2}}\mathbf{j} + \mathbf{k}$$

$$\int_S\int (\mathbf{curl\, F}) \cdot \mathbf{N}\, dS = \int_R\int \frac{yz}{\sqrt{4 - x^2 - y^2}}\, dA = \int_R\int \frac{y\sqrt{4 - x^2 - y^2}}{\sqrt{4 - x^2 - y^2}}\, dA = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} y\, dy\, dx = 0$$

16. $\mathbf{F}(x, y, z) = x^2\mathbf{i} + z^2\mathbf{j} - xyz\mathbf{k}$, S: $z = \sqrt{4 - x^2 - y^2}$

$$\mathbf{curl\ F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 & z^2 & -xyz \end{vmatrix} = (-xz - 2z)\mathbf{i} + yz\mathbf{j}$$

$$G(x, y, z) = z - \sqrt{4 - x^2 - y^2}$$

$$\nabla G(x, y, z) = \frac{x}{\sqrt{4 - x^2 - y^2}}\mathbf{i} + \frac{y}{\sqrt{4 - x^2 - y^2}}\mathbf{j} + \mathbf{k}$$

$$\int_S\int (\mathbf{curl\ F}) \cdot \mathbf{N}\, dS = \int_R\int \left[\frac{-z(x + 2)x}{\sqrt{4 - x^2 - y^2}} + \frac{y^2 z}{\sqrt{4 - x^2 - y^2}}\right] dA$$

$$= \int_R\int [-x(x + 2) + y^2]\, dA = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (-x^2 - 2x + y^2)\, dy\, dx$$

$$= \int_{-2}^{2} \left[-x^2 y - 2xy + \frac{y^3}{3}\right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx$$

$$= \int_{-2}^{2} \left[-2x^2\sqrt{4 - x^2} - 4x\sqrt{4 - x^2} + \frac{2}{3}(4 - x^2)\sqrt{4 - x^2}\right] dx$$

$$= \int_{-2}^{2} \left[-\frac{8}{3}x^2\sqrt{4 - x^2} - 4x\sqrt{4 - x^2} + \frac{8}{3}\sqrt{4 - x^2}\right] dx$$

$$= \left[-\frac{8}{3}\left(\frac{1}{8}\right)\left[x(2x^2 - 4)\sqrt{4 - x^2} + 16 \arcsin\frac{x}{2}\right] + \frac{4}{3}(4 - x^2)^{3/2} + \frac{8}{3}\left(\frac{1}{2}\right)\left[x\sqrt{4 - x^2} + 4 \arcsin\frac{x}{2}\right]\right]_{-2}^{2}$$

$$= \left[\left(-\frac{1}{3}\right)(8\pi) + \frac{4}{3}(2\pi) + \frac{1}{3}(-8\pi) - \frac{4}{3}(-2\pi)\right] = 0$$

17. $\mathbf{F}(x, y, z) = -\ln\sqrt{x^2 + y^2}\,\mathbf{i} + \arctan\dfrac{x}{y}\mathbf{j} + \mathbf{k}$

$$\mathbf{curl\ F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -1/2 \ln(x^2 + y^2) & \arctan x/y & 1 \end{vmatrix} = \left[\frac{(1/y)}{1 + (x^2/y^2)} + \frac{y}{x^2 + y^2}\right]\mathbf{k} = \left[\frac{2y}{x^2 + y^2}\right]\mathbf{k}$$

S: $z = 9 - 2x - 3y$ over one petal of $r = 2 \sin 2\theta$ in the first octant.

$$G(x, y, z) = 2x + 3y + z - 9$$

$$\nabla G(x, y, z) = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$$

$$\int_S\int (\mathbf{curl\ F}) \cdot \mathbf{N}\, dS = \int_R\int \frac{2y}{x^2 + y^2}\, dA$$

$$= \int_0^{\pi/2}\int_0^{2 \sin 2\theta} \frac{2r \sin\theta}{r^2} r\, dr\, d\theta$$

$$= \int_0^{\pi/2}\int_0^{4 \sin\theta \cos\theta} 2 \sin\theta\, dr\, d\theta$$

$$= \int_0^{\pi/2} 8 \sin^2\theta \cos\theta\, d\theta = \left[\frac{8 \sin^3\theta}{3}\right]_0^{\pi/2} = \frac{8}{3}$$

18. $\mathbf{F}(x, y, z) = yz\mathbf{i} + (2 - 3y)\mathbf{j} + (x^2 + y^2)\mathbf{k}$

$$\textbf{curl F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ yz & 2 - 3y & x^2 + y^2 \end{vmatrix} = 2y\mathbf{i} + (y - 2x)\mathbf{j} - z\mathbf{k}$$

S: the first octant portion of $x^2 + z^2 = 16$ over $x^2 + y^2 = 16$

$G(x, y, z) = z - \sqrt{16 - x^2}$

$\nabla G(x, y, z) = \dfrac{x}{\sqrt{16 - x^2}}\mathbf{i} + \mathbf{k}$

$$\begin{aligned} \int_S\!\!\int (\textbf{curl F}) \cdot \mathbf{N}\, dS &= \int_R\!\!\int \left[\frac{2xy}{\sqrt{16 - x^2}} - z\right] dA \\ &= \int_R\!\!\int \left[\frac{2xy}{\sqrt{16 - x^2}} - \sqrt{16 - x^2}\right] dA \\ &= \int_0^4 \int_0^{\sqrt{16-x^2}} \left[\frac{2xy}{\sqrt{16 - x^2}} - \sqrt{16 - x^2}\right] dy\, dx \\ &= \int_0^4 \left[\frac{x}{\sqrt{16 - x^2}}y^2 - \sqrt{16 - x^2}\,y\right]_0^{\sqrt{16-x^2}} dx \\ &= \int_0^4 \left[x\sqrt{16 - x^2} - (16 - x^2)\right] dx \\ &= \left[-\frac{1}{3}(16 - x^2)^{3/2} - 16x + \frac{x^3}{3}\right]_0^4 \\ &= \left(-64 + \frac{64}{3}\right) - \left(-\frac{64}{3}\right) = -\frac{64}{3} \end{aligned}$$

19. From Exercise 9, we have $\mathbf{N} = \dfrac{2x\mathbf{i} - \mathbf{k}}{\sqrt{1 + 4x^2}}$ and $dS = \sqrt{1 + 4x^2}\, dA$. Since $\textbf{curl F} = xy\mathbf{j} - xz\mathbf{k}$, we have

$$\int_S\!\!\int (\textbf{curl F}) \cdot \mathbf{N}\, dS = \int_R\!\!\int xz\, dA = \int_0^a \int_0^a x^3\, dy\, dx = \int_0^a ax^3\, dx = \left[\frac{ax^4}{4}\right]_0^a = \frac{a^5}{4}.$$

20. $\mathbf{F}(x, y, z) = xyz\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

$$\textbf{curl F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xyz & y & z \end{vmatrix} = xy\mathbf{j} - xz\mathbf{k}$$

S: the first octant portion of $z = x^2$ over $x^2 + y^2 = a^2$. We have

$\mathbf{N} = \dfrac{2x\mathbf{i} - \mathbf{k}}{\sqrt{1 + 4x^2}}$ and $dS = \sqrt{1 + 4x^2}\, dA$.

$$\begin{aligned} \int_S\!\!\int (\textbf{curl F}) \cdot \mathbf{N}\, dS &= \int_R\!\!\int xz\, dA = \int_R\!\!\int x^3\, dA \\ &= \int_0^a \int_0^{\sqrt{a^2-x^2}} x^3\, dy\, dx \\ &= \int_0^a x^3\sqrt{a^2 - x^2}\, dx \\ &= \left[-\frac{1}{3}x^2(a^2 - x^2)^{3/2} - \frac{2}{15}(a^2 - x^2)^{5/2}\right]_0^a \\ &= \frac{2}{15}a^5 \end{aligned}$$

21. $\mathbf{F}(x, y, z) = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$

$$\mathbf{curl\,F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 1 & 1 & -2 \end{vmatrix} = \mathbf{0}$$

Letting $\mathbf{N} = \mathbf{k}$, we have $\displaystyle\int_S\!\!\int (\mathbf{curl\,F}) \cdot \mathbf{N}\, dS = 0$.

22. $\mathbf{F}(x, y, z) = -z\mathbf{i} + y\mathbf{k}$

S: $x^2 + y^2 = 1$

$$\mathbf{curl\,F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -z & 0 & y \end{vmatrix} = \mathbf{i} - \mathbf{j}$$

Letting $\mathbf{N} = \mathbf{k}$, $\mathbf{curl\,F} \cdot \mathbf{N} = 0$ and

$$\int_S\!\!\int (\mathbf{curl\,F}) \cdot \mathbf{N}\, dS = 0.$$

23. See Theorem 15.13.

24. **curl F** measures the rotational tendency.

See page 1031.

25. If $\mathbf{curl}\,(\mathbf{F}) = \mathbf{0}$, then Stokes's Theorem gives

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_S\!\!\int (\mathbf{curl\,F}) \cdot \mathbf{N}\, dS = 0$$

The circulation is zero.

