CHAPTER 14
Multiple Integration

Section 14.1 Iterated Integrals and Area in the Plane

1. $\displaystyle\int_0^x (2x - y)\,dy = \left[2xy - \frac{1}{2}y^2\right]_0^x = \frac{3}{2}x^2$

2. $\displaystyle\int_x^{x^2} \frac{y}{x}\,dy = \left[\frac{1}{2}\frac{y^2}{x}\right]_x^{x^2} = \frac{1}{2}\left(\frac{x^4}{x} - \frac{x^2}{x}\right) = \frac{x}{2}(x^2 - 1)$

3. $\displaystyle\int_1^{2y} \frac{y}{x}\,dx = \left[y\ln x\right]_1^{2y} = y\ln 2y - 0 = y\ln 2y,\ (y > 0)$

4. $\displaystyle\int_0^{\cos y} y\,dx = \left[yx\right]_0^{\cos y} = y\cos y$

5. $\displaystyle\int_0^{\sqrt{4-x^2}} x^2y\,dy = \left[\frac{1}{2}x^2y^2\right]_0^{\sqrt{4-x^2}} = \frac{4x^2 - x^4}{2}$

6. $\displaystyle\int_{x^3}^{\sqrt{x}} (x^2 + 3y^2)\,dy = \left[x^2y + y^3\right]_{x^3}^{\sqrt{x}} = \left(x^2\sqrt{x} + (\sqrt{x})^3\right) - (x^2x^3 + (x^3)^3) = x^{5/2} + x^{3/2} - x^5 - x^9$

7. $\displaystyle\int_{e^y}^{y} \frac{y\ln x}{x}\,dx = \left[\frac{1}{2}y\ln^2 x\right]_{e^y}^{y} = \frac{1}{2}y[\ln^2 y - \ln^2 e^y] = \frac{y}{2}[(\ln y)^2 - y^2],\ (y > 0)$

8. $\displaystyle\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} (x^2 + y^2)\,dx = \left[\frac{1}{3}x^3 + y^2x\right]_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} = 2\left[\frac{1}{3}(1 - y^2)^{3/2} + y^2(1 - y^2)^{1/2}\right] = \frac{2\sqrt{1-y^2}}{3}(1 + 2y^2)$

9. $\displaystyle\int_0^{x^3} ye^{-y/x}\,dy = \left[-xye^{-y/x}\right]_0^{x^3} + x\int_0^{x^3} e^{-y/x}\,dy = -x^4e^{-x^2} - \left[x^2e^{-y/x}\right]_0^{x^3} = x^2(1 - e^{-x^2} - x^2e^{-x^2})$

$u = y,\ du = dy,\ dv = e^{-y/x}\,dy,\ v = -xe^{-y/x}$

10. $\displaystyle\int_y^{\pi/2} \sin^3 x\cos y\,dx = \int_y^{\pi/2} (1 - \cos^2 x)\sin x\cos y\,dx$

$\displaystyle= \left[\left(-\cos x + \frac{1}{3}\cos^3 x\right)\cos y\right]_y^{\pi/2} = \left(\cos y - \frac{1}{3}\cos^3 y\right)\cos y$

11. $\displaystyle\int_0^1\int_0^2 (x + y)\,dy\,dx = \int_0^1 \left[xy + \frac{1}{2}y^2\right]_0^2 dx = \int_0^1 (2x + 2)\,dx = \left[x^2 + 2x\right]_0^1 = 3$

12. $\displaystyle\int_{-1}^1\int_{-2}^2 (x^2 - y^2)\,dy\,dx = \int_{-1}^1 \left[x^2y - \frac{y^3}{3}\right]_{-2}^2 dx = \int_{-1}^1 \left[2x^2 - \frac{8}{3} + 2x^2 - \frac{8}{3}\right] dx$

$\displaystyle= \int_{-1}^1 \left(4x^2 - \frac{16}{3}\right) dx = \left[\frac{4x^3}{3} - \frac{16}{3}x\right]_{-1}^1 = \left(\frac{4}{3} - \frac{16}{3}\right) - \left(-\frac{4}{3} + \frac{16}{3}\right) = -8$

13. $\displaystyle\int_0^\pi\int_0^{\sin x} (1 + \cos x)\,dy\,dx = \int_0^\pi \left[(y + y\cos x)\right]_0^{\sin x} dx = \int_0^\pi [\sin x + \sin x\cos x]\,dx = \left[-\cos x + \frac{1}{2}\sin^2 x\right]_0^\pi = 1 + 1 = 2$

14. $\displaystyle\int_1^4\int_1^{\sqrt{x}} 2ye^{-x}\,dy\,dx = \int_1^4\Big[y^2e^{-x}\Big]_1^{\sqrt{x}} dx = \int_1^4 (xe^{-x} - e^{-x})\,dx = \Big[-xe^{-x}\Big]_1^4 = -4e^{-4} + e^{-1} = \frac{1}{e} - \frac{4}{e^4}$

15. $\displaystyle\int_0^1\int_0^x \sqrt{1-x^2}\,dy\,dx = \int_0^1\Big[y\sqrt{1-x^2}\Big]_0^x dx = \int_0^1 x\sqrt{1-x^2}\,dx = \left[-\frac{1}{2}\left(\frac{2}{3}\right)(1-x^2)^{3/2}\right]_0^1 = \frac{1}{3}$

16. $\displaystyle\int_{-4}^4\int_0^{x^2}\sqrt{64-x^3}\,dy\,dx = \int_{-4}^4\Big[y\sqrt{64-x^3}\Big]_0^{x^2} dx$

$\displaystyle= \int_{-4}^4 \sqrt{64-x^3}\,x^2\,dx = \left[-\frac{2}{9}(64-x^3)^{3/2}\right]_{-4}^4 = 0 + \frac{2}{9}(128)^{3/2} = \frac{2048}{9}\sqrt{2}$

17. $\displaystyle\int_1^2\int_0^4 (x^2-2y^2+1)\,dx\,dy = \int_1^2\left[\frac{1}{3}x^3 - 2xy^2 + x\right]_0^4 dy$

$\displaystyle= \int_1^2\left(\frac{64}{3} - 8y^2 + 4\right)dy = \frac{4}{3}\int_1^2 (19-6y^2)\,dy = \left[\frac{4}{3}(19y - 2y^3)\right]_1^2 = \frac{20}{3}$

18. $\displaystyle\int_0^2\int_y^{2y}(10+2x^2+2y^2)\,dx\,dy = \int_0^2\left[10x + \frac{2x^3}{3} + 2y^2x\right]_y^{2y} dy = \int_0^2\left[\left(20y + \frac{16}{3}y^3 + 4y^3\right) - \left(10y + \frac{2}{3}y^3 + 2y^3\right)\right]dy$

$\displaystyle= \int_0^2\left[10y + \frac{14}{3}y^3 + 2y^3\right]dy = \left[5y^2 + \frac{7y^4}{6} + \frac{y^4}{2}\right]_0^2 = 20 + \frac{56}{3} + 8 = \frac{140}{3}$

19. $\displaystyle\int_0^1\int_0^{\sqrt{1-y^2}}(x+y)\,dx\,dy = \int_0^1\left[\frac{1}{2}x^2 + xy\right]_0^{\sqrt{1-y^2}} dy$

$\displaystyle= \int_0^1\left[\frac{1}{2}(1-y^2) + y\sqrt{1-y^2}\right]dy = \left[\frac{1}{2}y - \frac{1}{6}y^3 - \frac{1}{2}\left(\frac{2}{3}\right)(1-y^2)^{3/2}\right]_0^1 = \frac{2}{3}$

20. $\displaystyle\int_0^2\int_{3y^2-6y}^{2y-y^2} 3y\,dx\,dy = \int_0^2\Big[3xy\Big]_{3y^2-6y}^{2y-y^2} dy = 3\int_0^2 (8y^2-4y^3)\,dy = \left[3\left(\frac{8}{3}y^3 - y^4\right)\right]_0^2 = 16$

21. $\displaystyle\int_0^2\int_0^{\sqrt{4-y^2}} \frac{2}{\sqrt{4-y^2}}\,dx\,dy = \int_0^2\left[\frac{2x}{\sqrt{4-y^2}}\right]_0^{\sqrt{4-y^2}} dy = \int_0^2 2\,dy = \Big[2y\Big]_0^2 = 4$

22. $\displaystyle\int_0^{\pi/2}\int_0^{2\cos\theta} r\,dr\,d\theta = \int_0^{\pi/2}\left[\frac{r^2}{2}\right]_0^{2\cos\theta} d\theta = \int_0^{\pi/2} 2\cos^2\theta\,d\theta = \left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\pi/2} = \frac{\pi}{2}$

23. $\displaystyle\int_0^{\pi/2}\int_0^{\sin\theta} \theta r\,dr\,d\theta = \int_0^{\pi/2}\left[\theta\frac{r^2}{2}\right]_0^{\sin\theta} d\theta = \int_0^{\pi/2}\frac{1}{2}\theta\sin^2\theta\,d\theta$

$\displaystyle= \frac{1}{4}\int_0^{\pi/2}(\theta - \theta\cos 2\theta)\,d\theta = \frac{1}{4}\left[\frac{\theta^2}{2} - \left(\frac{1}{4}\cos 2\theta + \frac{\theta}{2}\sin 2\theta\right)\right]_0^{\pi/2} = \frac{\pi^2}{32} + \frac{1}{8}$

24. $\displaystyle\int_0^{\pi/4}\int_0^{\cos\theta} 3r^2\sin\theta\,dr\,d\theta = \int_0^{\pi/4}\Big[r^3\sin\theta\Big]_0^{\cos\theta} d\theta$

$\displaystyle= \int_0^{\pi/4}\cos^3\sin\theta\,d\theta = \left[-\frac{\cos^4\theta}{4}\right]_0^{\pi/4} = -\frac{1}{4}\left[\left(\frac{1}{\sqrt{2}}\right)^4 - 1\right] = \frac{3}{16}$

25. $\displaystyle\int_1^\infty\int_0^{1/x} y\,dy\,dx = \int_1^\infty\left[\frac{y^2}{2}\right]_0^{1/x} dx = \frac{1}{2}\int_1^\infty \frac{1}{x^2}\,dx = \left[-\frac{1}{2x}\right]_1^\infty = 0 + \frac{1}{2} = \frac{1}{2}$

26. $\displaystyle\int_0^3\int_0^\infty \frac{x^2}{1+y^2}\,dy\,dx = \int_0^3 \Big[x^2 \arctan y\Big]_0^\infty dx = \int_0^3 x^2\left(\frac{\pi}{2}\right)dx = \left[\frac{\pi}{2}\cdot\frac{x^3}{3}\right]_0^3 = \frac{9\pi}{2}$

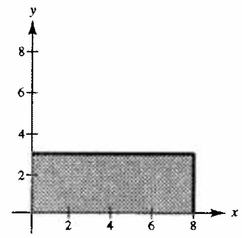
27. $\displaystyle\int_1^\infty\int_1^\infty \frac{1}{xy}\,dx\,dy = \int_1^\infty\left[\frac{1}{y}\ln x\right]_1^\infty dy = \int_1^\infty\left[\frac{1}{y}(\infty) - \frac{1}{y}(0)\right]dy$

Diverges

28. $\displaystyle\int_0^\infty\int_0^\infty xye^{-(x^2+y^2)}\,dx\,dy = \int_0^\infty\left[-\frac{1}{2}ye^{-(x^2+y^2)}\right]_0^\infty dy = \int_0^\infty \frac{1}{2}ye^{-y^2}\,dy = \left[-\frac{1}{4}e^{-y^2}\right]_0^\infty = \frac{1}{4}$

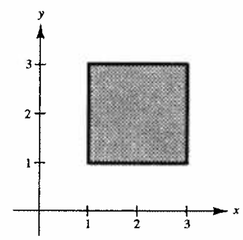
29. $\displaystyle A = \int_0^8\int_0^3 dy\,dx = \int_0^8\Big[y\Big]_0^3 dx = \int_0^8 3\,dx = \Big[3x\Big]_0^8 = 24$

$\displaystyle A = \int_0^3\int_0^8 dx\,dy = \int_0^3\Big[x\Big]_0^8 dy = \int_0^3 8\,dy = \Big[8y\Big]_0^3 = 24$



30. $\displaystyle A = \int_1^3\int_1^3 dy\,dx = \int_1^3\Big[y\Big]_1^3 dx = \int_1^3 2\,dx = \Big[2x\Big]_1^3 = 4$

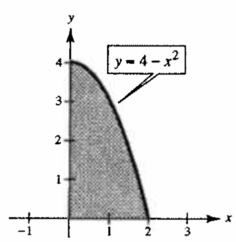
$\displaystyle A = \int_1^3\int_1^3 dx\,dy = \int_1^3\Big[x\Big]_1^3 dy = \int_1^3 2\,dy = \Big[2y\Big]_1^3 = 4$



31. $\displaystyle A = \int_0^2\int_0^{4-x^2} dy\,dx = \int_0^2\Big[y\Big]_0^{4-x^2} dx$

$\displaystyle = \int_0^2 (4-x^2)\,dx$

$\displaystyle = \left[4x - \frac{x^3}{3}\right]_0^2 = \frac{16}{3}$



$\displaystyle A = \int_0^4\int_0^{\sqrt{4-y}} dx\,dy$

$\displaystyle = \int_0^4\Big[x\Big]_0^{\sqrt{4-y}} dy = \int_0^4 \sqrt{4-y}\,dy = -\int_0^4 (4-y)^{1/2}(-1)\,dy = \left[-\frac{2}{3}(4-y)^{3/2}\right]_0^4 = \frac{2}{3}(8) = \frac{16}{3}$

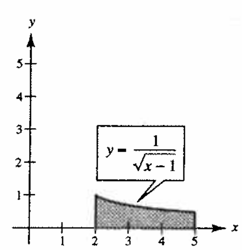
32. $\displaystyle A = \int_2^5\int_0^{1/\sqrt{x-1}} dy\,dx = \int_2^5\Big[y\Big]_0^{1/\sqrt{x-1}} dx = \int_2^5 \frac{1}{\sqrt{x-1}}\,dx = \Big[2\sqrt{x-1}\Big]_2^5 = 2$

$\displaystyle A = \int_0^{1/2}\int_2^5 dx\,dy + \int_{1/2}^1\int_2^{1+(1/y^2)} dx\,dy$

$\displaystyle = \int_0^{1/2}\Big[x\Big]_2^5 dy + \int_{1/2}^1\Big[x\Big]_2^{1+(1/y^2)} dy$

$\displaystyle = \int_0^{1/2} 3\,dy + \int_{1/2}^1\left(\frac{1}{y^2} - 1\right)dy$

$\displaystyle = \Big[3y\Big]_0^{1/2} + \left[-\frac{1}{y} - y\right]_{1/2}^1 = 2$



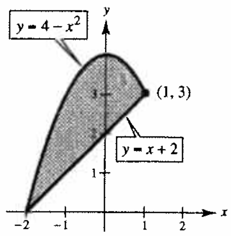
33. $A = \int_{-2}^{1}\int_{x+2}^{4-x^2} dy\,dx$

$= \int_{-2}^{1} \Big[y\Big]_{x+2}^{4-x^2} dx$

$= \int_{-2}^{1} (4 - x^2 - x - 2)\,dx$

$= \int_{-2}^{1} (2 - x - x^2)\,dx$

$= \left[2x - \frac{1}{2}x^2 - \frac{1}{3}x^3\right]_{-2}^{1} = \frac{9}{2}$



$A = \int_0^3\int_{-\sqrt{4-y}}^{y-2} dx\,dy + 2\int_3^4\int_0^{\sqrt{4-y}} dx\,dy$

$= \int_0^3 \Big[x\Big]_{-\sqrt{4-y}}^{y-2} dy + 2\int_3^4 \Big[x\Big]_0^{\sqrt{4-y}} dy$

$= \int_0^3 \left(y - 2 + \sqrt{4-y}\right) dy + 2\int_3^4 \sqrt{4-y}\,dy$

$= \left[\frac{1}{2}y^2 - 2y - \frac{2}{3}(4-y)^{3/2}\right]_0^3 - \left[\frac{4}{3}(4-y)^{3/2}\right]_3^4 = \frac{9}{2}$

34. $A = \int_0^2\int_0^{\sqrt{4-x^2}} dy\,dx$

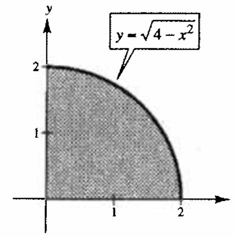
$= \int_0^2 \sqrt{4-x^2}\,dx$

$= 4\int_0^{\pi/2} \cos^2\theta\,d\theta$

$= 2\int_0^{\pi/2} (1 + \cos 2\theta)\,d\theta$

$= \left[2\left(\theta + \frac{1}{2}\sin 2\theta\right)\right]_0^{\pi/2} = \pi$

$\left(x = 2\sin\theta,\ dx = 2\cos\theta\,d\theta,\ \sqrt{4-x^2} = 2\cos\theta\right)$



$A = \int_0^2\int_0^{\sqrt{4-y^2}} dx\,dy = \int_0^2 \sqrt{4-y^2}\,dy$

$= 4\int_0^{\pi/2} \cos^2\theta\,d\theta = 2\int_0^{\pi/2} (1 + \cos 2\theta)\,d\theta$

$= \left[2\left(\theta + \frac{1}{2}\sin 2\theta\right)\right]_0^{\pi/2} = \pi$

$\left(y = 2\sin\theta,\ dy = 2\cos\theta\,d\theta,\ \sqrt{4-y^2} = 2\cos\theta\right)$

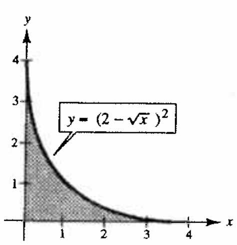
35. $\int_0^4\int_0^{(2-\sqrt{x})^2} dy\,dx = \int_0^4 \Big[y\Big]_0^{(2-\sqrt{x})^2} dx$

$= \int_0^4 \left(4 - 4\sqrt{x} + x\right) dx$

$= \left[4x - \frac{8}{3}x\sqrt{x} + \frac{x^2}{2}\right]_0^4 = \frac{8}{3}$

$\int_0^4\int_0^{(2-\sqrt{y})^2} dx\,dy = \frac{8}{3}$

Integration steps are similar to those above.