26. (a) $\displaystyle\int_C f\nabla g \cdot d\mathbf{r} = \int_S\!\!\int \mathbf{curl}[f\nabla g] \cdot \mathbf{N}\, dS$ (Stokes's Theorem)

$$f\nabla g = f\frac{\partial g}{\partial x}\mathbf{i} + f\frac{\partial g}{\partial y}\mathbf{j} + f\frac{\partial g}{\partial z}\mathbf{k}$$

$$\mathbf{curl}\,(f\nabla g) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ f(\partial g/\partial x) & f(\partial g/\partial y) & f(\partial g/\partial z) \end{vmatrix}$$

$$= \left[\left[f\left(\frac{\partial^2 g}{\partial y \partial z}\right) + \left(\frac{\partial f}{\partial y}\right)\left(\frac{\partial g}{\partial z}\right)\right] - \left[f\left(\frac{\partial^2 g}{\partial z \partial y}\right) + \left(\frac{\partial f}{\partial z}\right)\left(\frac{\partial g}{\partial y}\right)\right]\right]\mathbf{i}$$

$$- \left[\left[f\left(\frac{\partial^2 g}{\partial x \partial z}\right) + \left(\frac{\partial f}{\partial x}\right)\left(\frac{\partial g}{\partial z}\right)\right] - \left[f\left(\frac{\partial^2 g}{\partial z \partial x}\right) + \left(\frac{\partial f}{\partial z}\right)\left(\frac{\partial g}{\partial x}\right)\right]\right]\mathbf{j}$$

$$+ \left[\left[f\left(\frac{\partial^2 g}{\partial x \partial y}\right) + \left(\frac{\partial f}{\partial x}\right)\left(\frac{\partial g}{\partial y}\right)\right] - \left[f\left(\frac{\partial^2 g}{\partial y \partial x}\right) + \left(\frac{\partial f}{\partial y}\right)\left(\frac{\partial g}{\partial x}\right)\right]\right]\mathbf{k}$$

$$= \left[\left(\frac{\partial f}{\partial y}\right)\left(\frac{\partial g}{\partial z}\right) - \left(\frac{\partial f}{\partial z}\right)\left(\frac{\partial g}{\partial y}\right)\right]\mathbf{i} - \left[\left(\frac{\partial f}{\partial x}\right)\left(\frac{\partial g}{\partial z}\right) - \left(\frac{\partial f}{\partial z}\right)\left(\frac{\partial g}{\partial x}\right)\right]\mathbf{j} + \left[\left(\frac{\partial f}{\partial x}\right)\left(\frac{\partial g}{\partial y}\right) - \left(\frac{\partial f}{\partial y}\right)\left(\frac{\partial g}{\partial x}\right)\right]\mathbf{k}$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} & \dfrac{\partial f}{\partial z} \\ \dfrac{\partial g}{\partial x} & \dfrac{\partial g}{\partial y} & \dfrac{\partial g}{\partial z} \end{vmatrix} = \nabla f \times \nabla g$$

Therefore, $\displaystyle\int_C f\nabla g \cdot d\mathbf{r} = \int_S\!\!\int \mathbf{curl}[f\nabla g] \cdot \mathbf{N}\, dS = \int_S\!\!\int [\nabla f \times \nabla g] \cdot \mathbf{N}\, dS$.

(b) $\displaystyle\int_C (f\nabla f) \cdot d\mathbf{r} = \int_S\!\!\int (\nabla f \times \nabla f) \cdot \mathbf{N}\, dS$ (using part a)

$= 0$ since $\nabla f \times \nabla f = \mathbf{0}$.

—CONTINUED—

26. —CONTINUED—

(c) $$\int_C (f\nabla g + g\nabla f)\cdot d\mathbf{r} = \int_C (f\nabla g)\cdot d\mathbf{r} + \int_C (g\nabla f)\cdot d\mathbf{r}$$

$$= \int_S\int (\nabla f \times \nabla g)\cdot \mathbf{N}\,dS + \int_S\int (\nabla g \times \nabla f)\cdot \mathbf{N}\,dS \quad \text{(using part a)}$$

$$= \int_S\int (\nabla f \times \nabla g)\cdot \mathbf{N}\,dS + \int_S\int -(\nabla f \times \nabla g)\cdot \mathbf{N}\,dS = 0$$

27. $f(x, y, z) = xyz,\ g(x, y, z) = z,\ S\colon z = \sqrt{4 - x^2 - y^2}$

(a) $\nabla g(x, y, z) = \mathbf{k}$

$f(x, y, z)\nabla g(x, y, z) = xyz\mathbf{k}$

$\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j} + 0\mathbf{k},\ 0 \le t \le 2\pi$

$$\int_C [f(x, y, z)\nabla g(x, y, z)]\cdot d\mathbf{r} = 0$$

(b) $\nabla f(x, y, z) = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$

$\nabla g(x, y, z) = \mathbf{k}$

$$\nabla f \times \nabla g = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ yz & xz & xy \\ 0 & 0 & 1 \end{vmatrix} = xz\mathbf{i} - yz\mathbf{j}$$

$$\mathbf{N} = \frac{x}{\sqrt{4 - x^2 - y^2}}\mathbf{i} + \frac{y}{\sqrt{4 - x^2 - y^2}}\mathbf{j} + \mathbf{k}$$

$$dS = \sqrt{1 + \left(\frac{-x}{\sqrt{4 - x^2 - y^2}}\right)^2 + \left(\frac{-y}{\sqrt{4 - x^2 - y^2}}\right)^2}\,dA = \frac{2}{\sqrt{4 - x^2 - y^2}}\,dA$$

$$\int_S\int [\nabla f(x, y, z) \times \nabla g(x, y, z)]\cdot \mathbf{N}\,dS = \int_S\int \left[\frac{x^2 z}{\sqrt{4 - x^2 - y^2}} - \frac{y^2 z}{\sqrt{4 - x^2 - y^2}}\right]\frac{2}{\sqrt{4 - x^2 - y^2}}dA$$

$$= \int_S\int \frac{2(x^2 - y^2)}{\sqrt{4 - x^2 - y^2}}\,dA$$

$$= \int_0^2\int_0^{2\pi} \frac{2r^2(\cos^2\theta - \sin^2\theta)}{\sqrt{4 - r^2}}r\,d\theta\,dr$$

$$= \int_0^2 \left[\frac{2r^3}{\sqrt{4 - r^2}}\left(\frac{1}{2}\sin 2\theta\right)\right]_0^{2\pi} dr = 0$$

28. Let $\mathbf{C} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$, then

$$\frac{1}{2}\int_C (\mathbf{C} \times \mathbf{r})\cdot d\mathbf{r} = \frac{1}{2}\int_S\int \mathbf{curl}\,(\mathbf{C} \times \mathbf{r})\cdot \mathbf{N}\,dS = \frac{1}{2}\int_S\int 2\mathbf{C}\cdot \mathbf{N}\,dS = \int_S\int \mathbf{C}\cdot \mathbf{N}\,dS$$

since

$$\mathbf{C} \times \mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a & b & c \\ x & y & z \end{vmatrix} = (bz - cy)\mathbf{i} - (az - cx)\mathbf{j} + (ay - bx)\mathbf{k}$$

and

$$\mathbf{curl}(\mathbf{C} \times \mathbf{r}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ bz - cy & cx - az & ay - bx \end{vmatrix} = 2(a\mathbf{i} + b\mathbf{j} + c\mathbf{k}) = 2\mathbf{C}.$$

29. Let S be the upper portion of the ellipsoid

$$x^2 + 4y^2 + z^2 = 4, z \ge 0$$

Let C: $\mathbf{r}(t) = \langle 2\cos t, \sin t, 0\rangle$, $0 \le t \le 2\pi$, be the boundary of S.

If $\mathbf{F} = \langle M, N, P\rangle$ exists, then

$$0 = \int_S\int (\mathbf{curl\ F}) \cdot \mathbf{N}\, dS \qquad \text{(by (i))}$$

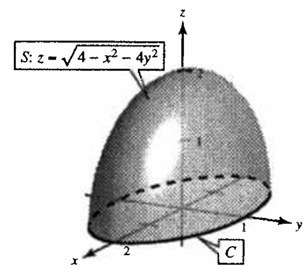
$$= \int_C \mathbf{F} \cdot d\mathbf{r} \qquad \text{(Stokes's Theorem)}$$

$$= \int_C \mathbf{G} \cdot d\mathbf{r} \qquad \text{(by (iii))}$$

$$= \int_0^{2\pi} \left\langle \frac{-\sin t}{4}, \frac{2\cos t}{4}, 0 \right\rangle \cdot \langle -2\sin t, \cos t, 0\rangle\, dt$$

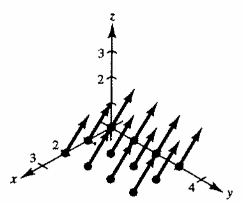
$$= \frac{1}{4}\int_0^{2\pi} (2\sin^2 t + 2\cos^2 t)\, dt = \pi$$

Hence, there is no such $\mathbf{F}$.