36. $A = \int_0^4\int_{x^{3/2}}^{2x} dy\,dx$

$= \int_0^4 \Big[y\Big]_{x^{3/2}}^{2x} dx$

$= \int_0^4 (2x - x^{3/2})\,dx$

$= \left[x^2 - \frac{2}{5}x^{5/2}\right]_0^4$

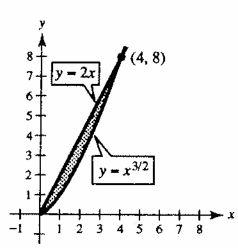
$= 16 - \frac{2}{5}(32) = \frac{16}{5}$

$A = \int_0^8\int_{y/2}^{y^{2/3}} dx\,dy$

$= \int_0^8 \left(y^{2/3} - \frac{y}{2}\right) dy$

$= \left[\frac{3}{5}y^{5/3} - \frac{y^2}{4}\right]_0^8$

$= \frac{3}{5}(32) - 16 = \frac{16}{5}$



37. $A = \int_0^3 \int_0^{2x/3} dy\,dx + \int_3^5 \int_0^{5-x} dy\,dx$

$= \int_0^3 \Big[y \Big]_0^{2x/3} dx + \int_3^5 \Big[y \Big]_0^{5-x} dx$

$= \int_0^3 \frac{2x}{3}\,dx + \int_3^5 (5 - x)\,dx$

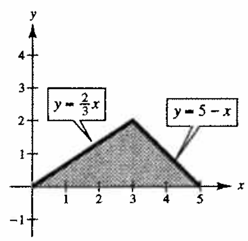
$= \Big[\frac{1}{3}x^2 \Big]_0^3 + \Big[5x - \frac{1}{2}x^2 \Big]_3^5 = 5$

$A = \int_0^2 \int_{3y/2}^{5-y} dx\,dy$

$= \int_0^2 \Big[x \Big]_{3y/2}^{5-y} dy$

$= \int_0^2 \left(5 - y - \frac{3y}{2}\right) dy$

$= \int_0^2 \left(5 - \frac{5y}{2}\right) dy = \Big[5y - \frac{5}{4}y^2 \Big]_0^2 = 5$



38. $A = \int_0^3 \int_0^{x} dy\,dx + \int_3^9 \int_0^{9/x} dy\,dx$

$= \int_0^3 \Big[y \Big]_0^{x} dx + \int_3^9 \Big[y \Big]_0^{9/x} dx = \int_0^3 x\,dx + \int_3^9 \frac{9}{x}\,dx$

$= \Big[\frac{1}{2}x^2 \Big]_0^3 + \Big[9 \ln x \Big]_3^9 = \frac{9}{2} + 9(\ln 9 - \ln 3)$

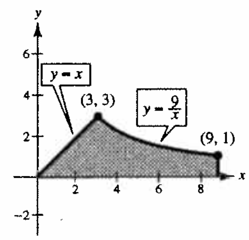
$= \frac{9}{2}(1 + \ln 9)$

$A = \int_0^1 \int_y^9 dx\,dy + \int_1^3 \int_y^{9/y} dx\,dy$

$= \int_0^1 \Big[x \Big]_y^9 dy + \int_1^3 \Big[x \Big]_y^{9/y} dy$

$= \int_0^1 (9 - y)\,dy + \int_1^3 \left(\frac{9}{y} - y\right) dy$

$= \Big[9y - \frac{1}{2}y^2 \Big]_0^1 + \Big[9 \ln y - \frac{1}{2}y^2 \Big]_1^3 = \frac{9}{2}(1 + \ln 9)$



39. $\frac{A}{4} = \int_0^a \int_0^{(b/a)\sqrt{a^2 - x^2}} dy\,dx = \int_0^a \Big[y \Big]_0^{(b/a)\sqrt{a^2-x^2}} dx$

$= \frac{b}{a}\int_0^a \sqrt{a^2 - x^2}\,dx = ab\int_0^{\pi/2} \cos^2 \theta\,d\theta$

$(x = a \sin \theta,\ dx = a \cos \theta\,d\theta)$

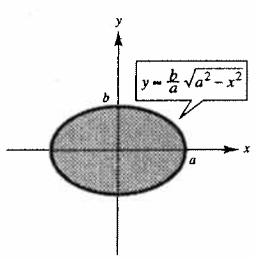
$= \frac{ab}{2}\int_0^{\pi/2} (1 + \cos 2\theta)\,d\theta = \Big[\frac{ab}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right) \Big]_0^{\pi/2}$

$= \frac{\pi ab}{4}$

Therefore, $A = \pi ab$.

$\frac{A}{4} = \int_0^b \int_0^{(a/b)\sqrt{b^2 - y^2}} dx\,dy = \frac{\pi ab}{4}$

Therefore, $A = \pi ab$. Integration steps are similar to those above.



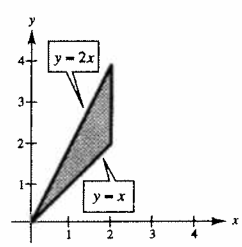
40. $A = \int_0^2 \int_{y/2}^{y} dx\,dy + \int_2^4 \int_{y/2}^{2} dx\,dy$

$= \int_0^2 \frac{y}{2}\,dy + \int_2^4 \left(2 - \frac{y}{2}\right) dy$

$= \Big[\frac{y^2}{4} \Big]_0^2 + \Big[2y - \frac{y^2}{4} \Big]_2^4$

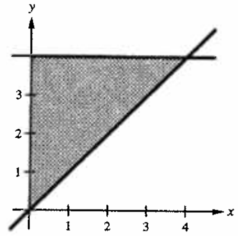
$= 1 + (4 - 3) = 2$

$A = \int_0^2 \int_x^{2x} dy\,dx = \int_0^2 (2x - x)\,dx = \Big[\frac{x^2}{2} \Big]_0^2 = 2$



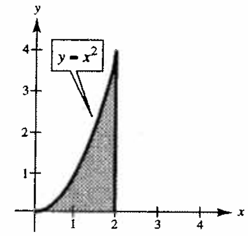
41. $\int_0^4 \int_0^y f(x, y)\, dx\, dy,\ 0 \le x \le y,\ 0 \le y \le 4$

$= \int_0^4 \int_x^4 f(x, y)\, dy\, dx$



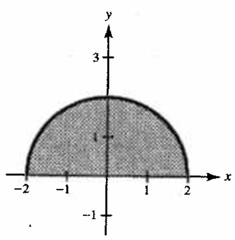
42. $\int_0^4 \int_{\sqrt{y}}^2 f(x, y)\, dx\, dy,\ \sqrt{y} \le x \le 2,\ 0 \le y \le 4$

$= \int_0^2 \int_0^{x^2} f(x, y)\, dy\, dx$



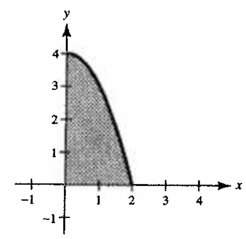
43. $\int_{-2}^2 \int_0^{\sqrt{4-x^2}} f(x, y)\, dy\, dx, 0 \le y \le \sqrt{4 - x^2}, -2 \le x \le 2$

$= \int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} dx\, dy$



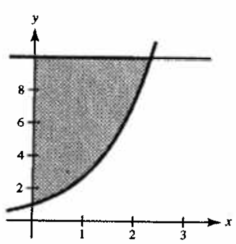
44. $\int_0^2 \int_0^{4-x^2} f(x, y)\, dy\, dx, 0 \le y \le 4 - x^2, 0 \le x \le 2$

$= \int_0^4 \int_0^{\sqrt{4-y}} f(x, y)\, dx\, dy$



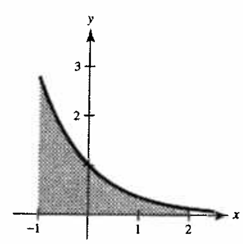
45. $\int_1^{10} \int_0^{\ln y} f(x, y)\, dx\, dy, 0 \le x \le \ln y, 1 \le y \le 10$

$= \int_0^{\ln 10} \int_{e^x}^{10} f(x, y)\, dy\, dx$



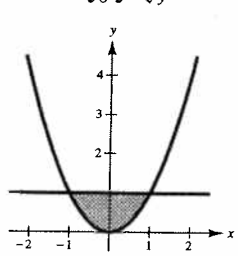
46. $\int_{-1}^2 \int_0^{e^{-x}} f(x, y)\, dy\, dx, 0 \le y \le e^{-x}, -1 \le x \le 2$

$= \int_0^{e^{-2}} \int_{-1}^2 f(x, y)\, dx\, dy + \int_{e^{-2}}^{e} \int_{-1}^{-\ln y} f(x, y)\, dx\, dy$



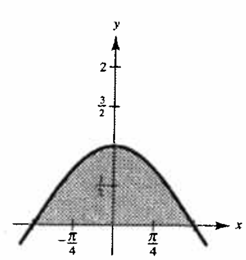
47. $\int_{-1}^1 \int_{x^2}^1 f(x, y)\, dy\, dx, x^2 \le y \le 1, -1 \le x \le 1$

$= \int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} f(x, y)\, dx\, dy$

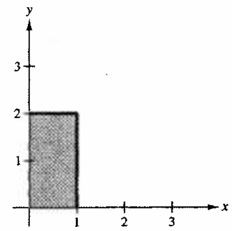


48. $\int_{-\pi/2}^{\pi/2} \int_0^{\cos x} f(x, y)\, dy\, dx, 0 \le y \le \cos x, -\frac{\pi}{2} \le x \le \frac{\pi}{2}$

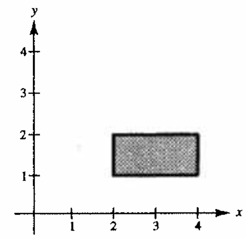
$= \int_0^1 \int_{-\arccos y}^{\arccos y} f(x, y)\, dx\, dy$



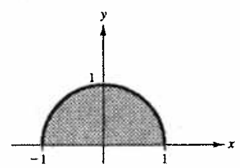
49. $\displaystyle\int_0^1\int_0^2 dy\,dx = \int_0^2\int_0^1 dx\,dy = 2$



50. $\displaystyle\int_1^2\int_2^4 dx\,dy = \int_2^4\int_1^2 dy\,dx = 2$

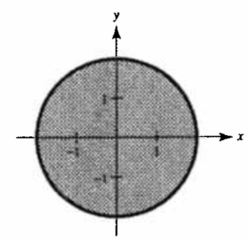


51. $\displaystyle\int_0^1\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} dx\,dy = \int_{-1}^{1}\int_0^{\sqrt{1-x^2}} dy\,dx = \frac{\pi}{2}$

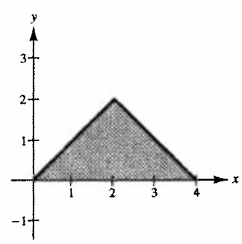


52. $\displaystyle\int_{-2}^2\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dy\,dx = \int_{-2}^{2}\left(\sqrt{4-x^2}+\sqrt{4-x^2}\right)dx = 4\pi$

$\displaystyle\int_{-2}^2\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} dx\,dy = 4\pi$

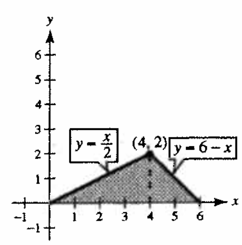


53. $\displaystyle\int_0^2\int_0^x dy\,dx + \int_2^4\int_0^{4-x} dy\,dx = \int_0^2\int_y^{4-y} dx\,dy = 4$

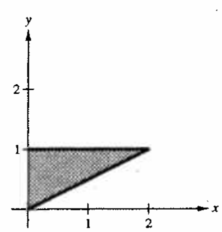


54. $\displaystyle\int_0^4\int_0^{x/2} dy\,dx + \int_4^6\int_0^{6-x} dy\,dx = \int_0^4 \frac{x}{2}\,dx + \int_4^6 (6-x)\,dx = 4 + 2 = 6$

$\displaystyle\int_0^2\int_{2y}^{6-y} dx\,dy = \int_0^2 (6-3y)\,dy = \left[6y - \frac{3y^2}{2}\right]_0^2 = 6$

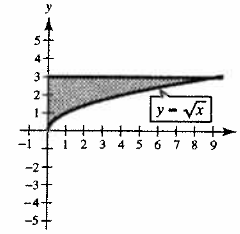


55. $\displaystyle\int_0^2\int_{x/2}^1 dy\,dx = \int_0^1\int_0^{2y} dx\,dy = 1$

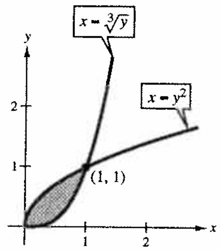


56. $\displaystyle\int_0^9\int_{\sqrt{x}}^3 dy\,dx = \int_0^9 (3-\sqrt{x})\,dx = \left[3x - \frac{2}{3}x^{3/2}\right]_0^9 = 27 - 18 = 9$

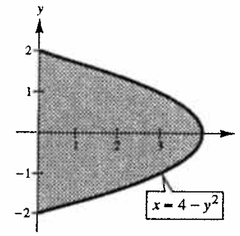
$\displaystyle\int_0^3\int_0^{y^2} dx\,dy = \int_0^3 y^2\,dy = \left[\frac{y^3}{3}\right]_0^3 = 9$



57. $\displaystyle\int_0^1\int_{y^2}^{\sqrt[3]{y}} dx\,dy = \int_0^1\int_{x^3}^{\sqrt{x}} dy\,dx = \frac{5}{12}$



58. $\displaystyle\int_{-2}^2\int_0^{4-y^2} dx\,dy = \int_0^4\int_{-\sqrt{4-x}}^{\sqrt{4-x}} dy\,dx = \frac{32}{3}$



59. The first integral arises using vertical representative rectangles. The second two integrals arise using horizontal representative rectangles.

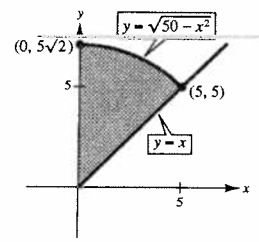
$$\int_0^5\int_x^{\sqrt{50-x^2}} x^2y^2\,dy\,dx = \int_0^5\left[\frac{1}{3}x^2(50-x^2)^{3/2} - \frac{1}{3}x^5\right]dx$$

$$= \frac{15{,}625}{24}\pi$$

$$\int_0^5\int_0^y x^2y^2\,dx\,dy + \int_5^{5\sqrt{2}}\int_0^{\sqrt{50-y^2}} x^2y^2\,dx\,dy = \int_0^5 \frac{1}{3}y^5\,dy + \int_5^{5\sqrt{2}} \frac{1}{3}(50-y^2)^{3/2}y^2\,dy$$

$$= \frac{15{,}625}{18} + \left(\frac{15{,}625}{18}\pi - \frac{15{,}625}{18}\right)$$

$$= \frac{15{,}625}{24}\pi$$



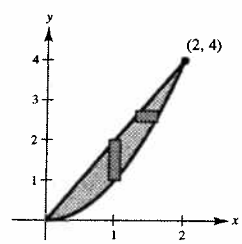
60. The first integral arises using vertical representative rectangles. The second integral arises using horizontal representative rectangles.

$$\int_0^2\int_{x^2}^{2x} x\sin y\,dy\,dx = \int_0^2 (-x\cos(2x) + x\cos(x^2))\,dx$$

$$= -\frac{1}{4}\cos(4) - \frac{1}{2}\sin(4) + \frac{1}{4}$$

$$\int_0^4\int_{y/2}^{\sqrt{y}} x\sin y\,dx\,dy = \int_0^4\left(\frac{1}{2}y\sin(y) - \frac{1}{8}y^2\sin(y)\right)dy$$

$$= -\frac{1}{4}\cos(4) - \frac{1}{2}\sin(4) + \frac{1}{4}$$



61. $\displaystyle\int_0^2\int_x^2 x\sqrt{1+y^3}\,dy\,dx = \int_0^2\int_0^y x\sqrt{1+y^3}\,dx\,dy = \int_0^2\left[\sqrt{1+y^3}\cdot\frac{x^2}{2}\right]_0^y dy$

$\displaystyle= \frac{1}{2}\int_0^2 \sqrt{1+y^3}\,y^2\,dy = \left[\frac{1}{2}\cdot\frac{1}{3}\cdot\frac{2}{3}(1+y^3)^{3/2}\right]_0^2 = \frac{1}{9}(27) - \frac{1}{9}(1) = \frac{26}{9}$

62. $\int_0^2 \int_x^2 e^{-y^2}\,dy\,dx = \int_0^2 \int_0^y e^{-y^2}\,dx\,dy$

$= \int_0^2 \left[xe^{-y^2}\right]_0^y dy = \int_0^2 ye^{-y^2}\,dy = \left[-\frac{1}{2}e^{-y^2}\right]_0^2 = -\frac{1}{2}(e^{-4}) + \frac{1}{2}e^0 = \frac{1}{2}\left(1 - \frac{1}{e^4}\right) \approx 0.4908$

63. $\int_0^1 \int_y^1 \sin(x^2)\,dx\,dy = \int_0^1 \int_0^x \sin(x^2)\,dy\,dx = \int_0^1 \left[y\sin(x^2)\right]_0^x dx$

$= \int_0^1 x\sin(x^2)\,dx = \left[-\frac{1}{2}\cos(x^2)\right]_0^1 = -\frac{1}{2}\cos 1 + \frac{1}{2}(1) = \frac{1}{2}(1 - \cos 1) \approx 0.2298$

64. $\int_0^2 \int_{y^2}^4 \sqrt{x}\sin x\,dx\,dy = \int_0^4 \int_0^{\sqrt{x}} \sqrt{x}\sin x\,dy\,dx$

$= \int_0^4 \left[y\sqrt{x}\sin x\right]_0^{\sqrt{x}} dx = \int_0^4 x\sin x\,dx = \left[\sin x - x\cos x\right]_0^4 = \sin 4 - 4\cos 4 \approx 1.858$

65. $\int_0^2 \int_{x^2}^{2x} (x^3 + 3y^2)\,dy\,dx = \frac{1664}{105} \approx 15.848$

66. $\int_0^1 \int_y^{2y} \sin(x + y)\,dx\,dy = \frac{\sin 2}{2} - \frac{\sin 3}{3} \approx 0.408$

67. $\int_0^4 \int_0^y \frac{2}{(x+1)(y+1)}\,dx\,dy = (\ln 5)^2 \approx 2.590$

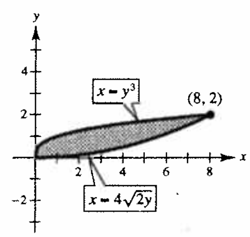
68. $\int_0^a \int_0^{a-x} (x^2 + y^2)\,dy\,dx = \frac{a^4}{6}$

69. (a) $x = y^3 \Leftrightarrow y = x^{1/3}$

$x = 4\sqrt{2y} \Leftrightarrow x^2 = 32y \Leftrightarrow y = \frac{x^2}{32}$

(b) $\int_0^8 \int_{x^2/32}^{x^{1/3}} (x^2y - xy^2)\,dy\,dx$

(c) Both integrals equal $67{,}520/693 \approx 97.43$

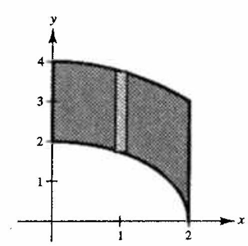


70. (a) $y = \sqrt{4 - x^2} \Leftrightarrow x = \sqrt{4 - y^2}$

$y = 4 - \frac{x^2}{4} \Leftrightarrow x = \sqrt{16 - 4y}$

(b) $\int_0^2 \int_{\sqrt{4-y^2}}^2 \frac{xy}{x^2 + y^2 + 1}\,dx\,dy + \int_2^3 \int_0^2 \frac{xy}{x^2 + y^2 + 1}\,dx\,dy + \int_3^4 \int_0^{\sqrt{16-4y}} \frac{xy}{x^2 + y^2 + 1}\,dx\,dy$

(c) Both orders of integration yield 1.11899.



71. $\int_0^2 \int_0^{4-x^2} e^{xy}\,dy\,dx \approx 20.5648$

72. $\int_0^2 \int_x^2 \sqrt{16 - x^3 - y^3}\,dy\,dx \approx 6.8520$

73. $\int_0^{2\pi} \int_0^{1+\cos\theta} 6r^2\cos\theta\,dr\,d\theta = \frac{15\pi}{2}$

74. $\int_0^{\pi/2} \int_0^{1+\sin\theta} 15\theta r\,dr\,d\theta = \frac{45\pi^2}{32} + \frac{135}{8} \approx 30.7541$

75. An iterated integral is a double integral of a function of two variables. First integrate with respect to one variable while holding the other variable constant. Then integrate with respect to the second variable.

76. A region is vertically simple if it is bounded on the left and right by vertical lines, and bounded on the top and bottom by functions of x. A region is horizontally simple if it is bounded on the top and bottom by horizontal lines, and bounded on the left and right by functions of y.

77. The region is a rectangle.

78. The integrations might be easier. See Exercise 59-62.

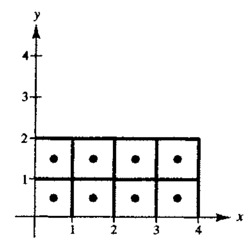
79. True

80. False, let $f(x, y) = x$.

Section 14.2 Double Integrals and Volume

For Exercise 1–4, $\Delta x_i = \Delta y_i = 1$ and the midpoints of the squares are

$$\left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{3}{2}, \frac{1}{2}\right), \left(\frac{5}{2}, \frac{1}{2}\right), \left(\frac{7}{2}, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{3}{2}\right), \left(\frac{3}{2}, \frac{3}{2}\right), \left(\frac{5}{2}, \frac{3}{2}\right), \left(\frac{7}{2}, \frac{3}{2}\right).$$



1. $f(x, y) = x + y$

$$\sum_{i=1}^{8} f(x_i, y_i)\,\Delta x_i \Delta y_i = 1 + 2 + 3 + 4 + 2 + 3 + 4 + 5 = 24$$

$$\int_0^4 \int_0^2 (x + y)\,dy\,dx = \int_0^4 \left[xy + \frac{y^2}{2}\right]_0^2 dx = \int_0^4 (2x + 2)\,dx = \Big[x^2 + 2x\Big]_0^4 = 24$$

2. $f(x, y) = \frac{1}{2}x^2y$

$$\sum_{i=1}^{\infty} f(x_i, y_i)\,\Delta x_i\,\Delta y_i = \frac{1}{16} + \frac{9}{16} + \frac{25}{16} + \frac{49}{16} + \frac{3}{16} + \frac{27}{16} + \frac{75}{16} + \frac{147}{16} = 21$$

$$\int_0^4 \int_0^2 \frac{1}{2}x^2y\,dy\,dx = \int_0^4 \left[\frac{x^2y^2}{4}\right]_0^2 dx = \int_0^4 x^2\,dx = \frac{x^3}{3}\Big]_0^4 = \frac{64}{3} \approx 21.3$$

3. $f(x, y) = x^2 + y^2$

$$\sum_{i=1}^{8} f(x_i, y_i)\,\Delta x_i\,\Delta y_i = \frac{2}{4} + \frac{10}{4} + \frac{26}{4} + \frac{50}{4} + \frac{10}{4} + \frac{18}{4} + \frac{34}{4} + \frac{58}{4} = 52$$

$$\int_0^4 \int_0^2 (x^2 + y^2)\,dy\,dx = \int_0^4 \left[x^2y + \frac{y^3}{3}\right]_0^2 dx = \int_0^4 \left(2x^2 + \frac{8}{3}\right) dx = \left[\frac{2x^3}{3} + \frac{8x}{3}\right]_0^4 = \frac{160}{3}$$

4. $f(x, y) = \dfrac{1}{(x + 1)(y + 1)}$

$$\sum_{i=1}^{8} f(x_i, y_i)\,\Delta x_i\,\Delta y_i = \frac{4}{9} + \frac{4}{15} + \frac{4}{21} + \frac{4}{27} + \frac{4}{15} + \frac{4}{25} + \frac{4}{35} + \frac{4}{45} = \frac{7936}{4725} \approx 1.680$$

$$\int_0^4 \int_0^2 \frac{1}{(x + 1)(y + 1)}\,dy\,dx = \int_0^4 \left[\frac{1}{x + 1}\ln(y + 1)\right]_0^2 dx$$

$$= \int_0^4 \frac{\ln 3}{x + 1}\,dx = \Big[\ln 3 \cdot \ln(x + 1)\Big]_0^4 = (\ln 3)(\ln 5) \approx 1.768$$

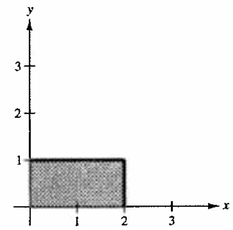
5. $\int_0^4\int_0^4 f(x, y)\,dy\,dx \approx (32 + 31 + 28 + 23) + (31 + 30 + 27 + 22) + (28 + 27 + 24 + 19) + (23 + 22 + 19 + 14)$

$= 400$

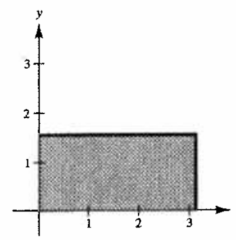
Using the corner of the ith square furthest from the origin, you obtain 272.

6. $\int_0^2\int_0^2 f(x, y)\,dy\,dx \approx 4 + 2 + 8 + 6 = 20$

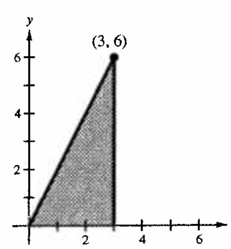
7. $$\int_0^2\int_0^1 (1 + 2x + 2y)\,dy\,dx = \int_0^2 \Big[y + 2xy + y^2\Big]_0^1 dx = \int_0^2 (2 + 2x)\,dx = \Big[2x + x^2\Big]_0^2 = 8$$



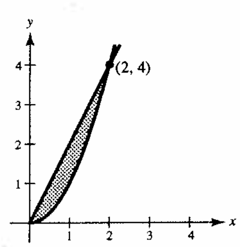
8. $$\int_0^{\pi}\int_0^{\pi/2} \sin^2 x\cos^2 y\,dy\,dx = \int_0^{\pi}\left[\frac{1}{2}\sin^2 x\left(y + \frac{1}{2}\sin 2y\right)\right]_0^{\pi/2} dx = \int_0^{\pi}\frac{1}{2}\sin^2 x\left(\frac{\pi}{2}\right)dx = \frac{\pi}{8}\int_0^{\pi}(1 - \cos 2x)\,dx = \left[\frac{\pi}{8}\left(x - \frac{1}{2}\sin 2x\right)\right]_0^{\pi} = \frac{\pi^2}{8}$$



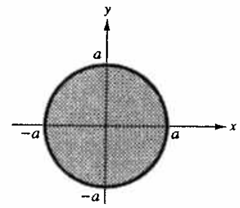
9. $$\int_0^6\int_{y/2}^3 (x + y)\,dx\,dy = \int_0^6\left[\frac{1}{2}x^2 + xy\right]_{y/2}^3 dy = \int_0^6\left(\frac{9}{2} + 3y - \frac{5}{8}y^2\right)dy = \left[\frac{9}{2}y + \frac{3}{2}y^2 - \frac{5}{24}y^3\right]_0^6 = 36$$



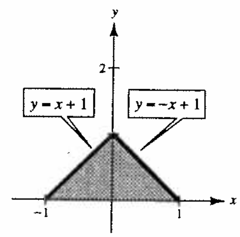
10. $$\int_0^4\int_{(1/2)y}^{\sqrt{y}} x^2y^2\,dx\,dy = \int_0^4\left[\frac{x^3y^2}{3}\right]_{(1/2)y}^{\sqrt{y}} dy = \int_0^4\left(\frac{y^{7/2}}{3} - \frac{y^5}{24}\right)dy = \left[\frac{2y^{9/2}}{27} - \frac{y^6}{144}\right]_0^4 = \frac{1024}{27} - \frac{256}{9} = \frac{256}{27}$$



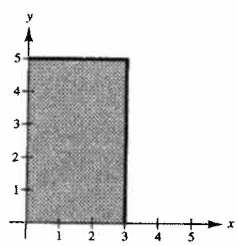
11. $$\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} (x + y)\,dy\,dx = \int_{-a}^{a}\left[xy + \frac{1}{2}y^2\right]_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dx$$
$$= \int_{-a}^{a} 2x\sqrt{a^2 - x^2}\,dx$$
$$= \left[-\frac{2}{3}(a^2 - x^2)^{3/2}\right]_{-a}^{a} = 0$$



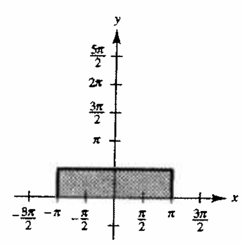
12. $$\int_0^1\int_{y-1}^{0} e^{x+y}\,dx\,dy + \int_0^1\int_0^{1-y} e^{x+y}\,dx\,dy = \int_0^1\left[e^{x+y}\right]_{y-1}^{0} dy + \int_0^1\left[e^{x+y}\right]_0^{1-y} dy$$
$$= \int_0^1 (e - e^{2y-1})\,dy$$
$$= \left[ey - \frac{1}{2}e^{2y-1}\right]_0^1$$
$$= \frac{1}{2}(e + e^{-1})$$



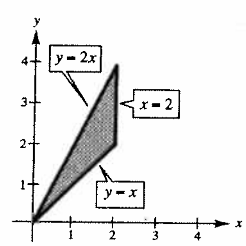
13. $$\int_0^5\int_0^3 xy\,dx\,dy = \int_0^3\int_0^5 xy\,dy\,dx$$
$$= \int_0^3\left[\frac{1}{2}xy^2\right]_0^5 dx$$
$$= \frac{25}{2}\int_0^3 x\,dx$$
$$= \left[\frac{25}{4}x^2\right]_0^3 = \frac{225}{4}$$



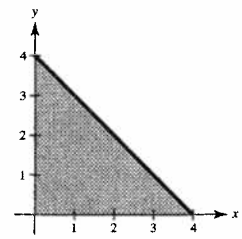
14. $$\int_0^{\pi/2}\int_{-\pi}^{\pi} \sin x \sin y\,dx\,dy = \int_{-\pi}^{\pi}\int_0^{\pi/2} \sin x \sin y\,dy\,dx$$
$$= \int_{-\pi}^{\pi}\left[-\sin x \cos y\right]_0^{\pi/2} dx$$
$$= \int_{-\pi}^{\pi} \sin x\,dx$$
$$= 0$$



15. $$\int_0^2\int_{y/2}^{y} \frac{y}{x^2 + y^2}\,dx\,dy + \int_2^4\int_{y/2}^{2} \frac{y}{x^2 + y^2}\,dx\,dy = \int_0^2\int_x^{2x} \frac{y}{x^2 + y^2}\,dy\,dx$$
$$= \frac{1}{2}\int_0^2\left[\ln(x^2 + y^2)\right]_x^{2x} dx$$
$$= \frac{1}{2}\int_0^2 (\ln 5x^2 - \ln 2x^2)\,dx$$
$$= \frac{1}{2}\ln\frac{5}{2}\int_0^2 dx$$
$$= \left[\frac{1}{2}\left(\ln\frac{5}{2}\right)x\right]_0^2 = \ln\frac{5}{2}$$



16. $\displaystyle\int_0^4\int_0^{4-x} xe^y\,dy\,dx = \int_0^4\int_0^{4-y} xe^y\,dx\,dy$



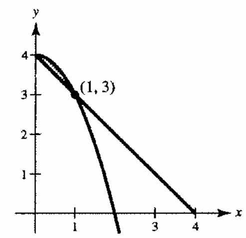
For the first integral, we obtain:

$$\int_0^4 \Big[xe^y\Big]_0^{4-x} dx = \int_0^4 (xe^{4-x} - x)\,dx$$

$$= \left[-e^{4-x}(1+x) - \frac{x^2}{2}\right]_0^4$$

$$= (-5 - 8) + e^4 = e^4 - 13.$$

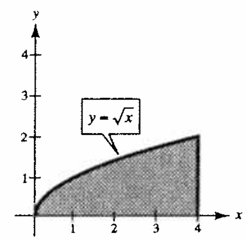
17. $\displaystyle\int_3^4\int_{4-y}^{\sqrt{4-y}} -2y\ln x\,dx\,dy = \int_0^1\int_{4-x}^{4-x^2} -2y\ln x\,dy\,dx$



$$= -\int_0^1 \Big[\ln x \cdot y^2\Big]_{4-x}^{4-x^2} dx$$

$$= -\int_0^1 [\ln x[(4-x^2)^2 - (4-x)^2]]\,dx = \frac{26}{25}$$

18. $\displaystyle\int_0^2\int_{y^2}^4 \frac{y}{1+x^2}\,dx\,dy = \int_0^4\int_0^{\sqrt{x}} \frac{y}{1+x^2}\,dy\,dx$

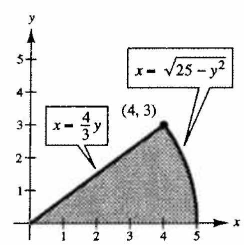


$$= \frac{1}{2}\int_0^4 \left[\frac{y^2}{1+x^2}\right]_0^{\sqrt{x}} dx$$

$$= \frac{1}{2}\int_0^4 \frac{x}{1+x^2}\,dx$$

$$= \left[\frac{1}{4}\ln(1+x^2)\right]_0^4 = \frac{1}{4}\ln(17)$$

19. $\displaystyle\int_0^4\int_0^{3x/4} x\,dy\,dx + \int_4^5\int_0^{\sqrt{25-x^2}} x\,dy\,dx = \int_0^3\int_{4y/3}^{\sqrt{25-y^2}} x\,dx\,dy$

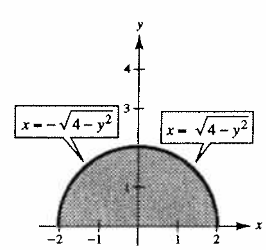


$$= \int_0^3 \left[\frac{1}{2}x^2\right]_{4y/3}^{\sqrt{25-y^2}} dy$$

$$= \frac{25}{18}\int_0^3 (9 - y^2)\,dy$$

$$= \left[\frac{25}{18}\left(9y - \frac{1}{3}y^3\right)\right]_0^3 = 25$$

20. $\displaystyle\int_0^2\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} (x^2 + y^2)\,dx\,dy$



$$= \int_{-2}^2\int_0^{\sqrt{4-x^2}} (x^2 + y^2)\,dy\,dx$$

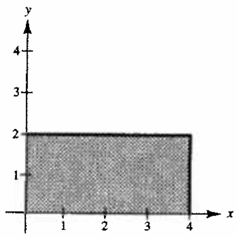
$$= \int_{-2}^2 \left[x^2y + \frac{1}{3}y^3\right]_0^{\sqrt{4-x^2}} dx$$

$$= \int_{-2}^2 \left[x^2\sqrt{4-x^2} + \frac{1}{3}(4-x^2)^{3/2}\right] dx$$

$$= \left[-\frac{x}{4}(4-x^2)^{3/2} + \frac{1}{2}\left(x\sqrt{4-x^2} + 4\arcsin\frac{x}{2}\right) + \frac{1}{12}\left[x(4-x^2)^{3/2} + 6x\sqrt{4-x^2} + 24\arctan\frac{x}{2}\right]\right]_{-2}^2 = 4\pi$$

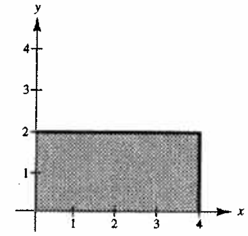
21. $\displaystyle\int_0^4\int_0^2 \frac{y}{2}\,dy\,dx = \int_0^4 \left[\frac{y^2}{4}\right]_0^2 dx$

$\displaystyle= \int_0^4 dx = 4$



22. $\displaystyle\int_0^4\int_0^2 (6-2y)\,dy\,dx = \int_0^4 \left[6y - y^2\right]_0^2 dx$

$\displaystyle= \int_0^4 8\,dx = 32$

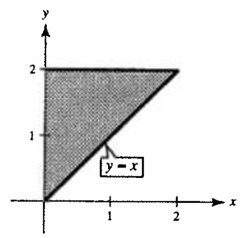


23. $\displaystyle\int_0^2\int_0^y (4-x-y)\,dx\,dy = \int_0^2 \left[4x - \frac{x^2}{2} - xy\right]_0^y dy$

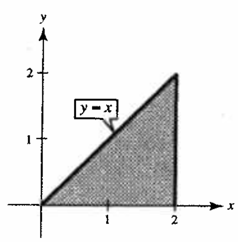
$\displaystyle= \int_0^2 \left(4y - \frac{y^2}{2} - y^2\right) dy$

$\displaystyle= \left[2y^2 - \frac{y^3}{6} - \frac{y^3}{3}\right]_0^2$

$\displaystyle= 8 - \frac{8}{6} - \frac{8}{3} = 4$



24. $\displaystyle\int_0^2\int_0^x 4\,dy\,dx = \int_0^2 4x\,dx = 2x^2\Big]_0^2 = 8$

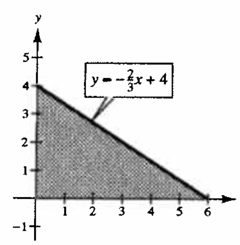


25. $\displaystyle\int_0^6\int_0^{(-2/3)x+4} \left(\frac{12-2x-3y}{4}\right) dy\,dx = \int_0^6 \left[\frac{1}{4}\left(12y - 2xy - \frac{3}{2}y^2\right)\right]_0^{(-2/3)x+4} dx$

$\displaystyle= \int_0^6 \left(\frac{1}{6}x^2 - 2x + 6\right) dx$

$\displaystyle= \left[\frac{1}{18}x^3 - x^2 + 6x\right]_0^6$

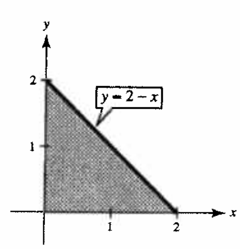
$= 12$



26. $\displaystyle\int_0^2\int_0^{2-x} (2-x-y)\,dy\,dx = \int_0^2 \left[2y - xy - \frac{y^2}{2}\right]_0^{2-x} dx$

$\displaystyle= \int_0^2 \frac{1}{2}(2-x)^2\,dx$

$\displaystyle= -\frac{1}{6}(x-2)^3\Big]_0^2 = \frac{4}{3}$

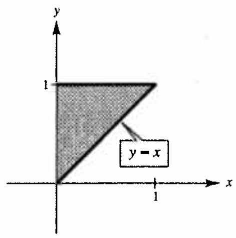


27. $\int_0^1 \int_0^y (1 - xy)\, dx\, dy = \int_0^1 \left[x - \frac{x^2y}{2}\right]_0^y dy$

$= \int_0^1 \left(y - \frac{y^3}{2}\right) dy$

$= \left[\frac{y^2}{2} - \frac{y^4}{8}\right]_0^1$

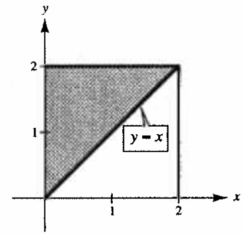
$= \frac{3}{8}$



28. $\int_0^2 \int_0^y (4 - y^2)\, dx\, dy = \int_0^2 (4y - y^3)\, dy$

$= \left[2y^2 - \frac{y^4}{4}\right]_0^2$

$= 4$



29. $\int_0^\infty \int_0^\infty \frac{1}{(x+1)^2(y+1)^2}\, dy\, dx = \int_0^\infty \left[-\frac{1}{(x+1)^2(y+1)}\right]_0^\infty dx = \int_0^\infty \frac{1}{(x+1)^2}\, dx = \left[-\frac{1}{(x+1)}\right]_0^\infty = 1$

30. $\int_0^\infty \int_0^\infty e^{-(x+y)/2}\, dy\, dx = \int_0^\infty \left[-2e^{-(x+y)/2}\right]_0^\infty dx = \int_0^\infty 2e^{-x/2}\, dx = \left[-4e^{-x/2}\right]_0^\infty = 4$

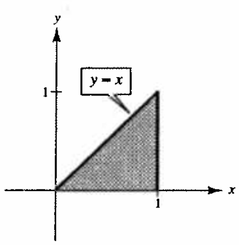
31. $4\int_0^2 \int_0^{\sqrt{4-x^2}} (4 - x^2 - y^2)\, dy\, dx = 8\pi$

32. $\int_0^1 \int_0^x \sqrt{1 - x^2}\, dy\, dx = \frac{1}{3}$

33. $V = \int_0^1 \int_0^x xy\, dy\, dx$

$= \int_0^1 \left[\frac{1}{2}xy^2\right]_0^x dx = \frac{1}{2}\int_0^1 x^3\, dx$

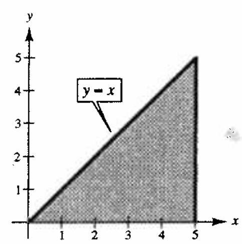
$= \left[\frac{1}{8}x^4\right]_0^1 = \frac{1}{8}$



34. $V = \int_0^5 \int_0^x x\, dy\, dx$

$= \int_0^5 \left[xy\right]_0^x dx = \int_0^5 x^2\, dx$

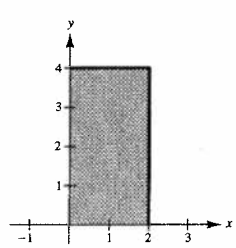
$= \left[\frac{1}{3}x^3\right]_0^5 = \frac{125}{3}$



35. $V = \int_0^2 \int_0^4 x^2\, dy\, dx$

$= \int_0^2 \left[x^2y\right]_0^4 dx = \int_0^2 4x^2\, dx$

$= \left[\frac{4x^3}{3}\right]_0^2 = \frac{32}{3}$



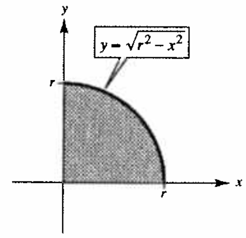
36. $V = 8\int_0^r \int_0^{\sqrt{r^2-x^2}} \sqrt{r^2 - x^2 - y^2}\, dy\, dx$

$= 4\int_0^r \left[\left[y\sqrt{r^2 - x^2 - y^2} + (r^2 - x^2)\arcsin \frac{y}{\sqrt{r^2 - x^2}}\right]\right]_0^{\sqrt{r^2-x^2}} dx$

$= 4\left(\frac{\pi}{2}\right)\int_0^r (r^2 - x^2)\, dx$

$= \left[2\pi\left(r^2x - \frac{1}{3}x^3\right)\right]_0^r$

$= \frac{4\pi r^3}{3}$



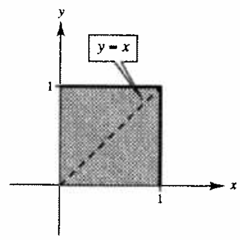
37. Divide the solid into two equal parts.