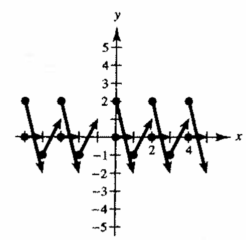


Review Exercises for Chapter 15

1. $\mathbf{F}(x, y, z) = x\mathbf{i} + \mathbf{j} + 2\mathbf{k}$



2. $\mathbf{F}(x, y) = \mathbf{i} - 2y\mathbf{j}$



3. $f(x, y, z) = 8x^2 + xy + z^2$

$\mathbf{F}(x, y, z) = (16x + y)\mathbf{i} + x\mathbf{j} + 2z\mathbf{k}$

4. $f(x, y, z) = x^2e^{yz}$

$\mathbf{F}(x, y, z) = 2xe^{yz}\,\mathbf{i} + x^2ze^{yz}\mathbf{j} + x^2ye^{yz}\,\mathbf{k}$

$= xe^{yz}(2\mathbf{i} + xz\mathbf{j} + xy\mathbf{k})$

5. Since $\partial M/\partial y = -1/y^2 \neq \partial N/\partial x$, $\mathbf{F}$ is not conservative.

6. Since $\partial M/\partial y = -1/x^2 = \partial N/\partial x$, $\mathbf{F}$ is conservative. From $M = \partial U/\partial x = -y/x^2$ and $N = \partial U/\partial y = 1/x$, partial integration yields $U = (y/x) + h(y)$ and $U = (y/x) + g(x)$ which suggests that $U(x, y) = (y/x) + C$.

7. Since $\partial M/\partial y = 12xy = \partial N/\partial x$, $\mathbf{F}$ is conservative. From $M = \partial U/\partial x = 6xy^2 - 3x^2$ and $N = \partial U/\partial y = 6x^2y + 3y^2 - 7$, partial integration yields $U = 3x^2y^2 - x^3 + h(y)$ and $U = 3x^2y^2 + y^3 - 7y + g(x)$ which suggests $h(y) = y^3 - 7y$, $g(x) = -x^3$, and $U(x, y) = 3x^2y^2 - x^3 + y^3 - 7y + C$.

8. Since $\partial M/\partial y = -6y^2 \sin 2x = \partial N/\partial x$, $\mathbf{F}$ is conservative. From $M = \partial U/\partial x = -2y^3 \sin 2x$ and $N = \partial U/\partial y = 3y^2(1 + \cos 2x)$, we obtain $U = y^3 \cos 2x + h(y)$ and $U = y^3(1 + \cos 2x) + g(x)$ which suggests that $h(y) = y^3$, $g(x) = C$, and $U(x, y) = y^3(1 + \cos 2x) + C$.

9. Since

$$\frac{\partial M}{\partial y} = 4x = \frac{\partial N}{\partial x},$$

$$\frac{\partial M}{\partial z} = 1 \neq \frac{\partial P}{\partial x}.$$

F is not conservative.

10. Since

$$\frac{\partial M}{\partial y} = 4x = \frac{\partial N}{\partial x},$$

$$\frac{\partial M}{\partial z} = 2z = \frac{\partial P}{\partial x},$$

$$\frac{\partial N}{\partial z} = 6y \neq \frac{\partial P}{\partial y}$$

F is not conservative.

11. Since

$$\frac{\partial M}{\partial y} = \frac{-1}{y^2z} = \frac{\partial N}{\partial x}, \quad \frac{\partial M}{\partial z} = \frac{-1}{yz^2} = \frac{\partial P}{\partial x}, \quad \frac{\partial N}{\partial z} = \frac{x}{y^2z^2} = \frac{\partial P}{\partial y},$$

F is conservative. From

$$M = \frac{\partial U}{\partial x} = \frac{1}{yz}, \quad N = \frac{\partial U}{\partial y} = \frac{-x}{y^2z}, \quad P = \frac{\partial U}{\partial z} = \frac{-x}{yz^2}$$

we obtain

$$U = \frac{x}{yz} + f(y, z), \quad U = \frac{x}{yz} + g(x, z), \quad U = \frac{x}{yz} + h(x, y) \implies f(x, y, z) = \frac{x}{yz} + K.$$

12. Since

$$\frac{\partial M}{\partial y} = \sin z = \frac{\partial N}{\partial x}, \quad \frac{\partial M}{\partial z} = y\cos z \neq \frac{\partial P}{\partial x},$$

F is not conservative.

13. Since $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$:

(a) div $\mathbf{F} = 2x + 2y + 2z$

(b) $\mathbf{curl\,F} = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\mathbf{i} - \left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z}\right)\mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k} = 0\mathbf{i} - 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}$

14. Since $\mathbf{F} = xy^2\mathbf{j} - zx^2\mathbf{k}$;

(a) div $\mathbf{F} = 2xy - x^2$

(b) $\mathbf{curl\,F} = 2xz\mathbf{j} + y^2\mathbf{k}$

15. Since $\mathbf{F} = (\cos y + y\cos x)\mathbf{i} + (\sin x - x\sin y)\mathbf{j} + xyz\mathbf{k}$:

(a) div $\mathbf{F} = -y\sin x - x\cos y + xy$

(b) $\mathbf{curl\,F} = xz\mathbf{i} - yz\mathbf{j} + (\cos x - \sin y + \sin y - \cos x)\mathbf{k} = xz\mathbf{i} - yz\mathbf{j}$

16. Since $\mathbf{F} = (3x - y)\mathbf{i} + (y - 2z)\mathbf{j} + (z - 3x)\mathbf{k}$:

(a) div $\mathbf{F} = 3 + 1 + 1 = 5$

(b) $\mathbf{curl\,F} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$

17. Since $\mathbf{F} = \arcsin x\mathbf{i} + xy^2\mathbf{j} + yz^2\mathbf{k}$:

(a) div $\mathbf{F} = \dfrac{1}{\sqrt{1 - x^2}} + 2xy + 2yz$

(b) $\mathbf{curl\,F} = z^2\mathbf{i} + y^2\mathbf{k}$

18. Since $\mathbf{F} = (x^2 - y)\mathbf{i} - (x + \sin^2 y)\mathbf{j}$:

(a) div $\mathbf{F} = 2x - 2\sin y\cos y$

(b) $\mathbf{curl\,F} = \mathbf{0}$

19. Since $\mathbf{F} = \ln(x^2+y^2)\mathbf{i} + \ln(x^2+y^2)\mathbf{j} + z\mathbf{k}$:

(a) $\text{div}\,\mathbf{F} = \dfrac{2x}{x^2+y^2} + \dfrac{2y}{x^2+y^2} + 1$

$= \dfrac{2x+2y}{x^2+y^2} + 1$

(b) $\mathbf{curl}\,\mathbf{F} = \dfrac{2x-2y}{x^2+y^2}\mathbf{k}$

20. Since $\mathbf{F} = \dfrac{z}{x}\mathbf{i} + \dfrac{z}{y}\mathbf{j} + z^2\mathbf{k}$:

(a) $\text{div}\,\mathbf{F} = -\dfrac{z}{x^2} - \dfrac{z}{y^2} + 2z = z\left(2 - \dfrac{1}{x^2} - \dfrac{1}{y^2}\right)$

(b) $\mathbf{curl}\,\mathbf{F} = -\dfrac{1}{y}\mathbf{i} + \dfrac{1}{x}\mathbf{j}$

21. (a) Let $x = t$, $y = t$, $-1 \le t \le 2$, then $ds = \sqrt{2}\,dt$.

$$\int_C (x^2+y^2)\,ds = \int_{-1}^{2} 2t^2\sqrt{2}\,dt = \left[2\sqrt{2}\left(\frac{t^3}{3}\right)\right]_{-1}^{2} = 6\sqrt{2}$$

(b) Let $x = 4\cos t$, $y = 4\sin t$, $0 \le t \le 2\pi$, then $ds = 4\,dt$.

$$\int_C (x^2+y^2)\,ds = \int_0^{2\pi} 16(4\,dt) = 128\pi$$

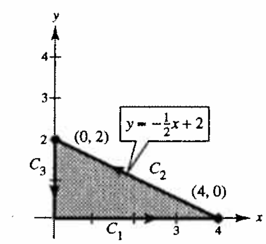
22. (a) Let $x = 5t$, $y = 4t$, $0 \le t \le 1$, then $ds = \sqrt{41}\,dt$.

$$\int_C xy\,ds = \int_0^1 20t^2\sqrt{41}\,dt = \frac{20\sqrt{41}}{3}$$

(b) C_1: $x = t$, $y = 0$, $0 \le t \le 4$, $ds = dt$

C_2: $x = 4 - 4t$, $y = 2t$, $0 \le t \le 1$, $ds = 2\sqrt{5}\,dt$

C_3: $x = 0$, $y = 2 - t$, $0 \le t \le 2$, $ds = dt$



Therefore, $\displaystyle\int_C xy\,ds = \int_0^4 0\,dt + \int_0^1 (8t - 8t^2)2\sqrt{5}\,dt + \int_0^2 0\,dt$

$$= 16\sqrt{5}\left[\frac{t^2}{2} - \frac{t^3}{3}\right]_0^1 = \frac{8\sqrt{5}}{3}.$$