$V = 2\int_0^1 \int_0^x \sqrt{1 - x^2}\, dy\, dx$

$= 2\int_0^1 \left[y\sqrt{1 - x^2}\right]_0^x dx$

$= 2\int_0^1 x\sqrt{1 - x^2}\, dx$

$= \left[-\frac{2}{3}(1 - x^2)^{3/2}\right]_0^1 = \frac{2}{3}$



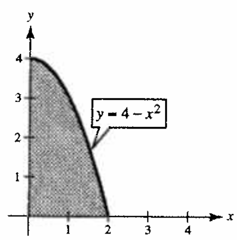
38. $V = \int_0^2 \int_0^{4-x^2} (4 - x^2)\, dy\, dx$

$= \int_0^2 (4 - x^2)(4 - x^2)\, dx$

$= \int_0^2 (16 - 8x^2 + x^4)\, dx$

$= \left[16x - 8\frac{x^3}{3} + \frac{x^5}{5}\right]_0^2$

$= 32 - \frac{64}{3} + \frac{32}{5} = \frac{256}{15}$



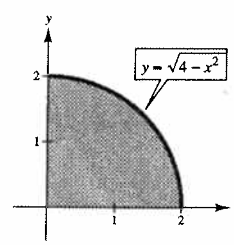
39. $V = \int_0^2 \int_0^{\sqrt{4-x^2}} (x + y)\, dy\, dx$

$= \int_0^2 \left[xy + \frac{1}{2}y^2\right]_0^{\sqrt{4-x^2}} dx$

$= \int_0^2 \left(x\sqrt{4 - x^2} + 2 - \frac{1}{2}x^2\right) dx$

$= \left[-\frac{1}{3}(4 - x^2)^{3/2} + 2x - \frac{1}{6}x^3\right]_0^2$

$= \frac{16}{3}$

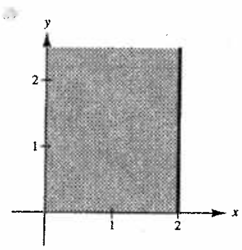


40. $V = \int_0^2 \int_0^{\infty} \frac{1}{1 + y^2}\, dy\, dx$

$= \int_0^2 \left[\arctan y\right]_0^{\infty} dx$

$= \int_0^2 \frac{\pi}{2}\, dx$

$= \left[\frac{\pi x}{2}\right]_0^2 = \pi$



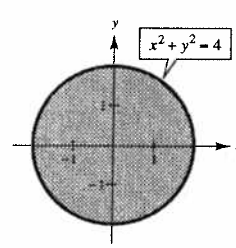
41. $V = 4\int_0^2 \int_0^{\sqrt{4-x^2}} (x^2 + y^2)\, dy\, dx$

$= 4\int_0^2 \left[x^2\sqrt{4 - x^2} + \frac{1}{3}(4 - x^2)^{3/2}\right] dx, \quad x = 2\sin\theta$

$= 4\int_0^{\pi/2} \left(16\cos^2\theta - \frac{32}{3}\cos^4\theta\right) d\theta$

$= 4\left[16\left(\frac{\pi}{4}\right) - \frac{32}{3}\left(\frac{3\pi}{16}\right)\right]$

$= 8\pi$

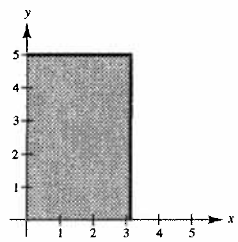


42. $V = \int_0^5 \int_0^{\pi} \sin^2 x \, dx \, dy$

$= \int_0^5 \frac{\pi}{2} \, dy$

$= \left[\frac{\pi}{2} y\right]_0^5$

$= \frac{5\pi}{2}$



43. $z = 9 - x^2 - y^2, z = 0$

$V = 4\int_0^3 \int_0^{\sqrt{9-x^2}} (9 - x^2 - y^2) \, dy \, dx = \frac{81\pi}{2}$

44. $V = \int_0^9 \int_0^{\sqrt{9-y}} \sqrt{9 - y} \, dx \, dy = \frac{81}{2}$

45. $V = \int_0^2 \int_0^{-0.5x+1} \frac{2}{1 + x^2 + y^2} \, dy \, dx \approx 1.2315$

46. $V = \int_0^{16} \int_0^{4-\sqrt{y}} \ln(1 + x + y) \, dx \, dy \approx 38.25$

47. f is a continuous function such that $0 \le f(x, y) \le 1$ over a region R of area 1. Let $f(m, n)$ = the minimum value of f over R and $f(M, N)$ = the maximum value of f over R. Then

$$f(m, n)\iint_R dA \le \iint_R f(x, y) \, dA \le f(M, N)\iint_R dA.$$

Since $\iint_R dA = 1$ and $0 \le f(m, n) \le f(M, N) \le 1$, we have $0 \le f(m, n)(1) \le \iint_R f(x, y) \, dA \le f(M, N)(1) \le 1$.

Therefore, $0 \le \iint_R f(x, y) \, dA \le 1$.

48. $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

$z = c\left(1 - \frac{x}{a} - \frac{y}{b}\right)$

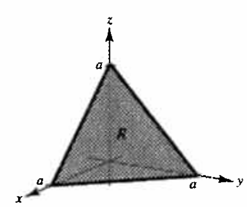
$V = \iint_R f(x, y) \, dA = \int_0^a \int_0^{b[1-(x/a)]} c\left(1 - \frac{x}{a} - \frac{y}{b}\right) dy \, dx$

$= c\int_0^a \left[y - \frac{xy}{a} - \frac{y^2}{2b}\right]_0^{b[1-(x/a)]} dx$

$= c\int_0^a \left[b\left(1 - \frac{x}{a}\right) - \frac{xb}{a}\left(1 - \frac{x}{a}\right) - \frac{b^2}{2b}\left(1 - \frac{x}{a}\right)^2\right] dx$

$= c\left[-\frac{ab}{2}\left(1 - \frac{x}{a}\right)^2 - \frac{x^2b}{2a} + \frac{x^3b}{3a^2} + \frac{ab}{6}\left(1 - \frac{x}{a}\right)^3\right]_0^a$

$= c\left[\left(-\frac{ab}{2} + \frac{ab}{3}\right) - \left(-\frac{ab}{2} + \frac{ab}{6}\right)\right] = \frac{abc}{6}$



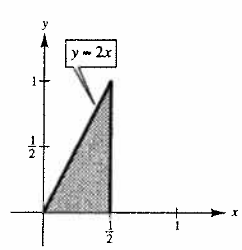
49. $\int_0^1 \int_{y/2}^{1/2} e^{-x^2} \, dx \, dy = \int_0^{1/2} \int_0^{2x} e^{-x^2} \, dy \, dx$

$= \int_0^{1/2} 2xe^{-x^2} \, dx$

$= \left[-e^{-x^2}\right]_0^{1/2}$

$= -e^{-1/4} + 1$

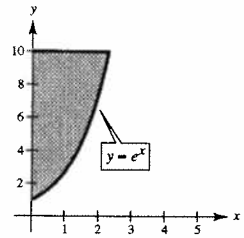
$= 1 - e^{-1/4} \approx 0.221$



50. $\displaystyle\int_0^{\ln 10}\int_{e^x}^{10} \frac{1}{\ln y}\,dy\,dx = \int_1^{10}\int_0^{\ln y} \frac{1}{\ln y}\,dx\,dy$

$$= \int_1^{10}\left[\frac{x}{\ln y}\right]_0^{\ln y} dy$$

$$= \int_1^{10} dy = \Big[y\Big]_1^{10} = 9$$

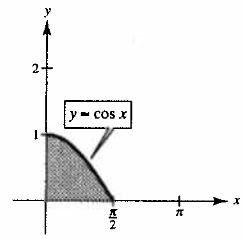


51. $\displaystyle\int_0^1\int_0^{\arccos y} \sin x\sqrt{1+\sin^2 x}\,dx\,dy$

$$= \int_0^{\pi/2}\int_0^{\cos x} \sin x\sqrt{1+\sin^2 x}\,dy\,dx$$

$$= \int_0^{\pi/2} (1+\sin^2 x)^{1/2}\sin x\cos x\,dx$$

$$= \left[\frac{1}{2}\cdot\frac{2}{3}(1+\sin^2 x)^{3/2}\right]_0^{\pi/2} = \frac{1}{3}\big[2\sqrt{2}-1\big]$$



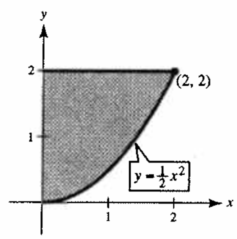
52. $\displaystyle\int_0^2\int_{(1/2)x^2}^{2} \sqrt{y}\cos y\,dy\,dx = \int_0^2\int_0^{\sqrt{2y}} \sqrt{y}\cos y\,dx\,dy$

$$= \int_0^2 \sqrt{y}\cos y\sqrt{2y}\,dy$$

$$= \sqrt{2}\int_0^2 y\cos y\,dy$$

$$= \sqrt{2}\Big[\cos y + y\sin y\Big]_0^2$$

$$= \sqrt{2}[\cos 2 + 2\sin 2 - 1]$$



53. $\displaystyle\text{Average} = \frac{1}{8}\int_0^4\int_0^2 x\,dy\,dx = \frac{1}{8}\int_0^4 2x\,dx = \left[\frac{x^2}{8}\right]_0^4 = 2$

54. $\displaystyle\text{Average} = \frac{1}{8}\int_0^4\int_0^2 xy\,dy\,dx = \frac{1}{8}\int_0^4 2x\,dx = \left[\frac{x^2}{8}\right]_0^4 = 2$

55. $\displaystyle\text{Average} = \frac{1}{4}\int_0^2\int_0^2 (x^2+y^2)\,dx\,dy$

$$= \frac{1}{4}\int_0^2\left[\frac{x^3}{3}+xy^2\right]_0^2 dy = \frac{1}{4}\int_0^2\left(\frac{8}{3}+2y^2\right)dy$$

$$= \left[\frac{1}{4}\left(\frac{8}{3}y+\frac{2}{3}y^3\right)\right]_0^2 = \frac{8}{3}$$

56. $\displaystyle\text{Average} = \frac{1}{1/2}\int_0^1\int_x^1 e^{x+y}\,dy\,dx = 2\int_0^1 e^{x+1}-e^{2x}\,dx$

$$= 2\left[e^{x+1}-\frac{1}{2}e^{2x}\right]_0^1 = 2\left[e^2-\frac{1}{2}e^2-e+\frac{1}{2}\right]$$

$$= e^2-2e+1$$

$$= (e-1)^2$$

57. See the definition on page 992.

58. The second is integrable. The first contains $\int \sin y^2\,dy$ which does not have an elementary antiderivation.

59. The value of $\int_R\int f(x, y)\,dA$ would be kB.

60. (a) The total snowfall in the county R.

(b) The average snowfall in R.

61. $\displaystyle\text{Average} = \frac{1}{1250}\int_{300}^{325}\int_{200}^{250} 100x^{0.6}y^{0.4}\,dx\,dy$

$$= \frac{1}{1250}\int_{300}^{325}\left[(100y^{0.4})\frac{x^{1.6}}{1.6}\right]_{200}^{250} dy = \frac{128{,}844.1}{1250}\int_{300}^{325} y^{0.4}\,dy = 103.0753\left[\frac{y^{1.4}}{1.4}\right]_{300}^{325} \approx 25{,}645.24$$

62. Average $= \dfrac{1}{150}\displaystyle\int_{45}^{60}\int_{40}^{50} [192x + 576y - x^2 - 5y^2 - 2xy - 5000]\,dx\,dy \approx 13{,}246.67$

63. $f(x, y) \geq 0$ for all (x, y) and

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\,dA = \int_0^5\int_0^2 \frac{1}{10}\,dy\,dx = \int_0^5 \frac{1}{5}\,dx = 1$$

$$P(0 \leq x \leq 2, 1 \leq y \leq 2) = \int_0^2\int_1^2 \frac{1}{10}\,dy\,dx = \int_0^2 \frac{1}{10}\,dx = \frac{1}{5}.$$

64. $f(x, y) \geq 0$ for all (x, y) and

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\,dA = \int_0^2\int_0^2 \frac{1}{4}xy\,dy\,dx = \int_0^2 \frac{x}{2}\,dx = 1$$

$$P(0 \leq x \leq 1, 1 \leq y \leq 2) = \int_0^1\int_1^2 \frac{1}{4}xy\,dy\,dx = \int_0^1 \frac{3x}{8}\,dx = \frac{3}{16}.$$

65. $f(x, y) \geq 0$ for all (x, y) and

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\,dA = \int_0^3\int_3^6 \frac{1}{27}(9 - x - y)\,dy\,dx$$

$$= \int_0^3 \frac{1}{27}\left[9y - xy - \frac{y^2}{2}\right]_3^6 dx = \int_0^3 \left(\frac{1}{2} - \frac{1}{9}x\right) dx = \left[\frac{x}{2} - \frac{x^2}{18}\right]_0^3 = 1$$

$$P(0 \leq x \leq 1, 4 \leq y \leq 6) = \int_0^1\int_4^6 \frac{1}{27}(9 - x - y)\,dy\,dx = \int_0^1 \frac{2}{27}(4 - x)\,dx = \frac{7}{27}.$$

66. $f(x, y) \geq 0$ for all (x, y) and

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\,dA = \int_0^{\infty}\int_0^{\infty} e^{-x-y}\,dy\,dx$$

$$= \int_0^{\infty} \lim_{b\to\infty}\left[-e^{-x-y}\right]_0^b dx = \int_0^{\infty} e^{-x}\,dx = \lim_{b\to\infty}\left[-e^{-x}\right]_0^b = 1$$

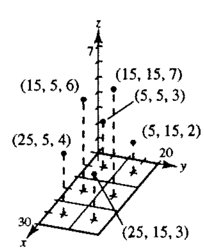
$$P(0 \leq x \leq 1, x \leq y \leq 1) = \int_0^1\int_x^1 e^{-x-y}\,dy\,dx = \int_0^1 \left[-e^{-x-y}\right]_x^1 dx = \int_0^1 (e^{-2x} - e^{-x-1})\,dx$$

$$= \left[-\frac{1}{2}e^{-2x} + e^{-x-1}\right]_0^1 = \frac{1}{2}e^{-2} - e^{-1} + \frac{1}{2} = \frac{1}{2}(e^{-1} - 1)^2 \approx 0.1998.$$

67. Divide the base into six squares, and assume the height at the center of each square is the height of the entire square.

Thus,

$V \approx (4 + 3 + 6 + 7 + 3 + 2)(100) = 2500\text{ m}^3.$



68. Sample Program for TI-82:

Program: DOUBLE

```
: Input A
: Input B
: Input M
: Input C
: Input D
: Input N
: 0 → V
: (B − A)/M → G
: (D − C)/N → H
: For (I, 1, M, 1)
: For (J, 1, N, 1)
: A + 0.5G(2I − 1) → x
: C + 0.5H(2J − 1) → y
: V + sin(√(x + y)) × G × H → V
: End
: End
: Disp V
```

69. $\int_0^1 \int_0^2 \sin \sqrt{x+y}\, dy\, dx \qquad m = 4, n = 8$

(a) 1.78435

(b) 1.7879

70. $\int_0^2 \int_0^4 20e^{-x^3/8}\, dy\, dx \qquad m = 10, n = 20$

(a) 129.2018

(b) 129.2756

71. $\int_4^6 \int_0^2 y \cos \sqrt{x}\, dx\, dy \qquad m = 4, n = 8$

(a) 11.0571

(b) 11.0414

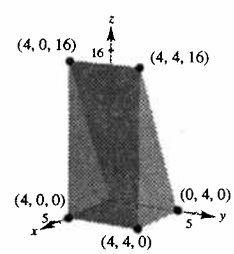
72. $\int_1^4 \int_1^2 \sqrt{x^3 + y^3}\, dx\, dy \qquad m = 6, n = 4$

(a) 13.956

(b) 13.9022

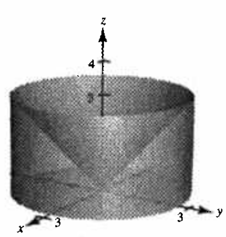
73. $V \approx 125$

Matches d.



74. $V \approx 50$

Matches a.



75. False

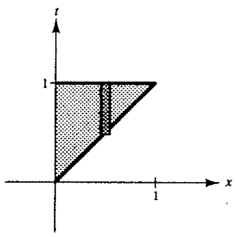
$$V = 8 \int_0^1 \int_0^{\sqrt{1-y^2}} \sqrt{1 - x^2 - y^2}\, dx\, dy$$

76. True

77. Average $= \int_0^1 f(x)\,dx = \int_0^1\int_1^x e^{t^2}\,dt\,dx = -\int_0^1\int_x^1 e^{t^2}\,dt\,dx$

$$= -\int_0^1\int_0^t e^{t^2}\,dx\,dt = -\int_0^1 te^{t^2}\,dt$$

$$= \left[-\frac{1}{2}e^{t^2}\right]_0^1 = -\frac{1}{2}(e-1) = \frac{1}{2}(1-e)$$



78. $\int_1^2 e^{-xy}\,dy = \left[-\frac{1}{x}e^{-xy}\right]_1^2 = \frac{e^{-x} - e^{-2x}}{x}$

Thus,

$$\int_0^\infty \frac{e^{-x} - e^{-2x}}{x}\,dx = \int_0^\infty\int_1^2 e^{-xy}\,dx\,dy$$

$$= \int_1^2\int_0^\infty e^{-xy}\,dx\,dy$$

$$= \int_1^2\left[-\frac{e^{-xy}}{y}\right]_0^\infty dy$$

$$= \int_1^2 \frac{1}{y}\,dy = \left[\ln y\right]_1^2 = \ln 2.$$

79. $z = 9 - x^2 - y^2$ is a paraboloid opening downward with vertex $(0, 0, 9)$. The double integral is maximized if $z \ge 0$. That is,

$$R = \{(x, y): x^2 + y^2 \le 9\}.$$

$\left[\text{The maximum value is } \iint_R (9 - x^2 - y^2)\,dA = \frac{81\pi}{2}.\right]$

80. $z = x^2 + y^2 - 4$ is a paraboloid opening upward with vertex $(0, 0, -4)$. The double integral is minimized if $z \le 0$. That is,

$$R = \{(x, y): x^2 + y^2 \le 4\}.$$

[The minimum value is -8π.]

81. $\int_0^2 (\tan^{-1}\pi x - \tan^{-1} x)\,dx$

$$= \int_0^2\int_x^{\pi x} \frac{1}{1+y^2}\,dy\,dx$$

$$= \int_0^2\int_{y/\pi}^{y} \frac{1}{1+y^2}\,dx\,dy + \int_2^{2\pi}\int_{y/\pi}^{2} \frac{1}{1+y^2}\,dx\,dy$$

$$= \int_0^2\left[\frac{x}{1+y^2}\right]_{y/\pi}^{y} dy + \int_2^{2\pi}\left[\frac{x}{1+y^2}\right]_{y/\pi}^{2} dy$$

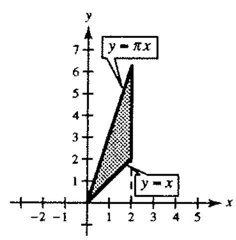
$$= \int_0^2\left[\frac{y}{1+y^2} - \frac{y/\pi}{1+y^2}\right]dy + \int_2^{2\pi}\left[\frac{2}{1+y^2} - \frac{y/\pi}{1+y^2}\right]dy$$

$$= \left[\frac{1}{2}\left(1 - \frac{1}{\pi}\right)\ln(1+y^2)\right]_0^2 + \left[2\tan^{-1}y - \frac{1}{2\pi}\ln(1+y^2)\right]_2^{2\pi}$$

$$= \frac{1}{2}\left(1 - \frac{1}{\pi}\right)\ln 5 + 2\tan^{-1}(2\pi) - \frac{1}{2\pi}\ln(1+4\pi^2) - 2\tan^{-1}(2) + \frac{1}{2\pi}\ln(5)$$

$$= \frac{1}{2}\ln 5 + 2\tan^{-1}(2\pi) - 2\tan^{-1}(2) - \frac{1}{2\pi}\ln(1+4\pi^2)$$

$$\approx 0.8274$$

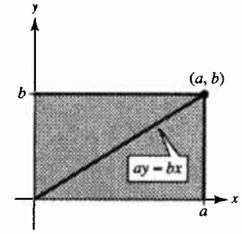


82. $\int_0^3\int_0^{\sqrt{9-y^2}} \sqrt{9 - x^2 - y^2}\,dx\,dy = \frac{9\pi}{2}$

because this double integral represents the portion of the sphere $x^2 + y^2 + z^2 = 9$ in the first octant.

$$V = \frac{1}{8}\frac{4}{3}\pi(3)^3 = \frac{9\pi}{2}$$

83. Let $I = \int_0^a \int_0^b e^{\max\{b^2x^2, a^2y^2\}} \, dy \, dx.$



Divide the rectangle into two parts by the diagonal line $ay = bx$.

On lower triangle, $b^2x^2 \ge a^2y^2$ because $y \le \frac{b}{a}x$.

$$I = \int_0^a \int_0^{bx/a} e^{b^2x^2} \, dy \, dx + \int_0^b \int_0^{ay/b} e^{a^2y^2} \, dx \, dy$$

$$= \int_0^a \frac{bx}{a} e^{b^2x^2} \, dx + \int_0^b \frac{ay}{b} e^{a^2y^2} \, dy$$

$$= \frac{1}{2ab}\left[e^{b^2x^2}\right]_0^a + \frac{1}{2ab}\left[e^{a^2y^2}\right]_0^b$$

$$= \frac{1}{2ab}[e^{b^2a^2} - 1 + e^{a^2b^2} - 1]$$

$$= \frac{e^{a^2b^2} - 1}{ab}$$

84. Assume such a function exists.

$$u(x) = 1 + \lambda \int_x^1 u(y)u(y - x) \, dy; \ \lambda > \frac{1}{2}, 0 \le x \le 1$$

$$\alpha = \int_0^1 u(x) \, dx = \int_0^1 dx + \lambda \int_0^1 \int_x^1 u(y)u(y - x) \, dy \, dx$$

Change the order of integration.

$$\alpha = \int_0^1 u(x) \, dx = 1 + \lambda \int_0^1 \int_0^y u(y)u(y - x) \, dx \, dy$$

$$= 1 + \lambda \int_0^1 u(y) \left[\int_0^y u(y - x) \, dx\right] dy$$

Hold y fixed and let $z = y - x$, $dz = -dx$.

$$\alpha = 1 + \lambda \int_0^1 u(y) \left[\int_y^0 u(z)(-dz)\right] dy$$

$$= 1 + \lambda \int_0^1 u(y) \left[\int_0^y u(z) \, dz\right] dy$$

Let $f(y) = \int_0^y u(z) \, dz$. Then $f'(y) = u(y), f(0) = 0, f(1) = \alpha$.

$$\alpha = 1 + \lambda \int_0^1 f'(y)f(y) \, dy$$

$$= 1 + \lambda \left[\frac{f(y)^2}{2}\right]_0^1$$

$$= 1 + \lambda \left[\frac{1}{2}f(1)^2 - \frac{1}{2}f(0)^2\right]$$

$$= 1 + \lambda \frac{1}{2} \alpha^2$$

$\lambda\alpha^2 - 2\alpha + 2 = 0.$

For α to exist, the discriminant of this quadratic must be nonnegative.

$$b^2 - 4ac = 4 - 8\lambda \ge 0 \implies \lambda \le \frac{1}{2}$$

But, $\lambda > \frac{1}{2}$, a contradiction.

Section 14.3 Change of Variables: Polar Coordinates

1. Rectangular coordinates

2. Polar coordinates

3. Polar coordinates

4. Rectangular coordinates

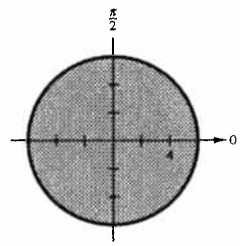
5. $R = \{(r, \theta): 0 \le r \le 8, 0 \le \theta \le \pi\}$

6. $R = \{(r, \theta): 0 \le r \le 4 \sin \theta, 0 \le \theta \le \pi\}$

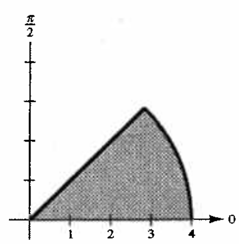
7. $R = \{(r, \theta): 0 \le r \le 3 + 3 \sin \theta, 0 \le \theta \le 2\pi\}$ Cardioid

8. $R = \{(r, \theta): 0 \le r \le 4 \cos 3\theta, 0 \le \theta \le \pi\}$

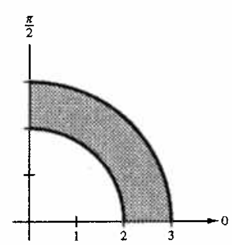
9.
$$\int_0^{2\pi}\int_0^6 3r^2 \sin\theta \, dr\, d\theta = \int_0^{2\pi}\Big[r^3 \sin\theta\Big]_0^6 d\theta = \int_0^{2\pi} 216 \sin\theta \, d\theta = \Big[-216\cos\theta\Big]_0^{2\pi} = 0$$



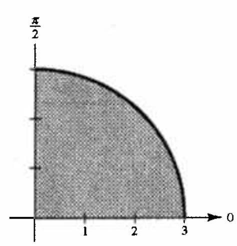
10.
$$\int_0^{\pi/4}\int_0^4 r^2 \sin\theta\cos\, dr\, d\theta = \int_0^{\pi/4}\left[\frac{r^3}{3}\sin\theta\cos\theta\right]_0^4 d\theta = \left[\left(\frac{64}{3}\right)\frac{\sin^2\theta}{2}\right]_0^{\pi/4} = \frac{16}{3}$$



11.
$$\int_0^{\pi/2}\int_2^3 \sqrt{9 - r^2}\, r\, dr\, d\theta = \int_0^{\pi/2}\left[-\frac{1}{3}(9 - r^2)^{3/2}\right]_2^3 d\theta = \left[\frac{5\sqrt{5}}{3}\theta\right]_0^{\pi/2} = \frac{5\sqrt{5}\pi}{6}$$

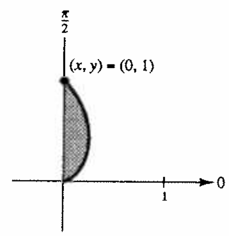


12.
$$\int_0^{\pi/2}\int_0^3 re^{-r^2}\, dr\, d\theta = \int_0^{\pi/2}\left[-\frac{1}{2}e^{-r^2}\right]_0^3 d\theta = \left[-\frac{1}{2}(e^{-9} - 1)\theta\right]_0^{\pi/2} = \frac{\pi}{4}\left(1 - \frac{1}{e^9}\right)$$



13.
$$\int_0^{\pi/2}\int_0^{1+\sin\theta} \theta r\, dr\, d\theta = \int_0^{\pi/2}\left[\frac{\theta r^2}{2}\right]_0^{1+\sin\theta} d\theta$$
$$= \int_0^{\pi/2} \frac{1}{2}\theta(1 + \sin\theta)^2\, d\theta$$
$$= \left[\frac{1}{8}\theta^2 + \sin\theta - \theta\cos\theta + \frac{1}{2}\theta\left(-\frac{1}{2}\cos\theta \cdot \sin\theta + \frac{1}{2}\theta\right) + \frac{1}{8}\sin^2\theta\right]_0^{\pi/2}$$
$$= \frac{3}{32}\pi^2 + \frac{9}{8}$$

14. $$\int_0^{\pi/2}\int_0^{1-\cos\theta}(\sin\theta)r\,dr\,d\theta = \int_0^{\pi/2}\left[(\sin\theta)\frac{r^2}{2}\right]_0^{1-\cos\theta}d\theta$$
$$= \int_0^{\pi/2}\frac{\sin\theta}{2}(1-\cos\theta)^2\,d\theta$$
$$= \left[\frac{1}{6}(1-\cos(\theta))^3\right]_0^{\pi/2} = \frac{1}{6}$$



15. $$\int_0^a\int_0^{\sqrt{a^2-y^2}} y\,dx\,dy = \int_0^{\pi/2}\int_0^a r^2\sin\theta\,dr\,d\theta = \frac{a^3}{3}\int_0^{\pi/2}\sin\theta\,d\theta = \left[\frac{a^3}{3}(-\cos\theta)\right]_0^{\pi/2} = \frac{a^3}{3}$$

16. $$\int_0^a\int_0^{\sqrt{a^2-x^2}} x\,dy\,dx = \int_0^{\pi/2}\int_0^a r^2\cos\theta\,dr\,d\theta = \frac{a^3}{3}\int_0^{\pi/2}\cos\theta\,d\theta = \left[\frac{a^3}{3}\sin\theta\right]_0^{\pi/2} = \frac{a^3}{3}$$

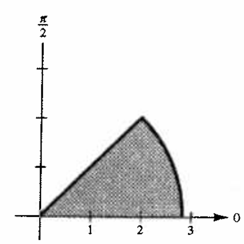
17. $$\int_0^3\int_0^{\sqrt{9-x^2}}(x^2+y^2)^{3/2}\,dy\,dx = \int_0^{\pi/2}\int_0^3 r^4\,dr\,d\theta = \frac{243}{5}\int_0^{\pi/2}d\theta = \frac{243\pi}{10}$$

18. $$\int_0^2\int_y^{\sqrt{8-y^2}}\sqrt{x^2+y^2}\,dx\,dy = \int_0^{\pi/4}\int_0^{2\sqrt{2}} r^2\,dr\,d\theta$$
$$= \int_0^{\pi/4}\frac{(2\sqrt{2})^3}{3}\,d\theta = \left[\frac{(2\sqrt{2})^3}{3}\theta\right]_0^{\pi/4} = \frac{(2\sqrt{2})^3}{3}\cdot\frac{\pi}{4} = \frac{4\sqrt{2}\pi}{3}$$

19. $$\int_0^2\int_0^{\sqrt{2x-x^2}} xy\,dy\,dx = \int_0^{\pi/2}\int_0^{2\cos\theta} r^3\cos\theta\sin\theta\,dr\,d\theta = 4\int_0^{\pi/2}\cos^5\theta\sin\,d\theta = \left[-\frac{4\cos^6\theta}{6}\right]_0^{\pi/2} = \frac{2}{3}$$

20. $$\int_0^4\int_0^{\sqrt{4y-y^2}} x^2\,dx\,dy = \int_0^{\pi/2}\int_0^{4\sin\theta} r^3\cos^2\theta\,dr\,d\theta = \int_0^{\pi/2} 64\sin^4\theta\cos^2 d\theta$$
$$= 64\int_0^{\pi/2}(\sin^4\theta - \sin^6\theta)\,d\theta = \frac{64}{6}\left[\sin^5\theta\cos\theta - \frac{\sin^3\theta\cos\theta}{4} + \frac{3}{8}(\theta - \sin\theta\cos\theta)\right]_0^{\pi/2} = 2\pi$$

21. $$\int_0^2\int_0^x\sqrt{x^2+y^2}\,dy\,dx + \int_2^{2\sqrt{2}}\int_0^{\sqrt{8-x^2}}\sqrt{x^2+y^2}\,dy\,dx = \int_0^{\pi/4}\int_0^{2\sqrt{2}} r^2\,dr\,d\theta$$
$$= \int_0^{\pi/4}\frac{16\sqrt{2}}{3}\,d\theta$$
$$= \frac{4\sqrt{2}\pi}{3}$$



22. $$\int_0^{(5\sqrt{2})/2}\int_0^x xy\,dy\,dx + \int_{(5\sqrt{2})/2}^5\int_0^{\sqrt{25-x^2}} xy\,dy\,dx = \int_0^{\pi/4}\int_0^5 r^3\sin\theta\cos\theta\,dr\,d\theta$$
$$= \int_0^{\pi/4}\frac{625}{4}\sin\theta\cos\,d\theta$$
$$= \left[\frac{625}{8}\sin^2\theta\right]_0^{\pi/4}$$
$$= \frac{625}{16}$$

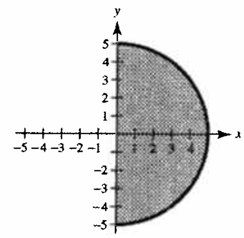
23. $\displaystyle\int_0^2\int_0^{\sqrt{4-x^2}} (x+y)\,dy\,dx = \int_0^{\pi/2}\int_0^2 (r\cos\theta + r\sin\theta)r\,dr\,d\theta = \int_0^{\pi/2}\int_0^2 (\cos\theta+\sin\theta)r^2\,dr\,d\theta$

$$= \frac{8}{3}\int_0^{\pi/2} (\cos\theta+\sin\theta)\,d\theta = \left[\frac{8}{3}(\sin\theta - \cos\theta)\right]_0^{\pi/2} = \frac{16}{3}$$

24. $\displaystyle\int_{-\pi/2}^{\pi/2}\int_0^5 e^{-r^2/2}r\,dr\,d\theta = \int_{-\pi/2}^{\pi/2}\left[-e^{-r^2/2}\right]_0^5 d\theta$

$$= \int_{-\pi/2}^{\pi/2} (1 - e^{-25/2})\,d\theta$$

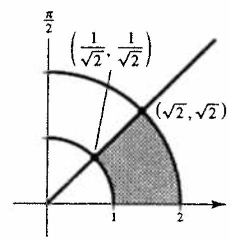
$$= \left[(1-e^{-25/2})\theta\right]_{-\pi/2}^{\pi/2} = \pi(1-e^{-25/2})$$



25. $\displaystyle\int_0^{1/\sqrt{2}}\int_{\sqrt{1-y^2}}^{\sqrt{4-y^2}} \arctan\frac{y}{x}\,dx\,dy + \int_{1/\sqrt{2}}^{\sqrt{2}}\int_y^{\sqrt{4-y^2}} \arctan\frac{y}{x}\,dx\,dy$

$$= \int_0^{\pi/4}\int_1^2 \theta r\,dr\,d\theta$$

$$= \int_0^{\pi/4} \frac{3}{2}\theta\,d\theta = \left[\frac{3\theta^2}{4}\right]_0^{\pi/4} = \frac{3\pi^2}{64}$$



26. $\displaystyle\int_0^3\int_0^{\sqrt{9-x^2}} (9-x^2-y^2)\,dy\,dx = \int_0^{\pi/2}\int_0^3 (9-r^2)r\,dr\,d\theta$

$$= \int_0^{\pi/2}\int_0^3 (9r - r^3)\,dr\,d\theta = \int_0^{\pi/2}\left[\frac{9}{2}r^2 - \frac{1}{4}r^4\right]_0^3 d\theta = \frac{81}{4}\int_0^{\pi/2} d\theta = \frac{81\pi}{8}$$

27. $\displaystyle V = \int_0^{\pi/2}\int_0^1 (r\cos\theta)(r\sin\theta)r\,dr\,d\theta$

$$= \frac{1}{2}\int_0^{\pi/2}\int_0^1 r^3\sin 2\theta\,dr\,d\theta = \frac{1}{8}\int_0^{\pi/2}\sin 2\theta\,d\theta = \left[-\frac{1}{16}\cos 2\theta\right]_0^{\pi/2} = \frac{1}{8}$$

28. $\displaystyle V = 4\int_0^{\pi/2}\int_0^1 (r^2+3)r\,dr\,d\theta = 4\int_0^{\pi/2}\left(\frac{r^4}{4} + \frac{3r^2}{2}\right)_0^1 d\theta$

$$= 4\int_0^{\pi/2} \frac{7}{4}\,d\theta = \frac{7\pi}{2}$$

29. $\displaystyle V = \int_0^{2\pi}\int_0^5 r^2\,dr\,d\theta = \int_0^{2\pi} \frac{125}{3}\,d\theta = \frac{250\pi}{3}$

30. $\displaystyle\int\int_R \ln(x^2+y^2)\,dA = \int_0^{2\pi}\int_1^2 (\ln r^2)r\,dr\,d\theta$

$$= 2\int_0^{2\pi}\int_1^2 r\ln r\,dr\,d\theta$$

$$= 2\int_0^{2\pi}\left[\frac{r^2}{4}(-1 + 2\ln r)\right]_1^2 d\theta$$

$$= 2\int_0^{2\pi}\left(\ln 4 - \frac{3}{4}\right)d\theta$$

$$= 4\pi\left(\ln 4 - \frac{3}{4}\right)$$

31. $V = 2\int_0^{\pi/2}\int_0^{4\cos\theta} \sqrt{16 - r^2}\, r\, dr\, d\theta = 2\int_0^{\pi/2}\left[-\frac{1}{3}\left(\sqrt{16 - r^2}\right)^3\right]_0^{4\cos\theta} d\theta = -\frac{2}{3}\int_0^{\pi/2} (64\sin^3\theta - 64)\, d\theta$

$= \frac{128}{3}\int_0^{\pi/2} [1 - \sin\theta(1 - \cos^2\theta)]\, d\theta = \frac{128}{3}\left[\theta + \cos\theta - \frac{\cos^3\theta}{3}\right]_0^{\pi/2} = \frac{64}{9}(3\pi - 4)$

32. $V = \int_0^{2\pi}\int_1^4 \sqrt{16 - r^2}\, r\, dr\, d\theta = \int_0^{2\pi}\left[-\frac{1}{3}\left(\sqrt{16 - r^2}\right)^3\right]_1^4 d\theta = \int_0^{2\pi} 5\sqrt{15}\, d\theta = 10\sqrt{15}\pi$

33. $V = \int_0^{2\pi}\int_a^4 \sqrt{16 - r^2}\, r\, dr\, d\theta = \int_0^{2\pi}\left[-\frac{1}{3}\left(\sqrt{16 - r^2}\right)^3\right]_a^4 d\theta = \frac{1}{3}\left(\sqrt{16 - a^2}\right)^3(2\pi)$

One-half the volume of the hemisphere is $(64\pi)/3$.

$$\frac{2\pi}{3}(16 - a^2)^{3/2} = \frac{64\pi}{3}$$

$$(16 - a^2)^{3/2} = 32$$

$$16 - a^2 = 32^{2/3}$$

$$a^2 = 16 - 32^{2/3} = 16 - 8\sqrt[3]{2}$$

$$a = \sqrt{4\left(4 - 2\sqrt[3]{2}\right)} = 2\sqrt{4 - 2\sqrt[3]{2}} \approx 2.4332$$

34. $x^2 + y^2 + z^2 = a^2 \implies z = \sqrt{a^2 - (x^2 + y^2)} = \sqrt{a^2 - r^2}$