23. $x = \cos t + t\sin t$, $y = \sin t - t\cos t$, $0 \le t \le 2\pi$, $\dfrac{dx}{dt} = t\cos t$, $\dfrac{dy}{dt} = t\sin t$

$$\int_C (x^2+y^2)\,ds = \int_0^{2\pi} [(\cos t + t\sin t)^2 + (\sin t - t\cos t)^2]\sqrt{t^2\cos^2 t + t^2\sin^2 t}\,dt = \int_0^{2\pi} [t^3 + t]\,dt$$

$$= 2\pi^2(1 + 2\pi^2)$$

24. $x = t - \sin t$, $y = 1 - \cos t$, $0 \le t \le 2\pi$, $\dfrac{dx}{dt} = 1 - \cos t$, $\dfrac{dy}{dt} = \sin t$

$$\int_C x\,ds = \int_0^{2\pi} (t - \sin t)\sqrt{(1-\cos t)^2 + (\sin t)^2}\,dt = \int_0^{2\pi} (t - \sin t)\sqrt{2 - 2\cos t}\,dt$$

$$= \sqrt{2}\int_0^{2\pi}\left[t\sqrt{1-\cos t} - \sin t\sqrt{1-\cos t}\right]dt = \sqrt{2}\left[-\frac{2}{3}(1-\cos t)^{3/2}\right]_0^{2\pi} + \sqrt{2}\int_0^{2\pi} t\sqrt{1-\cos t}\,dt$$

$$= \sqrt{2}\int_0^{2\pi} t\sqrt{1-\cos t}\,dt$$

$$= 8\pi$$

25. (a) Let $x = 2t, y = -3t, 0 \le t \le 1.$

$$\int_C (2x - y)\,dx + (x + 3y)\,dy = \int_0^1 [7t(2) + (-7t)(-3)]\,dt = \int_0^1 35t\,dt = \frac{35}{2}$$

(b) $x = 3\cos t, y = 3\sin t, dx = -3\sin t\,dt, dy = 3\cos t\,dt, 0 \le t \le 2\pi$

$$\int_C (2x - y)\,dx + (x + 3y)\,dy = \int_0^{2\pi} (9 + 9\sin t\cos t)\,dt = 18\pi$$

26. $x = \cos t + t\sin t, y = \sin t - t\sin t, 0 \le t \le \frac{\pi}{2}, dx = t\cos t\,dt, dy = (\cos t - t\cos t - \sin t)\,dt$

$$\int_C (2x - y)\,dx + (x + 3y)\,dy = \int_0^{\pi/2} [\sin t\cos t(5t^2 - 6t + 2) + \cos^2 t(t + 1) + \sin^2 t(2t - 3)]\,dt \approx 1.01$$

27. $\int_C (2x + y)\,ds, \mathbf{r}(t) = a\cos^3 t\,\mathbf{i} + a\sin^3 t\,\mathbf{j}, 0 \le t \le \frac{\pi}{2}$

$x'(t) = -3a \cdot \cos^2 t\sin t$

$y'(t) = 3a \cdot \sin^2 t\cos t$

$$\int_C (2x + y)\,ds = \int_0^{\pi/2} (2(a \cdot \cos^3 t) + a \cdot \sin^3 t)\sqrt{x'(t)^2 + y'(t)^2}\,dt = \frac{9a^2}{5}$$

28. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^{3/2}\mathbf{k}, 0 \le t \le 4$

$x'(t) = 1,\ y'(t) = 2t,\ z'(t) = \frac{3}{2}t^{1/2}$

$$\int_C (x^2 + y^2 + z^2)\,ds = \int_0^4 (t^2 + t^4 + t^3)\sqrt{1 + 4t^2 + \frac{9}{4}t}\,dt \approx 2080.59$$

29. $f(x, y) = 5 + \sin(x + y)$

C: $y = 3x$ from $(0, 0)$ to $(2, 6)$

$\mathbf{r}(t) = t\mathbf{i} + 3t\mathbf{j}, 0 \le t \le 2$

$\mathbf{r}'(t) = \mathbf{i} + 3\mathbf{j}$

$\|\mathbf{r}'(t)\| = \sqrt{10}$

Lateral surface area:

$$\int_C f(x, y)\,ds = \int_0^2 [5 + \sin(t + 3t)]\sqrt{10}\,dt = \sqrt{10}\int_0^2 (5 + \sin 4t)\,dt = \frac{\sqrt{10}}{4}(41 - \cos 8) \approx 32.528$$

30. $f(x, y) = 12 - x - y$

C: $y = x^2$ from $(0, 0)$ to $(2, 4)$

$\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}, 0 \le t \le 2$

$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j}$

$\|\mathbf{r}'(t)\| = \sqrt{1 + 4t^2}$

Lateral surface area:

$$\int_C f(x, y)\,ds = \int_0^2 (12 - t - t^2)\sqrt{1 + 4t^2}\,dt \approx 41.532$$

31. $d\mathbf{r} = (2t\mathbf{i} + 3t^2\mathbf{j})\,dt$

$\mathbf{F} = t^5\mathbf{i} + t^4\mathbf{j}, 0 \le t \le 1$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 5t^6\,dt = \frac{5}{7}$$

32. $d\mathbf{r} = [(-4\sin t)\mathbf{i} + 3\cos t\,\mathbf{j}]\,dt$

$\mathbf{F} = (4\cos t - 3\sin t)\mathbf{i} + (4\cos t + 3\sin t)\mathbf{j},\ 0 \le t \le 2\pi$

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} (12 - 7\sin t\cos t)\,dt = \left[12t - \frac{7\sin^2 t}{2}\right]_0^{2\pi} = 24\pi$$

33. $d\mathbf{r} = [(-2\sin t)\mathbf{i} + (2\cos t)\mathbf{j} + \mathbf{k}]\,dt$

$\mathbf{F} = (2\cos t)\mathbf{i} + (2\sin t)\mathbf{j} + t\mathbf{k},\ 0 \le t \le 2\pi$

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} t\,dt = 2\pi^2$$

34. $x = 2 - t,\ y = 2 - t,\ z = \sqrt{4t - t^2},\ 0 \le t \le 2$

$$d\mathbf{r} = \left[-\mathbf{i} - \mathbf{j} + \frac{2 - t}{\sqrt{4t - t^2}}\mathbf{k}\right]dt$$

$$\mathbf{F} = \left(4 - 2t - \sqrt{4t - t^2}\right)\mathbf{i} + \left(\sqrt{4t - t^2} - 2 + t\right)\mathbf{j} + 0\mathbf{k}$$

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^2 (t - 2)\,dt = \left[\frac{t^2}{2} - 2t\right]_0^2 = -2$$

35. Let $x = t,\ y = -t,\ z = 2t^2,\ -2 \le t \le 2,\ d\mathbf{r} = [\mathbf{i} - \mathbf{j} + 4t\mathbf{k}]\,dt$.

$\mathbf{F} = (-t - 2t^2)\mathbf{i} + (2t^2 - t)\mathbf{j} + (2t)\mathbf{k}$

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_{-2}^{2} 4t^2\,dt = \left[\frac{4t^3}{3}\right]_{-2}^{2} = \frac{64}{3}$$

36. Let $x = 2\sin t,\ y = -2\cos t,\ z = 4\sin^2 t,\ 0 \le t \le \pi$.

$d\mathbf{r} = [(2\cos t)\mathbf{i} + (2\sin t)\mathbf{j} + (8\sin t\cos t)\mathbf{k}]\,dt$

$\mathbf{F} = 0\mathbf{i} + 4\mathbf{j} + (2\sin t)\mathbf{k}$

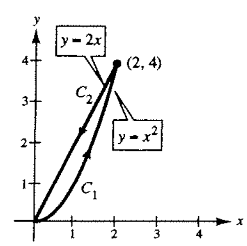
$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{\pi} (8\sin t + 16\sin^2 t\cos t)\,dt = \left[-8\cos t + \frac{16}{3}\sin^3 t\right]_0^{\pi} = 16$$

37. For $y = x^2$, $\mathbf{r}_1(t) = t\mathbf{i} + t^2\mathbf{j},\ 0 \le t \le 2$

For $y = 2x$, $\mathbf{r}_2(t) = (2 - t)\mathbf{i} + (4 - 2t)\mathbf{j},\ 0 \le t \le 2$

$$\int_C xy\,dx + (x^2 + y^2)\,dy = \int_{C_1} xy\,dx + (x^2 + y^2)\,dy + \int_{C_2} xy\,dx + (x^2 + y^2)\,dy$$

$$= \frac{100}{3} + (-32) = \frac{4}{3}$$



38. $$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C (2x - y)\,dx + (2y - x)\,dy$$

$\mathbf{r}(t) = (2\cos t + 2t\sin t)\mathbf{i} + (2\sin t - 2t\cos t)\mathbf{j},\ 0 \le t \le \pi$

$$\int_C \mathbf{F}\cdot d\mathbf{r} = 4\pi^2 + 4\pi$$

39. $\mathbf{F} = x\mathbf{i} - \sqrt{y}\mathbf{j}$ is conservative.

$$\text{Work} = \left[\frac{1}{2}x^2 - \frac{2}{3}y^{3/2}\right]_{(0,0)}^{(4,8)} = \frac{1}{2}(16) - \left(\frac{2}{3}\right)8^{3/2} = \frac{8}{3}\left(3 - 4\sqrt{2}\right)$$

40. $\mathbf{r}(t) = 10\sin t\,\mathbf{i} + 10\cos t\,\mathbf{j} + \dfrac{2000/5280}{\pi/2}t\mathbf{k},\ 0 \le t \le \dfrac{\pi}{2}$

$= 10\sin t\,\mathbf{i} + 10\cos t\,\mathbf{j} + \dfrac{25}{33\pi}t\mathbf{k}$

$\mathbf{F} = 20\mathbf{k}$

$d\mathbf{r} = \left(10\cos t\,\mathbf{i} - 10\sin t\,\mathbf{j} + \dfrac{25}{33\pi}\mathbf{k}\right)$

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{\pi/2} \frac{500}{33\pi}\,dt = \frac{250}{33}\text{ mi}\cdot\text{ton}$$

41. $$\int_C 2xyz\,dx + x^2z\,dy + x^2y\,dz = \Big[x^2yz\Big]_{(0,0,0)}^{(1,4,3)} = 12$$

42. $$\int_C y\,dx + x\,dy + \frac{1}{z}\,dz = \Big[xy + \ln|z|\Big]_{(0,0,1)}^{(4,4,4)} = 16 + \ln 4$$

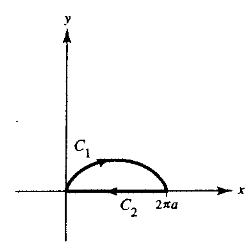
43. (a) $$\int_C y^2\,dx + 2xy\,dy = \int_0^1 [(1+t)^2(3) + 2(1+3t)(1+t)]\,dt$$
$$= \int_0^1 3(t^2 + 2t + 1) + 2(3t^2 + 4t + 1)]\,dt$$
$$= \int_0^1 (9t^2 + 14t + 5)\,dt$$
$$= \Big[3t^3 + 7t^2 + 5t\Big]_0^1 = 15$$

(b) $$\int_C y^2\,dx + 2xy\,dy = \int_1^4 \left[t(1) + 2(t)(\sqrt{t})\frac{1}{2\sqrt{t}}\right]dt$$
$$= \int_1^4 (t + t)\,dt$$
$$= \Big[t^2\Big]_1^4 = 15$$

(c) $\mathbf{F}(x, y) = y^2\mathbf{i} + 2xy\,\mathbf{j} = \nabla f$ where $f(x, y) = xy^2$.

Hence,
$$\int_C \mathbf{F}\cdot d\mathbf{r} = 4(2)^2 - 1(1)^2 = 15$$

44. $x = a(\theta - \sin\theta),\ y = a(1 - \cos\theta),\ 0 \le \theta \le 2\pi$



(a) $A = \dfrac{1}{2}\displaystyle\int_C x\,dy - y\,dx.$

Since these equations orient the curve backwards, we will use

$$A = \frac{1}{2}\int (y\,dx - x\,dy)$$
$$= \frac{1}{2}\int_0^{2\pi} [a^2(1-\cos\theta)(1-\cos\theta) - a^2(\theta - \sin\theta)(\sin\theta)]\,d\theta + \frac{1}{2}\int_0^{2\pi a} (0 - 0)\,d\theta$$
$$= \frac{a^2}{2}\int_0^{2\pi} [1 - 2\cos\theta + \cos^2\theta - \theta\sin\theta + \sin^2\theta]\,d\theta$$
$$= \frac{a^2}{2}\int_0^{2\pi} (2 - 2\cos\theta - \theta\sin\theta)\,d\theta = \frac{a^2}{2}(6\pi) = 3\pi a^2.$$

(b) By symmetry, $\bar{x} = \pi a$. From Section 15.4,

$$\bar{y} = -\frac{1}{2A}\int_C y^2\,dx = \frac{1}{2A}\int_0^{2\pi} a^3(1-\cos\theta)^2(1-\cos\theta)\,d\theta$$
$$= \frac{1}{2(3\pi a^2)}a^3(5\pi) = \frac{5}{6}a$$

45. $\displaystyle\int_C y\,dx + 2x\,dy = \int_0^2\int_0^2 (2-1)\,dy\,dx = \int_0^2 2\,dx = 4$

46. $\displaystyle\int_C xy\,dx + (x^2+y^2)\,dy = \int_0^2\int_0^2 (2x - x)\,dy\,dx$

$\displaystyle = \int_0^2 2x\,dx = 4$

47. $\displaystyle\int_C xy^2\,dx + x^2y\,dy = \iint_R (2xy - 2xy)\,dA = 0$

48. $\displaystyle\int_C (x^2-y^2)\,dx + 2xy\,dy = \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} 4y\,dy\,dx$

$\displaystyle = \int_{-a}^{a} 0\,dx = 0$

49. $\displaystyle\int_C xy\,dx + x^2\,dy = \int_0^1\int_{x^2}^{x} x\,dy\,dx = \int_0^1 (x^2 - x^3)\,dx = \frac{1}{12}$

50. $\displaystyle\int_C y^2\,dx + x^{4/3}\,dy = \int_{-1}^{1}\int_{-(1-x^{2/3})^{3/2}}^{(1-x^{2/3})^{3/2}} \left(\frac{4}{3}x^{1/3} - 2y\right) dy\,dx$

$\displaystyle = \int_{-1}^{1}\left[\frac{4}{3}x^{1/3}y - y^2\right]_{-(1-x^{2/3})^{3/2}}^{(1-x^{2/3})^{3/2}} dx$

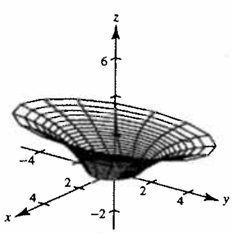
$\displaystyle = \int_{-1}^{1}\frac{8}{3}x^{1/3}(1-x^{2/3})^{3/2}\,dx$

$\displaystyle = \left[-\frac{8}{7}x^{2/3}(1-x^{2/3})^{5/2} - \frac{16}{35}(1-x^{2/3})^{5/2}\right]_{-1}^{1}$

$= 0$

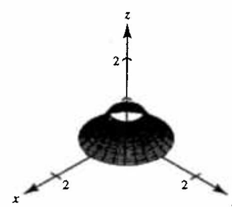
51. $\mathbf{r}(u, v) = \sec u \cos v\,\mathbf{i} + (1 + 2\tan u)\sin v\,\mathbf{j} + 2u\mathbf{k}$

$0 \le u \le \dfrac{\pi}{3}, \quad 0 \le v \le 2\pi$

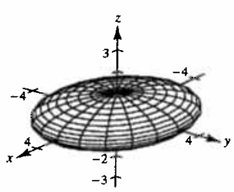


52. $\mathbf{r}(u, v) = e^{-u/4}\cos v\,\mathbf{i} + e^{-u/4}\sin v\,\mathbf{j} + \dfrac{u}{6}\mathbf{k}$

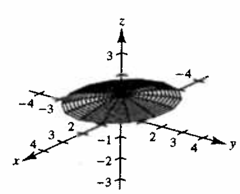
$0 \le u \le 4, \quad 0 \le v \le 2\pi$



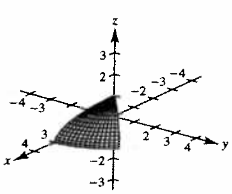
53. (a)



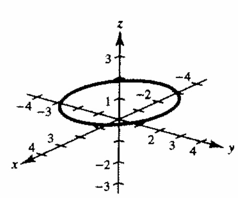
(b)



(c)



(d)



The space curve is a circle:

$\mathbf{r}\left(u, \dfrac{\pi}{4}\right) = \dfrac{3\sqrt{2}}{2}\cos u\,\mathbf{i} + \dfrac{3\sqrt{2}}{2}\sin u\,\mathbf{j} + \dfrac{\sqrt{2}}{2}\mathbf{k}$

—CONTINUED—

53. —CONTINUED—

(e) $\mathbf{r}_u = -3\cos v \sin u\,\mathbf{i} + 3\cos v \cos u\,\mathbf{j}$

$\mathbf{r}_v = -3\sin v \cos u\,\mathbf{i} - 3\sin v \sin u\,\mathbf{j} + \cos v\,\mathbf{k}$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3\cos v \sin u & 3\cos v \cos u & 0 \\ -3\sin v \cos u & -3\sin v \sin u & \cos v \end{vmatrix}$$

$$= (3\cos^2 v \cos u)\mathbf{i} + (3\cos^2 v \sin u)\mathbf{j} + (9\cos v \sin v \sin^2 u + 9\cos v \sin v \cos^2 u)\mathbf{k}$$

$$= (3\cos^2 v \cos u)\mathbf{i} + (3\cos^2 v \sin u)\mathbf{j} + (9\cos v \sin v)\mathbf{k}$$

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = \sqrt{9\cos^4 v \cos^2 u + 9\cos^4 v \sin^2 u + 81\cos^2 v \sin^2 v}$$

$$= \sqrt{9\cos^4 v + 81\cos^2 v \sin^2 v}$$

Using a Symbolic integration utility,

$$\int_{\pi/4}^{\pi/2}\int_0^{2\pi} \|\mathbf{r}_u \times \mathbf{r}_v\|\,du\,dv \approx 14.44.$$

(f) Similarly,

$$\int_0^{\pi/4}\int_0^{\pi/2} \|\mathbf{r}_u \times \mathbf{r}_v\|\,dv\,du \approx 4.27.$$