$V = 8\int_0^{\pi/2}\int_0^a \sqrt{a^2 - r^2}\, r\, dr\, d\theta$ (8 times the volume in the first octant)

$$= 8\int_0^{\pi/2}\left[-\frac{1}{2}\cdot\frac{2}{3}(a^2 - r^2)^{3/2}\right]_0^a d\theta$$

$$= 8\int_0^{\pi/2} \frac{a^3}{3}\, d\theta = \left[\frac{8a^3}{3}\theta\right]_0^{\pi/2} = \frac{4\pi a^3}{3}$$

35. Total volume $= V = \int_0^{2\pi}\int_0^4 25e^{-r^2/4} r\, dr\, d\theta$

$$= \int_0^{2\pi}\left[-50e^{-r^2/4}\right]_0^4 d\theta$$

$$= \int_0^{2\pi} -50(e^{-4} - 1)\, d\theta$$

$$= (1 - e^{-4})100\pi \approx 308.40524$$

Let c be the radius of the hole that is removed.

$$\frac{1}{10}V = \int_0^{2\pi}\int_0^c 25e^{-r^2/4} r\, dr\, d\theta = \int_0^{2\pi}\left[-50e^{-r^2/4}\right]_0^c d\theta$$

$$= \int_0^{2\pi} -50(e^{-c^2/4} - 1)\, d\theta \implies 30.84052 = 100\pi(1 - e^{-c^2/4})$$

$$\implies e^{-c^2/4} = 0.90183$$

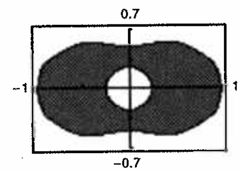
$$-\frac{c^2}{4} = -0.10333$$

$$c^2 = 0.41331$$

$$c = 0.6429$$

$$\implies \text{diameter} = 2c = 1.2858$$

36. $\dfrac{-9}{4(x^2+y^2+9)} \le z \le \dfrac{9}{4(x^2+y^2+9)}$; $\dfrac{1}{4} \le r \le \dfrac{1}{2}(1+\cos^2\theta)$



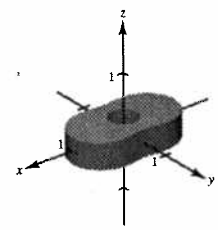
(a) $\dfrac{-9}{4r^2+36} \le z \le \dfrac{9}{4r^2+36}$

(b) Perimeter $= \displaystyle\int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$

$$r = \frac{1}{2}(1+\cos^2\theta) = \frac{1}{2} + \frac{1}{2}\cos^2\theta$$

$$\frac{dr}{d\theta} = -\cos\theta\sin\theta$$

$$\text{Perimeter} = 2\int_0^\pi \sqrt{\frac{1}{4}(1+\cos^2\theta)^2 + \cos^2\theta\sin^2\theta}\, d\theta \approx 5.21$$



(c) $V = 2\displaystyle\int_0^{2\pi}\int_{1/4}^{1/2(1+\cos^2\theta)} \frac{9}{4r^2+36}\, r\, dr\, d\theta \approx 0.8000$

37. $A = \displaystyle\int_0^\pi\int_0^{6\cos\theta} r\,dr\,d\theta = \int_0^\pi 18\cos^2\theta\,d\theta = 9\int_0^\pi (1+\cos 2\theta)\,d\theta = \left[9\left(\theta + \frac{1}{2}\sin 2\theta\right)\right]_0^\pi = 9\pi$

38. $A = \displaystyle\int_0^{2\pi}\int_2^4 r\,dr\,d\theta = \int_0^{2\pi} 6\,d\theta = 12\pi$

39. $\displaystyle\int_0^{2\pi}\int_0^{1+\cos\theta} r\,dr\,d\theta = \frac{1}{2}\int_0^{2\pi}(1+2\cos\theta+\cos^2\theta)\,d\theta$

$$= \frac{1}{2}\int_0^{2\pi}\left(1 + 2\cos\theta + \frac{1+\cos 2\theta}{2}\right)d\theta = \frac{1}{2}\left[\theta + 2\sin\theta + \frac{1}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right)\right]_0^{2\pi} = \frac{3\pi}{2}$$

40. $\displaystyle\int_0^{2\pi}\int_0^{2+\sin\theta} r\,dr\,d\theta = \frac{1}{2}\int_0^{2\pi}(2+\sin\theta)^2\,d\theta = \frac{1}{2}\int_0^{2\pi}(4+4\sin\theta+\sin^2\theta)\,d\theta = \frac{1}{2}\int_0^{2\pi}\left(4+4\sin\theta+\frac{1-\cos 2\theta}{2}\right)d\theta$

$$= \frac{1}{2}\left[4\theta - 4\cos\theta + \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right]_0^{2\pi} = \frac{1}{2}[8\pi - 4 + \pi + 4] = \frac{9\pi}{2}$$

41. $3\displaystyle\int_0^{\pi/3}\int_0^{2\sin 3\theta} r\,dr\,d\theta = \frac{3}{2}\int_0^{\pi/3} 4\sin^2 3\theta\,d\theta = 3\int_0^{\pi/3}(1-\cos 6\theta)\,d\theta = 3\left[\theta - \frac{1}{6}\sin 6\theta\right]_0^{\pi/3} = \pi$

42. $8\displaystyle\int_0^{\pi/4}\int_0^{3\cos 2\theta} r\,dr\,d\theta = 4\int_0^{\pi/4} 9\cos^2 2\theta\,d\theta = 18\int_0^{\pi/4}(1+\cos 4\theta)\,d\theta = 18\left[\theta + \frac{1}{4}\sin 4\theta\right]_0^{\pi/4} = \frac{9\pi}{2}$

43. Let R be a region bounded by the graphs of $r = g_1(\theta)$ and $r = g_2(\theta)$, and the lines $\theta = a$ and $\theta = b$.

When using polar coordinates to evaluate a double integral over R, R can be partitioned into small polar sectors.

44. See Theorem 14.3.

45. r-simple regions have fixed bounds for θ.

θ-simple regions have fixed bounds for r.

46. (a) Horizontal or polar representative elements

(b) Polar representative element

(c) Vertical or polar

47. You would need to insert a factor of r because of the $r\,dr\,d\theta$ nature of polar coordinate integrals. The plane regions would be sectors of circles.

48. (a) The volume of the subregion determined by the point $(5, \pi/16, 7)$ is base × height $= (5 \cdot 10 \cdot \pi/8)(7)$. Adding up the 20 volumes, ending with $(45 \cdot 10 \cdot \pi/8)(12)$, you obtain

$$V \approx 10 \cdot \frac{\pi}{8}[5(7 + 9 + 9 + 5) + 15(8 + 10 + 11 + 8) + 25(10 + 14 + 15 + 11) + 35(12 + 15 + 18 + 16) + 45(9 + 10 + 14 + 12)]$$
$$= \frac{5\pi}{4}[150 + 555 + 1250 + 2135 + 2025] \approx \frac{5\pi}{4}[6115] \approx 24013.5 \text{ ft}^3$$

(b) $(57)(24013.5) = 1{,}368{,}769.5$ pounds

(c) $(7.48)(24103.5) \approx 179{,}621$ gallons

49. $\displaystyle\int_{\pi/4}^{\pi/2}\int_0^5 r\sqrt{1 + r^3}\sin\sqrt{\theta}\,dr\,d\theta \approx 56.051$

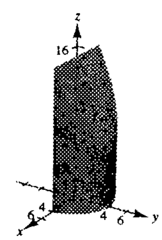
$\left[\textbf{Note:} \text{ This integral equals } \left(\int_{\pi/4}^{\pi/2} \sin\sqrt{\theta}\,d\theta\right)\left(\int_0^5 r\sqrt{1 + r^3}\,dr\right).\right]$

50. $\displaystyle\int_0^{\pi/4}\int_0^4 5e^{\sqrt{r\theta}}\,r\,dr\,d\theta \approx 87.130$

51. Volume = base × height

$\approx 8\pi \times 12 \approx 300$

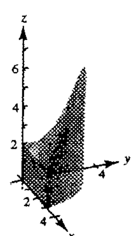
Answer (c)



52. Volume = base × height

$\approx \frac{9}{4}\pi \times 3 \approx 21$

Answer (a)



53. False

Let $f(r, \theta) = r - 1$ where R is the circular sector $0 \le r \le 6$ and $0 \le \theta \le \pi$. Then,

$$\int_R\int (r - 1)\,dA > 0 \quad \text{but} \quad r - 1 \not> 0 \text{ for all } r.$$

54. True

55. (a) $\displaystyle I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)/2}\,dA = 4\int_0^{\pi/2}\int_0^{\infty} e^{-r^2/2} r\,dr\,d\theta = 4\int_0^{\pi/2}\left[-e^{-r^2/2}\right]_0^{\infty} d\theta = 4\int_0^{\pi/2} d\theta = 2\pi$

(b) Therefore, $I = \sqrt{2\pi}$.

56. (a) Let $u = \sqrt{2}x$, then $\displaystyle\int_{-\infty}^{\infty} e^{-x^2}\,dx = \int_{-\infty}^{\infty} e^{-u^2/2}\frac{1}{\sqrt{2}}\,du = \frac{1}{\sqrt{2}}\left(\sqrt{2\pi}\right) = \sqrt{\pi}$.

(b) Let $u = 2x$, then $\displaystyle\int_{-\infty}^{\infty} e^{-4x^2}\,dx = \int_{-\infty}^{\infty} e^{-u^2}\frac{1}{2}\,du = \frac{1}{2}\sqrt{\pi}$.

57. $\displaystyle\int_{-7}^{7}\int_{-\sqrt{49-x^2}}^{\sqrt{49-x^2}} 4000e^{-0.01(x^2+y^2)}\,dy\,dx = \int_0^{2\pi}\int_0^7 4000e^{-0.01r^2}\,r\,dr\,d\theta = \int_0^{2\pi}\Big[-200{,}000e^{-0.01r^2}\Big]_0^7\,d\theta$

$= 2\pi(-200{,}000)(e^{-0.49} - 1) = 400{,}000\pi(1 - e^{-0.49}) \approx 486{,}788$

58. $\displaystyle\int_0^{\infty}\int_0^{\infty} ke^{-(x^2+y^2)}\,dy\,dx = \int_0^{\pi/2}\int_0^{\infty} ke^{-r^2}\,r\,dr\,d\theta = \int_0^{\pi/2}\Big[-\frac{k}{2}e^{-r^2}\Big]_0^{\infty}\,d\theta = \int_0^{\pi/2}\frac{k}{2}\,d\theta = \frac{k\pi}{4}$

For $f(x, y)$ to be a probability density function,

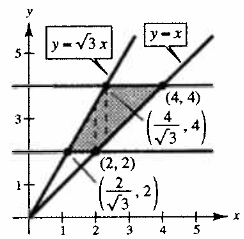
$\dfrac{k\pi}{4} = 1$

$k = \dfrac{4}{\pi}.$

59. (a) $\displaystyle\int_2^4\int_{y/\sqrt{3}}^{y} f\,dx\,dy$

(b) $\displaystyle\int_{2/\sqrt{3}}^{2}\int_2^{\sqrt{3}x} f\,dy\,dx + \int_2^{4/\sqrt{3}}\int_x^{\sqrt{3}x} f\,dy\,dx + \int_{4/\sqrt{3}}^{4}\int_x^{4} f\,dy\,dx$

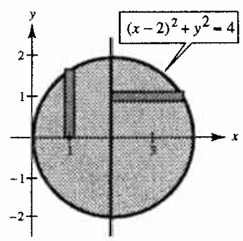
(c) $\displaystyle\int_{\pi/4}^{\pi/3}\int_{2\csc\theta}^{4\csc\theta} fr\,dr\,d\theta$



60. (a) $\displaystyle 4\int_0^2\int_2^{2+\sqrt{4-y^2}} f\,dx\,dy$

(b) $\displaystyle 4\int_0^2\int_0^{\sqrt{4-(x-2)^2}} f\,dy\,dx$

(c) $\displaystyle 2\int_0^{\pi/2}\int_0^{4\cos\theta} fr\,dr\,d\theta$



61. $A = \dfrac{\Delta\theta r_2^2}{2} - \dfrac{\Delta\theta r_1^2}{2} = \Delta\theta\left(\dfrac{r_1 + r_2}{2}\right)(r_2 - r_1) = r\,\Delta r\,\Delta\theta$

Section 14.4 Center of Mass and Moments of Inertia

1. $m = \displaystyle\int_0^4\int_0^3 xy\,dy\,dx = \int_0^4\left[\frac{xy^2}{2}\right]_0^3 dx$

$= \displaystyle\int_0^4 \frac{9}{2}x\,dx = \left[\frac{9x^2}{4}\right]_0^4 = 36$

2. $m = \displaystyle\int_0^3\int_0^{9-x^2} xy\,dy\,dx = \int_0^3\left[\frac{xy^2}{2}\right]_0^{9-x^2} dx$

$= \displaystyle\int_0^3 \frac{x(9 - x^2)^2}{2}\,dx$

$= \left[-\dfrac{1}{4}\dfrac{(9 - x^2)^3}{3}\right]_0^3$

$= -\dfrac{1}{12}(0 - 9^3) = \dfrac{243}{4}$

3. $m = \displaystyle\int_0^{\pi/2}\int_0^2 (r\cos\theta)(r\sin\theta)r\,dr\,d\theta = \int_0^{\pi/2}\int_0^2 \cos\theta\sin\theta\cdot r^3\,dr\,d\theta$

$= \displaystyle\int_0^{\pi/2} 4\cos\theta\sin\theta\,d\theta$

$= \left[4\dfrac{\sin^2\theta}{2}\right]_0^{\pi/2} = 2$

4. $m = \int_0^3 \int_3^{3+\sqrt{9-x^2}} xy\,dy\,dx = \int_0^3 \left[x\frac{y^2}{2}\right]_3^{3+\sqrt{9-x^2}} dx$

$= \int_0^3 \frac{x}{2}\left(\left(3 + \sqrt{9-x^2}\right) - 9\right) dx$

$= \frac{1}{2}\int_0^3 \left[6x\sqrt{9-x^2} + 9x - x^3\right] dx$

$= \frac{1}{2}\left[-2(9-x^2)^{3/2} + \frac{9x^2}{2} - \frac{x^4}{4}\right]_0^3$

$= \frac{1}{2}\left[\frac{81}{2} - \frac{81}{4} + 54\right] = \frac{297}{8}$

5. (a) $m = \int_0^a \int_0^b k\,dy\,dx = kab$

$M_x = \int_0^a \int_0^b ky\,dy\,dx = \frac{kab^2}{2}$

$M_y = \int_0^a \int_0^b kx\,dy\,dx = \frac{ka^2b}{2}$

$\bar{x} = \frac{M_y}{m} = \frac{ka^2b/2}{kab} = \frac{a}{2}$

$\bar{y} = \frac{M_x}{m} = \frac{kab^2/2}{kab} = \frac{b}{2}$

$(\bar{x}, \bar{y}) = \left(\frac{a}{2}, \frac{b}{2}\right)$

(b) $m = \int_0^a \int_0^b ky\,dy\,dx = \frac{kab^2}{2}$

$M_x = \int_0^a \int_0^b ky^2\,dy\,dx = \frac{kab^3}{3}$

$M_y = \int_0^a \int_0^b kxy\,dy\,dx = \frac{ka^2b^2}{4}$

$\bar{x} = \frac{M_y}{m} = \frac{ka^2b^2/4}{kab^2/2} = \frac{a}{2}$

$\bar{y} = \frac{M_x}{m} = \frac{kab^3/3}{kab^2/2} = \frac{2}{3}b$

$(\bar{x}, \bar{y}) = \left(\frac{a}{2}, \frac{2}{3}b\right)$

(c) $m = \int_0^a \int_0^b kx\,dy\,dx = k\int_0^a xb\,dx = \frac{1}{2}ka^2b$

$M_x = \int_0^a \int_0^b kxy\,dy\,dx = \frac{ka^2b^2}{4}$

$M_y = \int_0^a \int_0^b kx^2\,dy\,dx = \frac{ka^3b}{3}$

$\bar{x} = \frac{M_y}{m} = \frac{ka^3b/3}{ka^2b/2} = \frac{2}{3}a$

$\bar{y} = \frac{M_x}{m} = \frac{ka^2b^2/4}{ka^2b/2} = \frac{b}{2}$

$(\bar{x}, \bar{y}) = \left(\frac{2}{3}a, \frac{b}{2}\right)$

6. (a) $m = \int_0^a \int_0^b kxy\,dy\,dx = \frac{ka^2b^2}{4}$

$M_x = \int_0^a \int_0^b kxy^2\,dy\,dx = \frac{ka^2b^3}{6}$

$M_y = \int_0^a \int_0^b kx^2y\,dy\,dx = \frac{ka^3b^2}{6}$

$\bar{x} = \frac{M_y}{m} = \frac{ka^3b^2/6}{ka^2b^2/4} = \frac{2}{3}a$

$\bar{y} = \frac{M_x}{m} = \frac{ka^2b^3/6}{ka^2b^2/4} = \frac{2}{3}b$

(b) $m = \int_0^a \int_0^b k(x^2 + y^2)\,dy\,dx = \frac{kab}{3}(a^2 + b^2)$

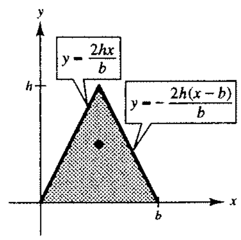
$M_x = \int_0^a \int_0^b k(x^2y + y^3)\,dy\,dx = \frac{kab^2}{12}(2a^2 + 3b^2)$

$M_y = \int_0^a \int_0^b k(x^3 + xy^2)\,dy\,dx = \frac{ka^2b}{12}(3a^2 + 2b^2)$

$\bar{x} = \frac{M_y}{m} = \frac{(ka^2b/12)(3a^2 + 2b^2)}{(kab/3)(a^2 + b^2)} = \frac{a}{4}\left(\frac{3a^2 + 2b^2}{a^2 + b^2}\right)$

$\bar{y} = \frac{M_x}{m} = \frac{(kab^2/12)(2a^2 + 3b^2)}{(kab/3)(a^2 + b^2)} = \frac{b}{4}\left(\frac{2a^2 + 3b^2}{a^2 + b^2}\right)$

7. (a) $m = \dfrac{k}{2}bh$



$\bar{x} = \dfrac{b}{2}$ by symmetry

$$M_x = \int_0^{b/2}\int_0^{2hx/b} ky\,dy\,dx + \int_{b/2}^{b}\int_0^{-2h(x-b)/b} ky\,dy\,dx$$

$$= \frac{kbh^2}{12} + \frac{kbh^2}{12} = \frac{kbh^2}{6}$$

$$\bar{y} = \frac{M_x}{m} = \frac{kbh^2/6}{kbh/2} = \frac{h}{3}$$

$$(\bar{x}, \bar{y}) = \left(\frac{b}{2}, \frac{h}{3}\right)$$

(b) $$m = \int_0^{b/2}\int_0^{2hx/b} ky\,dy\,dx + \int_{b/2}^{b}\int_0^{-2h(x-b)/b} ky\,dy\,dx = \frac{kbh^2}{6}$$

$$M_x = \int_0^{b/2}\int_0^{2hx/b} ky^2\,dy\,dx + \int_{b/2}^{b}\int_0^{-2h(x-b)/b} ky^2\,dy\,dx = \frac{kbh^3}{12}$$

$$M_y = \int_0^{b/2}\int_0^{2hx/b} kxy\,dy\,dx + \int_{b/2}^{b}\int_0^{-2h(x-b)/b} kxy\,dy\,dx = \frac{kb^2h^2}{12}$$

$$\bar{x} = \frac{M_y}{m} = \frac{kb^2h^2/12}{kbh^2/6} = \frac{b}{2}$$

$$\bar{y} = \frac{M_x}{m} = \frac{kbh^3/12}{kbh^2/6} = \frac{h}{2}$$

$$(\bar{x}, \bar{y}) = \left(\frac{b}{2}, \frac{h}{2}\right)$$

(c) $$m = \int_0^{b/2}\int_0^{2hx/b} kx\,dy\,dx + \int_{b/2}^{b}\int_0^{-2h(x-b)/b} kx\,dy\,dx$$

$$= \frac{1}{12}kb^2h + \frac{1}{6}kb^2h = \frac{1}{4}kb^2h$$

$$M_x = \int_0^{b/2}\int_0^{2hx/b} kxy\,dy\,dx + \int_{b/2}^{b}\int_0^{-2h(x-b)/b} kxy\,dy\,dx$$

$$= \frac{1}{32}kh^2b^2 + \frac{5}{96}kh^2b^2 = \frac{1}{12}kh^2b^2$$

$$M_y = \int_0^{b/2}\int_0^{2hx/b} kx^2\,dy\,dx + \int_{b/2}^{b}\int_0^{-2h(x-b)/b} kx^2\,dy\,dx$$

$$= \frac{1}{32}kb^3h + \frac{11}{96}kb^3h = \frac{7}{48}kb^3h$$

$$\bar{x} = \frac{M_y}{m} = \frac{7kb^3h/48}{kb^2h/4} = \frac{7}{12}b$$

$$\bar{y} = \frac{M_x}{m} = \frac{kh^2b^2/12}{kb^2h/4} = \frac{h}{3}$$

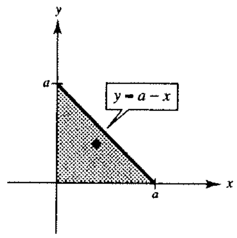
$$(\bar{x}, \bar{y}) = \left(\frac{7}{12}b, \frac{h}{3}\right)$$