54. S: $\mathbf{r}(u, v) = (u + v)\mathbf{i} + (u - v)\mathbf{j} + \sin v\mathbf{k}, \quad 0 \le u \le 2, 0 \le v \le \pi$

$\mathbf{r}_u(u, v) = \mathbf{i} + \mathbf{j}$

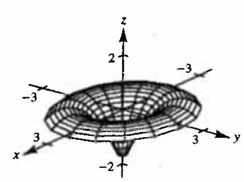
$\mathbf{r}_v(u, v) = \mathbf{i} - \mathbf{j} + \cos v\mathbf{k}$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 1 & -1 & \cos v \end{vmatrix} = \cos v\mathbf{i} - \cos v\mathbf{j} - 2\mathbf{k}$$

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = \sqrt{2\cos^2 v + 4}$$

$$\iint_S z\,dS = \int_0^{\pi}\int_0^2 \sin v\sqrt{2\cos^2 v + 4}\,du\,dv = 2\left[\sqrt{6} + \sqrt{2}\ln\left(\frac{\sqrt{6} + \sqrt{2}}{\sqrt{6} - \sqrt{2}}\right)\right]$$

55. S: $\mathbf{r}(u, v) = u\cos v\mathbf{i} + u\sin v\mathbf{j} + (u - 1)(2 - u)\mathbf{k}, \ 0 \le u \le 2, 0 \le v \le 2\pi$



$\mathbf{r}_u(u, v) = \cos v\mathbf{i} + \sin v\mathbf{j} + (3 - 2u)\mathbf{k}$

$\mathbf{r}_v(u, v) = -u\sin v\mathbf{i} + u\cos v\mathbf{j}$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos v & \sin v & 3 - 2u \\ -u\sin v & u\cos v & 0 \end{vmatrix} = (2u - 3)u\cos v\mathbf{i} + (2u - 3)u\sin v\mathbf{j} + u\mathbf{k}$$

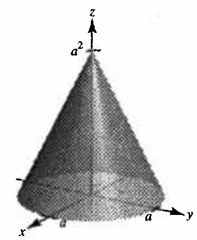
$$\|\mathbf{r}_u \times \mathbf{r}_v\| = u\sqrt{(2u - 3)^2 + 1}$$

$$\iint_S (x + y)\,dS = \int_0^{2\pi}\int_0^2 (u\cos v + u\sin v)\,u\sqrt{(2u - 3)^2 + 1}\,du\,dv$$

$$= \int_0^2\int_0^{2\pi} (\cos v + \sin v)u^2\sqrt{(2u - 3)^2 + 1}\,dv\,du = 0$$

56. (a) $z = a\left(a - \sqrt{x^2 + y^2}\right), 0 \le z \le a^2$

$z = 0 \Longrightarrow x^2 + y^2 = a^2$



(b) S: $g(x, y) = z = a^2 - a\sqrt{x^2 + y^2}$

$\rho(x, y) = k\sqrt{x^2 + y^2}$

$$m = \iint_S e(x, y, z)\, dS$$

$$= \iint_R k\sqrt{x^2 + y^2}\sqrt{1 + {g_x}^2 + {g_y}^2}\, dA$$

$$= k\iint_R \sqrt{x^2 + y^2}\sqrt{1 + \frac{a^2x^2}{x^2 + y^2} + \frac{a^2y^2}{x^2 + y^2}}\, dA$$

$$= k\iint_R \sqrt{a^2 + 1}\left(\sqrt{x^2 + y^2}\right) dA$$

$$= k\sqrt{a^2 + 1}\int_0^{2\pi}\int_0^a r^2\, dr\, d\theta$$

$$= k\sqrt{a^2 + 1}\int_0^{2\pi}\frac{a^3}{3}\, d\theta$$

$$= \frac{2}{3}k\sqrt{a^2 + 1}\, a^3\pi$$

57. $\mathbf{F}(x, y, z) = x^2\mathbf{i} + xy\mathbf{j} + z\mathbf{k}$

Q: solid region bounded by the coordinates planes and the plane $2x + 3y + 4z = 12$

Surface Integral: There are four surfaces for this solid.

$z = 0 \quad \mathbf{N} = -\mathbf{k}, \quad \mathbf{F}\cdot\mathbf{N} = -z, \quad \iint_{S_1} 0\, dS = 0$

$y = 0, \quad \mathbf{N} = -\mathbf{j}, \quad \mathbf{F}\cdot\mathbf{N} = -xy, \quad \iint_{S_2} 0\, dS = 0$

$x = 0, \quad \mathbf{N} = -\mathbf{i}, \quad \mathbf{F}\cdot\mathbf{N} = -x^2, \quad \iint_{S_3} 0\, dS = 0$

$2x + 3y + 4z = 12, \mathbf{N} = \dfrac{2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}}{\sqrt{29}}, dS = \sqrt{1 + \left(\dfrac{1}{4}\right) + \left(\dfrac{9}{16}\right)}dA = \dfrac{\sqrt{29}}{4}\, dA$

$$\iint_{S_4} \mathbf{F}\cdot\mathbf{N}\, dS = \frac{1}{4}\iint_R (2x^2 + 3xy + 4z)\, dA$$

$$= \frac{1}{4}\int_0^6\int_0^{4-(2x/3)} (2x^2 + 3xy + 12 - 2x - 3y)\, dy\, dx$$

$$= \frac{1}{4}\int_0^6\left[2x^2\left(\frac{12 - 2x}{3}\right) + \frac{3x}{2}\left(\frac{12 - 2x}{3}\right)^2 + 12\left(\frac{12 - 2x}{3}\right) - 2x\left(\frac{12 - 2x}{3}\right) - \frac{3}{2}\left(\frac{12 - 2x}{3}\right)^2\right] dx$$

$$= \frac{1}{6}\int_0^6 (-x^3 + x^2 + 24x + 36)\, dx$$

$$= \frac{1}{6}\left[-\frac{x^4}{4} + \frac{x^3}{3} + 12x^2 + 36x\right]_0^6 = 66$$

—CONTINUED—

57. —CONTINUED—

Divergence Theorem: Since div $\mathbf{F} = 2x + x + 1 = 3x + 1$, Divergence Theorem yields

$$\iiint_Q \text{div}\,\mathbf{F}\,dV = \int_0^6\int_0^{(12-2x)/3}\int_0^{(12-2x-3y)/4} (3x+1)\,dz\,dy\,dx$$

$$= \int_0^6\int_0^{(12-2x)/3} (3x+1)\left(\frac{12-2x-3y}{4}\right)dy\,dx$$

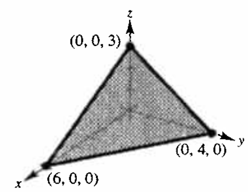
$$= \frac{1}{4}\int_0^6 (3x+1)\left[12y - 2xy - \frac{3}{2}y^2\right]_0^{(12-2x)/3} dx$$

$$= \frac{1}{4}\int_0^6 (3x+1)\left[4(12-2x) - 2x\left(\frac{12-2x}{3}\right) - \frac{3}{2}\left(\frac{12-2x}{3}\right)^2\right]dx$$

$$= \frac{1}{4}\int_0^6 \frac{2}{3}(3x^3 - 35x^2 + 96x + 36)\,dx = \frac{1}{6}\left[\frac{3x^4}{4} - \frac{35x^3}{3} + 48x^2 + 36x\right]_0^6 = 66.$$

58. $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

Q: solid region bounded by the coordinate planes and the plane $2x + 3y + 4z = 12$



Surface Integral: There are four surfaces for this solid.

$z = 0 \quad \mathbf{N} = -\mathbf{k}, \quad \mathbf{F}\cdot\mathbf{N} = -z, \quad \displaystyle\iint_{S_1} 0\,dS = 0$

$y = 0, \quad \mathbf{N} = -\mathbf{j}, \quad \mathbf{F}\cdot\mathbf{N} = -y, \quad \displaystyle\iint_{S_2} 0\,dS = 0$

$x = 0, \quad \mathbf{N} = -\mathbf{i}, \quad \mathbf{F}\cdot\mathbf{N} = -x, \quad \displaystyle\iint_{S_3} 0\,dS = 0$

$$2x + 3y + 4z = 12, \mathbf{N} = \frac{2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}}{\sqrt{29}}, dS = \sqrt{1 + \left(\frac{1}{4}\right) + \left(\frac{9}{16}\right)}dA = \frac{\sqrt{29}}{4}dA$$

$$\iint_{S_4} \mathbf{N}\cdot\mathbf{F}\,dS = \frac{1}{4}\iint_R (2x + 3y + 4z)\,dy\,dx$$

$$= \frac{1}{4}\int_0^6\int_0^{(12-2x)/3} 12\,dy\,dx = 3\int_0^6\left(4 - \frac{2x}{3}\right)dx = 3\left[4x - \frac{x^2}{3}\right]_0^6 = 36$$

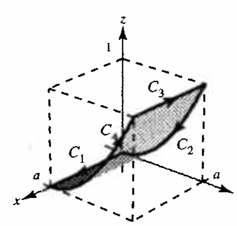
Triple Integral: Since div $\mathbf{F} = 3$, the Divergence Theorem yields.