8. (a) $m = \frac{a^2k}{2}$

$M_x = \int_0^a \int_0^{a-x} ky\,dy\,dx = \frac{ka^3}{6}$

$M_y = M_x$ by symmetry

$\bar{x} = \bar{y} = \frac{M_x}{m} = \frac{ka^3/6}{ka^2/2} = \frac{a}{3}$



(b) $m = \int_0^a \int_0^{a-x} (x^2 + y^2)\,dy\,dx$

$= \int_0^a \left[x^2y + \frac{y^3}{3}\right]_0^{a-x} dx = \int_0^a \left[ax^2 - x^3 + \frac{1}{3}(a-x)^3\right] dx = \frac{a^4}{6}$

$M_y = \int_0^a \int_0^{a-x} (x^3 + xy^2)\,dy\,dx$

$= \int_0^a \left(ax^3 - x^4 + \frac{1}{3}a^3x - a^2x^2 + ax^3 - \frac{1}{3}x^4\right) dx = \frac{1}{3}\int_0^a (a^3x - 3a^2x^2 + 6ax^3 - 4x^4)\,dx = \frac{a^5}{15}$

$\bar{x} = \frac{M_y}{m} = \frac{a^5/15}{a^4/6} = \frac{2a}{5}$

$\bar{y} = \frac{2a}{5}$ by symmetry

9. (a) The x-coordinate changes by 5: $(\bar{x}, \bar{y}) = \left(\frac{a}{2} + 5, \frac{b}{2}\right)$

(b) The x-coordinate changes by 5: $(\bar{x}, \bar{y}) = \left(\frac{a}{2} + 5, \frac{2b}{3}\right)$

(c) $m = \int_5^{a+5} \int_0^b kx\,dy\,dx = \frac{1}{2}k(a+5)^2b - \frac{25}{2}kb$

$M_x = \int_5^{a+5} \int_0^b kxy\,dy\,dx = \frac{1}{4}k(a+5)^2b^2 - \frac{25}{4}kb^2$

$M_y = \int_5^{a+5} \int_0^b kx^2\,dy\,dx = \frac{1}{3}k(a+5)^3\,b - \frac{125}{3}kb$

$\bar{x} = \frac{M_y}{m} = \frac{2(a^2 + 15a + 75)}{3(a + 10)}$

$\bar{y} = \frac{M_x}{m} = \frac{b}{2}$

10. The x-coordinate changes by h units horizontally and k units vertically. This is not true for variable densities.

11. (a) $\bar{x} = 0$ by symmetry

$m = \frac{\pi a^2 k}{2}$

$M_x = \int_{-a}^a \int_0^{\sqrt{a^2-x^2}} yk\,dy\,dx = \frac{2a^3k}{3}$

$\bar{y} = \frac{M_x}{m} = \frac{2a^3k}{3} \cdot \frac{2}{\pi a^2 k} = \frac{4a}{3\pi}$

(b) $m = \int_{-a}^a \int_0^{\sqrt{a^2-x^2}} k(a-y)y\,dy\,dx = \frac{a^4k}{24}(16 - 3\pi)$

$M_x = \int_{-a}^a \int_0^{\sqrt{a^2-x^2}} k(a-y)y^2\,dy\,dx = \frac{a^5k}{120}(15\pi - 32)$

$M_y = \int_{-a}^a \int_0^{\sqrt{a^2-x^2}} kx(a-y)y\,dy\,dx = 0$

$\bar{x} = \frac{M_y}{m} = 0$

$\bar{y} = \frac{M_x}{m} = \frac{a}{5}\left[\frac{15\pi - 32}{16 - 3\pi}\right]$

12. (a) $m = \int_0^a \int_0^{\sqrt{a^2-x^2}} k\,dy\,dx = \frac{k\pi a^2}{4}$

$M_y = \int_0^a \int_0^{\sqrt{a^2-x^2}} kx\,dy\,dx$

$= k\int_0^a x\sqrt{a^2 - x^2}\,dx$

$= \left[-\frac{k}{3}(a^2 - x^2)^{3/2}\right]_0^a = \frac{ka^3}{3}$

$\bar{x} = \frac{M_y}{m} = \frac{ka^3/3}{k\pi a^2/4} = \frac{4a}{3\pi}$

$\bar{y} = \frac{4a}{3\pi}$ by symmetry

(b) $m = \int_0^a \int_0^{\sqrt{a^2-x^2}} k(x^2 + y^2)\,dy\,dx$

$= \int_0^{\pi/2} \int_0^a kr^3\,dr\,d\theta = \frac{ka^4\pi}{8}$

$M_x = \int_0^a \int_0^{\sqrt{a^2-x^2}} k(x^2 + y^2)y\,dy\,dx$

$= \int_0^{\pi/2} \int_0^a kr^4 \sin\theta\,dr\,d\theta = \frac{ka^5}{5}$

$M_y = M_x$ by symmetry

$\bar{x} = \bar{y} = \frac{M_y}{m} = \frac{ka^5}{5} \cdot \frac{8}{ka^4\pi} = \frac{8a}{5\pi}$

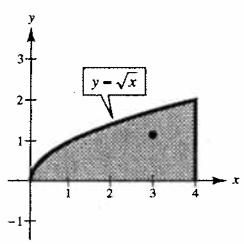
13. $m = \int_0^4 \int_0^{\sqrt{x}} kxy\,dy\,dx = \frac{32k}{3}$

$M_x = \int_0^4 \int_0^{\sqrt{x}} kxy^2\,dy\,dx = \frac{256k}{21}$

$M_y = \int_0^4 \int_0^{\sqrt{x}} kx^2y\,dy\,dx = 32k$

$\bar{x} = \frac{M_y}{m} = \frac{32k}{1} \cdot \frac{3}{32k} = 3$

$\bar{y} = \frac{M_x}{m} = \frac{256k}{21} \cdot \frac{3}{32k} = \frac{8}{7}$



14. $m = \int_0^2 \int_0^{x^3} kx\,dy\,dx = \int_0^2 kx^4\,dx = \frac{32k}{5}$

$M_x = \int_0^2 \int_0^{x^3} kxy\,dy\,dx = 16k$

$M_y = \int_0^2 \int_0^{x^3} kx^2\,dy\,dx = \frac{32k}{3}$

$\bar{x} = \frac{M_y}{m} = \frac{32k}{3} \cdot \frac{5}{32k} = \frac{5}{3}$

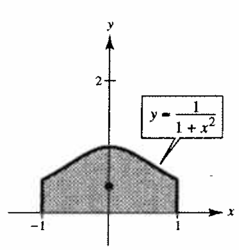
$\bar{y} = \frac{M_x}{m} = \frac{16k}{32k}(5) = \frac{5}{2}$

15. $\bar{x} = 0$ by symmetry

$m = \int_{-1}^1 \int_0^{1/(1+x^2)} k\,dy\,dx = \frac{k\pi}{2}$

$M_x = \int_{-1}^1 \int_0^{1/(1+x^2)} ky\,dy\,dx = \frac{k}{8}(2 + \pi)$

$\bar{y} = \frac{M_x}{m} = \frac{k}{8}(2 + \pi) \cdot \frac{2}{k\pi} = \frac{2 + \pi}{4\pi}$



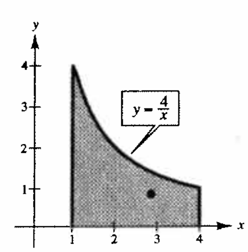
16. $m = \int_1^4 \int_0^{4/x} kx^2\,dy\,dx = 30k$

$M_x = \int_1^4 \int_0^{4/x} kx^2y\,dy\,dx = 24k$

$M_y = \int_1^4 \int_0^{4/x} kx^3\,dy\,dx = 84k$

$\bar{x} = \frac{M_y}{m} = \frac{84k}{30k} = \frac{14}{5}$

$\bar{y} = \frac{M_y}{m} = \frac{24k}{30k} = \frac{4}{5}$

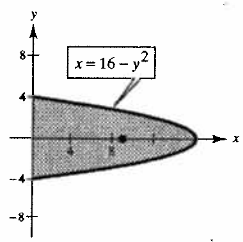


17. $\bar{y} = 0$ by symmetry

$$m = \int_{-4}^{4}\int_{0}^{16-y^2} kx\,dx\,dy = \frac{8192k}{15}$$

$$M_y = \int_{-4}^{4}\int_{0}^{16-y^2} kx^2\,dx\,dy = \frac{524{,}288k}{105}$$

$$\bar{x} = \frac{M_y}{m} = \frac{524{,}288k}{105}\cdot\frac{15}{8192k} = \frac{64}{7}$$

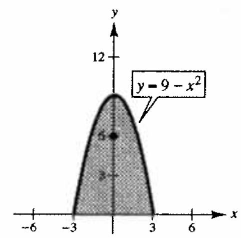


18. $\bar{x} = 0$ by symmetry

$$m = \int_{-3}^{3}\int_{0}^{9-x^2} ky^2\,dy\,dx = \frac{23{,}328k}{35}$$

$$M_x = \int_{-3}^{3}\int_{0}^{9-x^2} ky^3\,dy\,dx = \frac{139{,}968k}{35}$$

$$\bar{y} = \frac{M_x}{m} = \frac{139{,}968k}{35}\cdot\frac{35}{23{,}328k} = 6$$

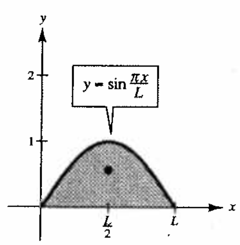


19. $\bar{x} = \frac{L}{2}$ by symmetry

$$m = \int_{0}^{L}\int_{0}^{\sin \pi x/L} ky\,dy\,dx = \frac{kL}{4}$$

$$M_x = \int_{0}^{L}\int_{0}^{\sin \pi x/L} ky^2\,dy\,dx = \frac{4kL}{9\pi}$$

$$\bar{y} = \frac{M_x}{m} = \frac{4kL}{9\pi}\cdot\frac{4}{kL} = \frac{16}{9\pi}$$



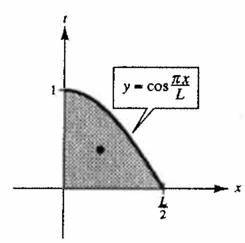
20. $$m = \int_{0}^{L/2}\int_{0}^{\cos \pi x/L} k\,dy\,dx = \frac{kL}{\pi}$$

$$M_x = \int_{0}^{L/2}\int_{0}^{\cos \pi x/L} ky\,dy\,dx = \frac{kL}{8}$$

$$M_y = \int_{0}^{L/2}\int_{0}^{\cos \pi x/L} kx\,dy\,dx = \frac{L^2(\pi-2)k}{2\pi^2}$$

$$\bar{x} = \frac{M_y}{m} = \frac{L^2(\pi-2)k}{2\pi^2}\cdot\frac{\pi}{kL} = \frac{L(\pi-2)}{2\pi}$$

$$\bar{y} = \frac{M_x}{m} = \frac{kL}{8}\cdot\frac{\pi}{kL} = \frac{\pi}{8}$$



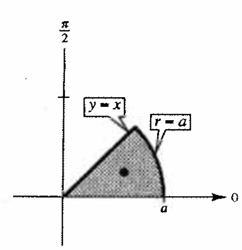
21. $$m = \frac{\pi a^2 k}{8}$$

$$M_x = \iint_R ky\,dA = \int_{0}^{\pi/4}\int_{0}^{a} kr^2\sin\theta\,dr\,d\theta = \frac{ka^3(2-\sqrt{2})}{6}$$

$$M_y = \iint_R kx\,dA = \int_{0}^{\pi/4}\int_{0}^{a} kr^2\cos\theta\,dr\,d\theta = \frac{ka^3\sqrt{2}}{6}$$

$$\bar{x} = \frac{M_y}{m} = \frac{ka^3\sqrt{2}}{6}\cdot\frac{8}{\pi a^2 k} = \frac{4a\sqrt{2}}{3\pi}$$

$$\bar{y} = \frac{M_x}{m} = \frac{ka^3(2-\sqrt{2})}{6}\cdot\frac{8}{\pi a^2 k} = \frac{4a(2-\sqrt{2})}{3\pi}$$



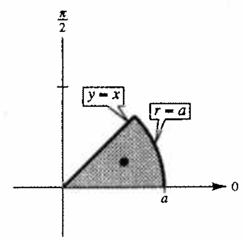
22. $m = \int\int_R k\sqrt{x^2 + y^2}\, dA = \int_0^{\pi/4} \int_0^a kr^2\, dr\, d\theta = \frac{ka^3\pi}{12}$

$M_x = \int\int_R k\sqrt{x^2 + y^2} y\, dA = \int_0^{\pi/4} \int_0^a kr^3 \sin\theta\, d\theta = \frac{ka^4(2 - \sqrt{2})}{8}$

$M_y = \int\int_R k\sqrt{x^2 + y^2}\, dA = \int_0^{\pi/4} \int_0^a kr^3 \cos\theta\, d\theta = \frac{ka^4\sqrt{2}}{8}$

$\bar{x} = \frac{M_y}{m} = \frac{ka^4\sqrt{2}}{8} \cdot \frac{12}{ka^3\pi} = \frac{3\sqrt{2}a}{2\pi}$

$\bar{y} = \frac{M_x}{m} = \frac{ka^4(2 - \sqrt{2})}{8} \cdot \frac{12}{ka^3\pi} = \frac{3(2 - \sqrt{2})a}{2\pi}$



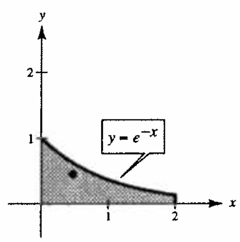
23. $m = \int_0^2 \int_0^{e^{-x}} ky\, dy\, dx = \frac{k}{4}(1 - e^{-4})$

$M_x = \int_0^2 \int_0^{e^{-x}} ky^2\, dy\, dx = \frac{k}{9}(1 - e^{-6})$

$M_y = \int_0^2 \int_0^{e^{-x}} kxy\, dy\, dx = \frac{k(1 - 5e^{-4})}{8}$

$\bar{x} = \frac{M_y}{m} = \frac{k(e^4 - 5)}{8e^4} \cdot \frac{4e^4}{k(e^4 - 1)} = \frac{e^4 - 5}{2(e^4 - 1)} \approx 0.46$

$\bar{y} = \frac{M_x}{m} = \frac{k(e^6 - 1)}{9e^6} \cdot \frac{4e^4}{k(e^4 - 1)} = \frac{4}{9}\left[\frac{e^6 - 1}{e^6 - e^2}\right] \approx 0.45$



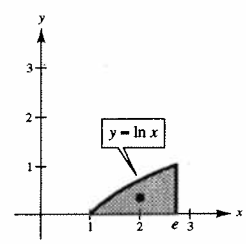
24. $m = \int_1^e \int_0^{\ln x} \frac{k}{x}\, dy\, dx = \frac{k}{2}$

$M_x = \int_1^e \int_0^{\ln x} \frac{k}{x} y\, dy\, dx = \frac{k}{6}$

$M_y = \int_1^e \int_0^{\ln x} \frac{k}{x} x\, dy\, dx = k$

$\bar{x} = \frac{M_y}{m} = \frac{k}{1} \cdot \frac{2}{k} = 2$

$\bar{y} = \frac{M_x}{m} = \frac{k}{6} \cdot \frac{2}{k} = \frac{1}{3}$



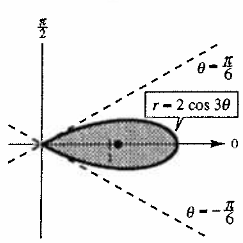
25. $\bar{y} = 0$ by symmetry

$m = \int\int_R k\, dA = \int_{-\pi/6}^{\pi/6} \int_0^{2\cos 3\theta} kr\, dr\, d\theta = \frac{k\pi}{3}$

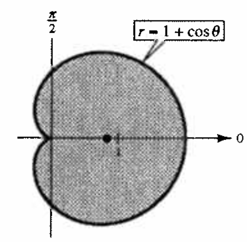
$M_y = \int\int_R kx\, dA$

$= \int_{-\pi/6}^{\pi/6} \int_0^{2\cos 3\theta} kr^2 \cos\theta\, dr\, d\theta = \frac{27\sqrt{3}}{40}k \approx 1.17k$

$\bar{x} = \frac{M_y}{m} = \frac{81\sqrt{3}}{40\pi} \approx 1.12$



26. $\bar{y} = 0$ by symmetry



$$m = \iint_R k\,dA = \int_0^{2\pi}\int_0^{1+\cos\theta} kr\,dr\,d\theta = \frac{3\pi k}{2}$$

$$M_y = \iint_R kx\,dA = \int_0^{2\pi}\int_0^{1+\cos\theta} kr^2\cos\theta\,dr\,d\theta$$

$$= \frac{k}{3}\int_0^{2\pi} \cos\theta(1 + 3\cos\theta + 3\cos^2\theta + \cos^3\theta)\,d\theta$$

$$= \frac{k}{3}\int_0^{2\pi}\left[\cos\theta + \frac{3}{2}(1+\cos^2\theta) + 3\cos\theta(1-\sin^2\theta) + \frac{1}{4}(1+\cos 2\theta)^2\,d\theta\right.$$

$$= \frac{5k\pi}{4}$$

$$\bar{x} = \frac{M_y}{m} = \frac{5k\pi}{4}\cdot\frac{2}{3k\pi} = \frac{5}{6}$$

27. $m = bh$

$$I_x = \int_0^b\int_0^h y^2\,dy\,dx = \frac{bh^3}{3}$$

$$I_y = \int_0^b\int_0^h x^2 dy\,dx = \frac{b^3h}{3}$$

$$\bar{\bar{x}} = \sqrt{\frac{I_y}{m}} = \sqrt{\frac{b^3h}{3}\cdot\frac{1}{bh}} = \sqrt{\frac{b^2}{3}} = \frac{b}{\sqrt{3}} = \frac{\sqrt{3}}{3}b$$

$$\bar{\bar{y}} = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{bh^3}{3}\cdot\frac{1}{bh}} = \sqrt{\frac{h^2}{3}} = \frac{h}{\sqrt{3}} = \frac{\sqrt{3}}{3}h$$

28. $m = \int_0^b\int_0^{h-(hx/b)} dy\,dx = \frac{bh}{2}$

$$I_x = \int_0^b\int_0^{h-(hx/b)} y^2\,dy\,dx = \frac{bh^3}{12}$$

$$I_y = \int_0^b\int_0^{h-(hx/b)} x^2\,dy\,dx = \frac{b^3h}{12}$$

$$\bar{\bar{x}} = \sqrt{\frac{I_y}{m}} = \sqrt{\frac{b^3h/12}{bh/2}} = \frac{b}{\sqrt{6}} = \frac{\sqrt{6}}{6}b$$

$$\bar{\bar{y}} = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{bh^3/12}{bh/2}} = \frac{h}{\sqrt{6}} = \frac{\sqrt{6}}{6}h$$