$$\iiint_Q \text{div}\,\mathbf{F}\,dV = \iiint_Q 3\,dV = 3(\text{Volume of solid}) = 3\left[\frac{1}{3}(\text{Area of base})(\text{Height})\right] = \frac{1}{2}(6)(4)(3) = 36.$$

59. $\mathbf{F}(x, y, z) = (\cos y + y\cos x)\mathbf{i} + (\sin x - x\sin y)\mathbf{j} + xyz\mathbf{k}$

S: portion of $z = y^2$ over the square in the xy-plane with vertices $(0, 0)$, $(a, 0)$, (a, a), $(0, a)$

Line Integral: Using the line integral we have:



C_1: $y = 0, \quad dy = 0$

C_2: $x = 0, \quad dx = 0, \quad z = y^2, \quad dz = 2y\,dy$

C_3: $y = a, \quad dy = 0, \quad z = a^2, \quad dz = 0$

C_4: $x = a, \quad dx = 0, \quad z = y^2, \quad dz = 2y\,dy$

—CONTINUED—

59. —CONTINUED—

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C (\cos y + y\cos x)\,dx + (\sin x - x\sin y)\,dy + xyz\,dz$$

$$= \int_{C_1} dx + \int_{C_2} 0 + \int_{C_3} (\cos a + a\cos x)\,dx + \int_{C_4} (\sin a - a\sin y)\,dy + ay^3(2y\,dy)$$

$$= \int_0^a dx + \int_a^0 (\cos a + a\cos x)\,dx + \int_0^a (\sin a - a\sin y)\,dy + \int_0^a 2ay^4\,dy$$

$$= a + \Big[x\cos a + a\sin x\Big]_a^0 + \Big[y\sin a + a\cos y\Big]_0^a + \left[2a\frac{y^5}{5}\right]_0^a$$

$$= a - a\cos a - a\sin a + a\sin a + a\cos a - a + \frac{2a^6}{5} = \frac{2a^6}{5}$$

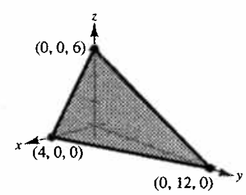
Double Integral: Considering $f(x, y, z) = z - y^2$, we have:

$$\mathbf{N} = \frac{\nabla f}{\|\nabla f\|} = \frac{-2y\mathbf{j} + \mathbf{k}}{\sqrt{1 + 4y^2}}, \quad dS = \sqrt{1 + 4y^2}\,dA, \text{ and } \mathbf{curl\ F} = xz\mathbf{i} - yz\mathbf{j}.$$

Hence,

$$\iint_S (\mathbf{curl\ F}) \cdot \mathbf{N}\,dS = \int_0^a \int_0^a 2y^2z\,dy\,dx = \int_0^a \int_0^a 2y^4\,dy\,dx = \int_0^a \frac{2a^5}{5}\,dx = \frac{2a^6}{5}.$$

60. $\mathbf{F}(x, y, z) = (x - z)\mathbf{i} + (y - z)\mathbf{j} + x^2\mathbf{k}$



S: first octant portion of the plane $3x + y + 2z = 12$

Line Integral:

C_1: $y = 0, \quad dy = 0, \quad z = \dfrac{12 - 3x}{2}, \quad dz = -\dfrac{3}{2}\,dx$

C_2: $x = 0, \quad dx = 0, \quad z = \dfrac{12 - y}{2}, \quad dz = -\dfrac{1}{2}\,dy$

C_3: $z = 0, \quad dz = 0, \quad y = 12 - 3x, \quad dy = -3\,dx$

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C (x - z)\,dx + (y - z)\,dy + x^2\,dz$$

$$= \int_{C_1} \left[x - \frac{12 - 3x}{2} + x^2\left(-\frac{3}{2}\right)\right] dx + \int_{C_2} \left[y - \frac{12 - y}{2}\right] dy + \int_{C_3} [x + (12 - 3x)(-3)]\,dx$$

$$= \int_4^0 \left(-\frac{3}{2}x^2 + \frac{5}{2}x - 6\right) dx + \int_0^{12} \left(\frac{3}{2}y - 6\right) dy + \int_0^4 (10x - 36)\,dx = 8$$

Double Integral: $G(x, y, z) = \dfrac{12 - 3x - y}{2} - z$

$$\nabla G(x, y, z) = -\frac{3}{2}\mathbf{i} - \frac{1}{2}\mathbf{j} - \mathbf{k}$$

$$\mathbf{curl\ F} = \mathbf{i} - (2x + 1)\mathbf{j}$$

$$\iint_S (\mathbf{curl\ F}) \cdot \mathbf{N}\,dS = \int_0^4 \int_0^{12-3x} (x - 1)\,dy\,dx = \int_0^4 (-3x^2 + 15x - 12)\,dx = 8$$

61. If **curl** $(\mathbf{F}) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, then div(**curl F**) $= 1 + 1 + 1 = 3$, contradicting Theorem 15.3.

Problem Solving for Chapter 15

1. (a) $\nabla T = \dfrac{-25}{(x^2 + y^2 + z^2)^{3/2}}[x\mathbf{i} + y\mathbf{i} + z\mathbf{k}]$

$\mathbf{N} = x\mathbf{i} + \sqrt{1 - x^2}\mathbf{k}$

$dS = \dfrac{1}{\sqrt{1 - x^2}}\,dA$

$$\text{Flux} = \int_S\int -k\nabla T \cdot \mathbf{N}\, dS$$

$$= 25k\int_R\int\left[\frac{x^2}{(x^2 + y^2 + z^2)^{3/2}(1 - x^2)^{1/2}} + \frac{z}{(x^2 + y^2 + z^2)^{3/2}}\right] dA$$

$$= 25k\int_{-1/2}^{1/2}\int_0^1\left[\frac{x^2}{(x^2 + y^2 + z^2)^{3/2}(1 - x^2)^{1/2}} + \frac{1 - x^2}{(x^2 + y^2 + z^2)^{3/2}(1 - x^2)^{1/2}}\right] dy\, dx$$

$$= 25k\int_{-1/2}^{1/2}\int_0^1 \frac{1}{(1 + y^2)^{3/2}(1 - x^2)^{1/2}}\, dy\, dx$$

$$= 25k\int_0^1 \frac{1}{(1 + y^2)^{3/2}}\, dy \int_{-1/2}^{1/2}\frac{1}{(1 - x^2)^{1/2}}\, dx$$

$$= 25k\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\pi}{3}\right) = 25k\frac{\sqrt{2}\pi}{6}$$

(b) $\mathbf{r}(u, v) = \langle \cos u, v, \sin u\rangle$

$\mathbf{r}_u = \langle -\sin u, 0, \cos u\rangle, \mathbf{r}_v = \langle 0, 1, 0\rangle$

$\mathbf{r}_u \times \mathbf{r}_v = \langle -\cos u, 0, -\sin u\rangle$

$$\nabla T = \frac{-25}{(x^2 + y^2 + z^2)^{3/2}}[x\mathbf{i} + y\mathbf{j} + z\mathbf{k}] = \frac{-25}{(v^2 + 1)^{3/2}}[\cos u\mathbf{i} + v\mathbf{j} + \sin u\mathbf{k}]$$

$$\nabla T \cdot (\mathbf{r}_u \times \mathbf{r}_v) = \frac{-25}{(v^2 + 1)^{3/2}}(-\cos^2 u - \sin^2 u) = \frac{25}{(v^2 + 1)^{3/2}}$$

$$\text{Flux} = \int_0^1\int_{\pi/3}^{2\pi/3} \frac{25k}{(v^2 + 1)^{3/2}}\, du\, dv = 25k\frac{\sqrt{2}\pi}{6}$$

2. (a) $z = \sqrt{1 - x^2 - y^2}, \dfrac{\partial z}{\partial x} = \dfrac{-x}{\sqrt{1 - x^2 - y^2}}, \dfrac{\partial z}{\partial y} = \dfrac{-y}{\sqrt{1 - x^2 - y^2}}$

$$dS = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dA = \frac{1}{\sqrt{1 - x^2 - y^2}}$$

$$\nabla T = \frac{-25}{(x^2 + y^2 + z^2)^{3/2}}(x\mathbf{i} + y\mathbf{i} + z\mathbf{k}) = -25(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$$

$$\mathbf{N} = \frac{-\dfrac{\partial z}{\partial x}\mathbf{i} - \dfrac{\partial z}{\partial y}\mathbf{j} + \mathbf{k}}{\sqrt{\left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2 + 1}} = \left(\frac{x}{\sqrt{1 - x^2 - y^2}}\mathbf{i} + \frac{y}{\sqrt{1 - x^2 - y^2}}\mathbf{j} + \mathbf{k}\right)\sqrt{1 - x^2 - y^2}$$

$$= x\mathbf{i} + y\mathbf{j} + \sqrt{1 - x^2 - y^2}\mathbf{k} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

$$\text{Flux} = \int_S\int -k\nabla T \cdot N\, dS = k\int_R\int 25(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \cdot (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})\frac{1}{\sqrt{1 - x^2 - y^2}}\, dA$$

$$= k\int_R\int \frac{25}{\sqrt{1 - x^2 - y^2}}\, dA = 25k\int_0^{2\pi}\int_0^1 \frac{1}{\sqrt{1 - r^2}}\, r\, dr\, d\theta = 50\pi k$$

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2. —CONTINUED—

(b) $\mathbf{r}(u, v) = \langle \sin u \cos v, \sin u \sin v, \cos u\rangle$

$\mathbf{r_u} = \langle \cos u \cos v, \cos u \sin v, -\sin u\rangle$

$\mathbf{r_v} = \langle -\sin u \sin v, \sin u \cos v, 0\rangle$

$\mathbf{r_u} \times \mathbf{r_v} = \langle \sin^2 u \cos v, \sin^2 u \sin v, \sin u \cos u \sin^2 v + \sin u \cos u \cos^2 v\rangle$

$\|\mathbf{r_u} \times \mathbf{r_v}\| = \sin u$

$$\text{Flux} = 25k\int_0^{2\pi}\int_0^{\pi/2} \sin u \, du \, dv = 50\pi k$$

3. $\mathbf{r}(t) = \langle 3\cos t, 3\sin t, 2t\rangle$

$\mathbf{r}'(t) = \langle -3\sin t, 3\cos t, 2\rangle, \|\mathbf{r}'(t)\| = \sqrt{13}$

$$I_x = \int_C (y^2 + z^2)\rho \, ds = \int_0^{2\pi} (9\sin^2 t + 4t^2)\sqrt{13}\, dt = \frac{1}{3}\sqrt{13}\pi(32\pi^2 + 27)$$

$$I_y = \int_C (x^2 + z^2)\rho \, ds = \int_0^{2\pi} (9\cos^2 t + 4t^2)\sqrt{13}\, dt = \frac{1}{3}\sqrt{13}\pi(32\pi^2 + 27)$$

$$I_z = \int_C (x^2 + y^2)\rho \, ds = \int_0^{2\pi} (9\cos^2 t + 9\sin^2 t)\sqrt{13}\, dt = 18\pi\sqrt{13}$$

4. $\mathbf{r}(t) = \left\langle \frac{t^2}{2}, t, \frac{2\sqrt{2}t^{3/2}}{3}\right\rangle$

$\mathbf{r}'(t) = \langle t, 1, \sqrt{2}t^{1/2}\rangle, \|\mathbf{r}'(t)\| = t + 1$

$$\rho \, ds = \frac{1}{1+t}(t+1)\, dt = 1$$

$$I_y = \int_C (x^2 + z^2)\rho \, ds = \int_0^1 \left(\frac{t^4}{4} + \frac{8}{9}t^3\right) dt = \frac{49}{180}$$

$$I_x = \int_C (y^2 + z^2)\rho \, ds = \int_0^1 \left(t^2 + \frac{8}{9}t^3\right) dt = \frac{5}{9}$$

$$I_z = \int_C (x^2 + y^2)\rho \, ds = \int_0^1 \left(\frac{t^4}{4} + t^2\right) dt = \frac{23}{60}$$

5.
$$\begin{aligned}\frac{1}{2}\int_C x\,dy - y\,dx &= \frac{1}{2}\int_0^{2\pi}[a(\theta - \sin\theta)(a\sin\theta)\,d\theta - a(1-\cos\theta)(a(1-\cos\theta))\,d\theta] \\ &= \frac{1}{2}a^2\int_0^{2\pi}[\theta\sin\theta - \sin^2\theta - 1 + 2\cos\theta - \cos^2\theta]\,d\theta \\ &= \frac{1}{2}a^2\int_0^{2\pi}(\theta\sin\theta + 2\cos\theta - 2)\,d\theta \\ &= -3\pi a^2\end{aligned}$$

Hence, the area is $3\pi a^2$.

6. $$\frac{1}{2}\int_C x\,dy - y\,dx = 2\int_0^{\pi/2}\left[\frac{1}{2}\sin 2t\cos t - \sin t\cos 2t\right]dt = 2\left(\frac{2}{3}\right)$$

Hence, the area is $4/3$.

7. (a) $\mathbf{r}(t) = t\mathbf{j}, 0 \le t \le 1$

$\mathbf{r}'(t) = \mathbf{j}$

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (t\mathbf{i} + \mathbf{j}) \cdot \mathbf{j}\, dt = \int_0^1 dt = 1$$

(b) $\mathbf{r}(t) = (t - t^2)\mathbf{i} + t\mathbf{j}, 0 \le t \le 1$

$\mathbf{r}'(t) = (1 - 2t)\mathbf{i} + \mathbf{j}$

$$W = \mathbf{F} \cdot d\mathbf{r} = \int_0^1 ((2t - t^2)\mathbf{i} + [(t - t^2)^2 + 1]\mathbf{j}) \cdot ((1 - 2t)\mathbf{i} + \mathbf{j})\, dt$$

$$= \int_0^1 [(1 - 2t)(2t - t^2) + (t^4 - 2t^3 + t^2 + 1)]\, dt$$

$$= \int_0^1 (t^4 - 4t^2 + 2t + 1)\, dt = \frac{13}{15}$$

(c) $\mathbf{r}(t) = c(t - t^2)\mathbf{i} + t\mathbf{j}, 0 \le t \le 1$

$\mathbf{r}'(t) = c(1 - 2t)\mathbf{i} + \mathbf{j}$

$$\mathbf{F} \cdot d\mathbf{r} = (c(t - t^2) + t)(c(1 - 2t)) + (c^2(t - t^2)^2 + 1)(1)$$

$$= c^2t^4 - 2c^2t^2 + c^2t - 2ct^2 + ct + 1$$

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \frac{1}{30}c^2 - \frac{1}{6}c + 1$$

$$\frac{dW}{dc} = \frac{1}{15}c - \frac{1}{6} = 0 \implies c = \frac{5}{2}$$

$$\frac{d^2W}{dc^2} = \frac{1}{15} > 0 \quad c = \frac{5}{2} \text{ minimum.}$$

8. $F(x, y) = 3x^2y^2\mathbf{i} + 2x^3y\mathbf{j}$ is conservative.

$f(x, y) = x^3y^2$ potential function.

Work $= f(2, 4) - f(1, 1) = 8(16) - 1 = 127$

9. $\mathbf{v} \times \mathbf{r} = \langle a_1, a_2, a_3 \rangle \times \langle x, y, z \rangle$

$$= \langle a_2z - a_3y, -a_1z + a_3x, a_1y - a_2x \rangle$$

$\mathbf{curl}(\mathbf{v} \times \mathbf{r}) = \langle 2a_1, 2a_2, 2a_3 \rangle = 2\mathbf{v}$

By Stokes's Theorem,

$$\int_C (\mathbf{v} \times \mathbf{r})\, d\mathbf{r} = \iint_S \mathbf{curl}(\mathbf{v} \times \mathbf{r}) \cdot \mathbf{N}\, dS$$

$$= \iint_S 2\mathbf{v} \cdot \mathbf{N}\, dS.$$

10. Area $= \pi ab$

$\mathbf{r}(t) = a \cos t\mathbf{i} + b \sin t\mathbf{j}, 0 \le t \le 2\pi$

$\mathbf{r}'(t) = -a \sin t\mathbf{i} + b \cos t\mathbf{j}$

$$\mathbf{F} = -\frac{1}{2}b \sin t\mathbf{i} + \frac{1}{2}a \cos t\mathbf{j}$$

$$\mathbf{F} \cdot d\mathbf{r} = \left[\frac{1}{2}ab \sin^2 t + \frac{1}{2}ab \cos^2 t\right] dt = \frac{1}{2}ab$$

$$W = \int_0^{2\pi} \mathbf{F} \cdot d\mathbf{r} = \frac{1}{2}ab(2\pi) = \pi ab$$

Same as area.

11. $\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j} = \dfrac{m}{(x^2 + y^2)^{5/2}}[3xy\mathbf{i} + (2y^2 - x^2)\mathbf{j}]$

$$M = \frac{3mxy}{(x^2 + y^2)^{5/2}} = 3mxy(x^2 + y^2)^{-5/2}$$

$$\frac{\partial M}{\partial y} = 3mxy\left[-\frac{5}{2}(x^2 + y^2)^{-7/2}(2y)\right] + (x^2 + y^2)^{-5/2}(3mx)$$

$$= 3mx(x^2 + y^2)^{-7/2}[-5y^2 + (x^2 + y^2)] = \frac{3mx(x^2 - 4y^2)}{(x^2 + y^2)^{7/2}}$$

$$N = \frac{m(2y^2 - x^2)}{(x^2 + y^2)^{5/2}} = m(2y^2 - x^2)(x^2 + y^2)^{-5/2}$$

$$\frac{\partial N}{\partial x} = m(2y^2 - x^2)\left[-\frac{5}{2}(x^2 + y^2)^{-7/2}(2x)\right] + (x^2 + y^2)^{-5/2}(-2mx)$$

$$= mx(x^2 + y^2)^{-7/2}[(2y^2 - x^2)(-5) + (x^2 + y^2)(-2)]$$

$$= mx(x^2 + y^2)^{-7/2}(3x^2 - 12y^2) = \frac{3mx(x^2 - 4y^2)}{(x^2 + y^2)^{7/2}}$$

Therefore, $\dfrac{\partial N}{\partial x} = \dfrac{\partial M}{\partial y}$ and $\mathbf{F}$ is conservative.