29. $m = \pi a^2$

$$I_x = \iint_R y^2\,dA = \int_0^{2\pi}\int_0^a r^3\sin^2\theta\,dr\,d\theta = \frac{a^4\pi}{4}$$

$$I_y = \iint_R x^2\,dA = \int_0^{2\pi}\int_0^a r^3\cos^2\theta\,dr\,d\theta = \frac{a^4\pi}{4}$$

$$I_0 = I_x + I_y = \frac{a^4\pi}{4} + \frac{a^4\pi}{4} = \frac{a^4\pi}{2}$$

$$\bar{\bar{x}} = \bar{\bar{y}} = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{a^4\pi}{4}\cdot\frac{1}{\pi a^2}} = \frac{a}{2}$$

30. $m = \frac{\pi a^2}{2}$

$$I_x = \iint_R y^2\,dA = \int_0^{\pi}\int_0^a r^3\sin^2\theta\,dr\,d\theta = \frac{a^4\pi}{8}$$

$$I_y = \iint_R x^2\,dA = \int_0^{\pi}\int_0^a r^3\cos^2\theta\,dr\,d\theta = \frac{a^4\pi}{8}$$

$$I_0 = I_x + I_x = \frac{a^4\pi}{8} + \frac{a^4\pi}{8} = \frac{a^4\pi}{4}$$

$$\bar{\bar{x}} = \bar{\bar{y}} = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{a^4\pi}{8}\cdot\frac{2}{\pi a^2}} = \frac{a}{2}$$

31. $m = \frac{\pi a^2}{4}$

$$I_x = \iint_R y^2\,dA = \int_0^{\pi/2}\int_0^a r^3\sin^2\theta\,dr\,d\theta = \frac{\pi a^4}{16}$$

$$I_y = \iint_R x^2\,dA = \int_0^{\pi/2}\int_0^a r^3\cos^2\theta\,dr\,d\theta = \frac{\pi a^4}{16}$$

$$I_0 = I_x + I_y = \frac{\pi a^4}{16} + \frac{\pi a^4}{16} = \frac{\pi a^4}{8}$$

$$\bar{\bar{x}} = \bar{\bar{y}} = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{\pi a^4}{16}\cdot\frac{4}{\pi a^2}} = \frac{a}{2}$$

32. $m = \pi ab$

$$I_x = 4\int_0^a \int_0^{(b/a)\sqrt{a^2-x^2}} y^2\,dy\,dx$$

$$= 4\int_0^a \frac{b^3}{3a^3}(a^2 - x^2)^{3/2}\,dx = \frac{4b^3}{3a^3}\int_0^a \left[a^2\sqrt{a^2 - x^2} - x^2\sqrt{a^2 - x^2}\right] dx$$

$$= \frac{4b^3}{3a^3}\left[\frac{a^2}{2}\left(x\sqrt{a^2 - x^2} + a^2 \arcsin\frac{x}{a}\right) - \frac{1}{8}\left[x(2x^2 - a^2)\sqrt{a^2 - x^2} + a^4 \arcsin\frac{x}{a}\right]\right]_0^a = \frac{ab^3\pi}{4}$$

$$I_y = 4\int_0^b \int_0^{(a/b)\sqrt{b^2-y^2}} x^2\,dx\,dy = \frac{a^3b\pi}{4}$$

$$I_0 = I_y + I_x = \frac{a^3b\pi}{4} + \frac{ab^3\pi}{4} = \frac{ab\pi}{4}(a^2 + b^2)$$

$$\bar{\bar{x}} = \sqrt{\frac{I_y}{m}} = \sqrt{\frac{a^3b\pi}{4}\cdot\frac{1}{\pi ab}} = \frac{a}{2}$$

$$\bar{\bar{y}} = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{ab^3\pi}{4}\cdot\frac{1}{\pi ab}} = \frac{b}{2}$$

33. $\rho = ky$

$$m = k\int_0^a \int_0^b y\,dy\,dx = \frac{kab^2}{2}$$

$$I_x = k\int_0^a \int_0^b y^3\,dy\,dx = \frac{kab^4}{4}$$

$$I_y = k\int_0^a \int_0^b x^2 y\,dy\,dx = \frac{ka^3b^2}{6}$$

$$I_0 = I_x + I_y = \frac{3kab^4 + 2kb^2a^3}{12}$$

$$\bar{\bar{x}} = \sqrt{\frac{I_y}{m}} = \sqrt{\frac{ka^3b^2/6}{kab^2/2}} = \sqrt{\frac{a^2}{3}} = \frac{a}{\sqrt{3}} = \frac{\sqrt{3}}{3}a$$

$$\bar{\bar{y}} = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{kab^4/4}{kab^2/2}} = \sqrt{\frac{b^2}{2}} = \frac{b}{\sqrt{2}} = \frac{\sqrt{2}}{2}b$$

34. $\rho = ky$

$$m = 2k\int_0^a \int_0^{\sqrt{a^2-x^2}} y\,dy\,dx$$

$$= k\int_0^a (a^2 - x^2)\,dx = \frac{2ka^3}{3}$$

$$I_x = k\int_{-a}^a \int_0^{\sqrt{a^2-x^2}} y^3\,dy\,dx = \frac{4ka^5}{15}$$

$$I_y = k\int_{-a}^a \int_0^{\sqrt{a^2-x^2}} x^2y\,dy\,dx = \frac{2ka^5}{15}$$

$$I_0 = I_x + I_y = \frac{2ka^5}{5}$$

$$\bar{\bar{x}} = \sqrt{\frac{I_y}{m}} = \sqrt{\frac{2ka^5/15}{2ka^3/3}} = \sqrt{\frac{a^2}{5}} = \frac{a}{\sqrt{5}}$$

$$\bar{\bar{y}} = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{4ka^5/15}{2ka^3/3}} = \sqrt{\frac{2a^2}{5}} = \frac{2a}{\sqrt{10}}$$

35. $\rho = kx$

$$m = k\int_0^2 \int_0^{4-x^2} x\,dy\,dx = 4k$$

$$I_x = k\int_0^2 \int_0^{4-x^2} xy^2\,dy\,dx = \frac{32k}{3}$$

$$I_y = k\int_0^2 \int_0^{4-x^2} x^3\,dy\,dx = \frac{16k}{3}$$

$$I_0 = I_x + I_y = 16k$$

$$\bar{\bar{x}} = \sqrt{\frac{I_y}{m}} = \sqrt{\frac{16k/3}{4k}} = \sqrt{\frac{4}{3}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

$$\bar{\bar{y}} = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{32k/3}{4k}} = \sqrt{\frac{8}{3}} = \frac{4}{\sqrt{6}} = \frac{2\sqrt{6}}{3}$$

36. $\rho = kxy$

$$m = k\int_0^1 \int_{x^2}^x xy\,dy\,dx = \frac{k}{2}\int_0^1 (x^3 - x^5)\,dx = \frac{k}{24}$$

$$I_x = k\int_0^1 \int_{x^2}^x xy^3\,dy\,dx = \frac{k}{4}\int_0^1 (x^5 - x^9)\,dx = \frac{k}{60}$$

$$I_y = k\int_0^1 \int_{x^2}^x x^3y\,dy\,dx = \frac{k}{2}\int_0^1 (x^5 - x^7)\,dx = \frac{k}{48}$$

$$I_0 = I_x + I_y = \frac{9k}{240} = \frac{3k}{80}$$

$$\bar{\bar{x}} = \sqrt{\frac{I_y}{m}} = \sqrt{\frac{k/48}{k/24}} = \frac{1}{\sqrt{2}}$$

$$\bar{\bar{y}} = \sqrt{\frac{I_x}{m}}\sqrt{\frac{k/60}{k/24}} = \sqrt{\frac{2}{5}}$$

37. $\rho = kxy$

$$m = \int_0^4 \int_0^{\sqrt{x}} kxy\,dy\,dx = \frac{32k}{3}$$

$$I_x = \int_0^4 \int_0^{\sqrt{x}} kxy^3\,dy\,dx = 16k$$

$$I_y = \int_0^4 \int_0^{\sqrt{x}} kx^3y\,dy\,dx = \frac{512k}{5}$$

$$I_0 = I_x + I_y = \frac{592k}{5}$$

$$\bar{\bar{x}} = \sqrt{\frac{I_y}{m}} = \sqrt{\frac{512k}{5} \cdot \frac{3}{32k}} = \sqrt{\frac{48}{5}} = \frac{4\sqrt{15}}{5}$$

$$\bar{\bar{y}} = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{16k}{1} \cdot \frac{3}{32k}} = \sqrt{\frac{3}{2}} = \frac{\sqrt{6}}{2}$$

38. $\rho = x^2 + y^2$

$$m = \int_0^1 \int_{x^2}^{\sqrt{x}} (x^2 + y^2)\,dy\,dx = \frac{6}{35}$$

$$I_x = \int_0^1 \int_{x^2}^{\sqrt{x}} (x^2 + y^2)y^2\,dy\,dx = \frac{158}{2079}$$

$$I_y = \int_0^1 \int_{x^2}^{\sqrt{x}} (x^2 + y^2)x^2\,dy\,dx = \frac{158}{2079}$$

$$I_0 = I_x + I_y = \frac{316}{2079}$$

$$\bar{\bar{x}} = \sqrt{\frac{I_y}{m}} = \sqrt{\frac{158}{2079} \cdot \frac{35}{6}} = \sqrt{\frac{395}{891}}$$

$$\bar{\bar{y}} = \sqrt{\frac{I_x}{m}} = \bar{\bar{x}} = \sqrt{\frac{395}{891}}$$

39. $\rho = kx$

$$m = \int_0^1 \int_{x^2}^{\sqrt{x}} kx\,dy\,dx = \frac{3k}{20}$$

$$I_x = \int_0^1 \int_{x^2}^{\sqrt{x}} kxy^2\,dy\,dx = \frac{3k}{56}$$

$$I_y = \int_0^1 \int_{x^2}^{\sqrt{x}} kx^3\,dy\,dx = \frac{k}{18}$$

$$I_0 = I_x + I_y = \frac{55k}{504}$$

$$\bar{\bar{x}} = \sqrt{\frac{I_y}{m}} = \sqrt{\frac{k}{18} \cdot \frac{20}{3k}} = \frac{\sqrt{30}}{9}$$

$$\bar{\bar{y}} = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{3k}{56} \cdot \frac{20}{3k}} = \frac{\sqrt{70}}{14}$$

40. $\rho = ky$

$$m = 2\int_0^2 \int_{x^3}^{4x} ky\,dy\,dx = \frac{512k}{21}$$

$$I_x = 2\int_0^2 \int_{x^3}^{4x} ky^3\,dy\,dx = \frac{32{,}768k}{65}$$

$$I_y = 2\int_0^2 \int_{x^3}^{4x} kx^2y\,dy\,dx = \frac{2048k}{45}$$

$$I_0 = I_x + I_y = \frac{321{,}536k}{585}$$

$$\bar{\bar{x}} = \sqrt{\frac{I_y}{m}} = \sqrt{\frac{2048k}{45} \cdot \frac{21}{512k}} = \sqrt{\frac{28}{15}} = \frac{2\sqrt{105}}{15}$$

$$\bar{\bar{y}} = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{32{,}768k}{65} \cdot \frac{21}{512k}} = \frac{8\sqrt{1365}}{65}$$

41. $$I = 2k\int_{-b}^{b} \int_0^{\sqrt{b^2-x^2}} (x-a)^2\,dy\,dx = 2k\int_{-b}^{b} (x-a)^2\sqrt{b^2-x^2}\,dx$$

$$= 2k\left[\int_{-b}^{b} x^2\sqrt{b^2-x^2}\,dx - 2a\int_{-b}^{b} x\sqrt{b^2-x^2}\,dx + a^2\int_{-b}^{b} \sqrt{b^2-x^2}\,dx\right]$$

$$= 2k\left[\frac{\pi b^4}{8} + 0 + \frac{\pi a^2 b^2}{2}\right] = \frac{k\pi b^2}{4}(b^2 + 4a^2)$$

42. $$I = \int_0^4 \int_0^2 k(x-6)^2\,dy\,dx = \int_0^4 2k(x-6)^2\,dx = \left[\frac{2k}{3}(x-6)^3\right]_0^4 = \frac{416k}{3}$$

43. $$I = \int_0^4 \int_0^{\sqrt{x}} kx(x-6)^2\,dy\,dx = \int_0^4 kx\sqrt{x}(x^2 - 12x + 36)\,dx = k\left[\frac{2}{9}x^{9/2} - \frac{24}{7}x^{7/2} + \frac{72}{5}x^{5/2}\right]_0^4 = \frac{42{,}752k}{315}$$

44. $I = \int_{-a}^{a}\int_{0}^{\sqrt{a^2-x^2}} ky(y-a)^2\,dy\,dx$

$$= \int_{-a}^{a} k\left[\frac{y^4}{4} - \frac{2ay^3}{3} + \frac{a^2y^2}{2}\right]_0^{\sqrt{a^2-x^2}} dx$$

$$= \int_{-a}^{a} k\left[\frac{1}{4}(a^4 - 2a^2x^2 + x^4) - \frac{2a}{3}\left(a^2\sqrt{a^2-x^2} - x^2\sqrt{a^2-x^2}\right) + \frac{a^2}{2}(a^2 - x^2)\right] dx$$

$$= k\left[\frac{1}{4}\left(a^4x - \frac{2a^2x^3}{3} + \frac{x^5}{5}\right) - \frac{2a}{3}\left[\frac{a^2}{2}\left(x\sqrt{a^2-x^2} + a^2\arcsin\frac{x}{a}\right)\right.\right.$$

$$\left.\left. - \frac{1}{8}\left(x(2x^2 - a^2)\sqrt{a^2-x^2} + a^4\arcsin\frac{x}{a}\right)\right] + \frac{a^2}{2}\left(a^2x - \frac{x^3}{3}\right)\right]_{-a}^{a}$$

$$= 2k\left[\frac{1}{4}\left(a^5 - \frac{2}{3}a^5 + \frac{1}{5}a^5\right) - \frac{2a}{3}\left(\frac{a^4\pi}{4} - \frac{a^4\pi}{16}\right) + \frac{a^2}{2}\left(a^3 - \frac{a^3}{3}\right)\right] = 2k\left(\frac{7a^5}{15} - \frac{a^5\pi}{8}\right) = ka^5\left(\frac{56 - 15\pi}{60}\right)$$

45. $I = \int_0^a\int_0^{\sqrt{a^2-x^2}} k(a-y)(y-a)^2\,dy\,dx = \int_0^a\int_0^{\sqrt{a^2-x^2}} k(a-y)^3\,dy\,dx = \int_0^a\left[-\frac{k}{4}(a-y)^4\right]_0^{\sqrt{a^2-x^2}} dx$

$$= -\frac{k}{4}\int_0^a\left[a^4 - 4a^3y + 6a^2y^2 - 4ay^3 + y^4\right]_0^{\sqrt{a^2-x^2}} dx$$

$$= -\frac{k}{4}\int_0^a\left[a^4 - 4a^3\sqrt{a^2-x^2} + 6a^2(a^2-x^2) - 4a(a^2-x^2)\sqrt{a^2-x^2} + (a^4 - 2a^2x^2 + x^4) - a^4\right] dx$$

$$= -\frac{k}{4}\int_0^a\left[7a^4 - 8a^2x^2 + x^4 - 8a^3\sqrt{a^2-x^2} + 4ax^2\sqrt{a^2-x^2}\right] dx$$

$$= -\frac{k}{4}\left[7a^4x - \frac{8a^2}{3}x^3 + \frac{x^5}{5} - 4a^3\left(x\sqrt{a^2-x^2} + a^2\arcsin\frac{x}{a}\right) + \frac{a}{2}\left(x(2x^2-a^2)\sqrt{a^2-x^2} + a^4\arcsin\frac{x}{a}\right)\right]_0^a$$

$$= -\frac{k}{4}\left(7a^5 - \frac{8}{3}a^5 + \frac{1}{5}a^5 - 2a^5\pi + \frac{1}{4}a^5\pi\right) = a^5k\left(\frac{7\pi}{16} - \frac{17}{15}\right)$$

46. $I = \int_{-2}^{2}\int_0^{4-x^2} k(y-2)^2\,dy\,dx = \int_{-2}^{2}\left[\frac{k}{3}(y-1)^3\right]_0^{4-x^2} dx = \int_{-2}^{2}\frac{k}{3}[(2-x^2) + 8]\,dx$

$$= \frac{k}{3}\int_{-2}^{2}(16 - 12x^2 + 6x^4 - x^6)\,dx = \left[\frac{k}{3}\left(16x - 4x^3 + \frac{6}{5}x^5 - \frac{1}{7}x^7\right)\right]_{-2}^{2}$$

$$= \frac{2k}{3}\left(32 - 32 + \frac{192}{5} - \frac{128}{7}\right) = \frac{1408k}{105}$$

47. $\rho(x, y) = ky$

$\bar{y}$ will increase.

48. $\rho(x, y) = k|2 - x|$

$(\bar{x}, \bar{y})$ will be the same.

49. $\rho(x, y) = kxy$

Both $\bar{x}$ and $\bar{y}$ will increase.

50. $\rho(x, y) = k(4 - x)(4 - y)$

Both $\bar{x}$ and $\bar{y}$ will decrease.

51. Let $\rho(x, y)$ be a continuous density function on the planar lamina R.

The movements of mass with respect to the x- and y-axes are

$$M_x = \iint_R y\rho(x, y)\, dA \text{ and } M_y = \iint_R x\rho(x, y)\, dA.$$

If m is the mass of the lamina, then the center of mass is

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m}\right).$$

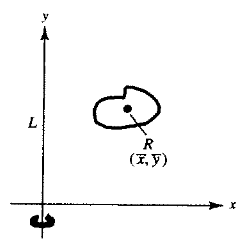
52. $I_x = \displaystyle\iint_R y^2\rho(x, y)\, dA$, Moment of inertia about x-axis

$I_y = \displaystyle\iint_R x^2\rho(x, y)\, dA$, Moment of inertia about y-axis

53. See the definition on page 1014.

54. Orient the xy-coordinate system so that L is along the y-axis and R is in the first quadrant. Then the volume of the solid is

$$\begin{aligned} V &= \iint_R 2\pi x\, dA \\ &= 2\pi \iint_R x\, dA \\ &= 2\pi\left(\frac{\iint_R x\, dA}{\iint_R dA}\right)\iint_R dA \\ &= 2\pi \bar{x} A. \end{aligned}$$



By our positioning, $\bar{x} = r$. Therefore, $V = 2\pi rA$.

55. $\bar{y} = \dfrac{L}{2}, A = bL, h = \dfrac{L}{2}$

$$\begin{aligned} I_{\bar{y}} &= \int_0^b \int_0^L \left(y - \frac{L}{2}\right)^2 dy\, dx \\ &= \int_0^b \left[\frac{[y - (L/2)]^3}{3}\right]_0^L dx = \frac{L^3b}{12} \end{aligned}$$

$$y_a = \bar{y} - \frac{I_{\bar{y}}}{hA} = \frac{L}{2} - \frac{L^3b/12}{(L/2)(bL)} = \frac{L}{3}$$

56. $\bar{y} = \dfrac{a}{2}, A = ab, h = L - \dfrac{a}{2}$

$$I_{\bar{y}} = \int_0^b \int_0^a \left(y - \frac{a}{2}\right)^2 dy\, dx = \frac{a^3b}{12}$$

$$y_a = \frac{a}{2} - \frac{a^3b/12}{[L - (a/2)]ab} = \frac{a(3L - 2a)}{3(2L - a)}$$

57. $\bar{y} = \dfrac{2L}{3}, A = \dfrac{bL}{2}, h = \dfrac{L}{3}$

$$\begin{aligned} I_{\bar{y}} &= 2\int_0^{b/2} \int_{2Lx/b}^{L} \left(y - \frac{2L}{3}\right)^2 dy\, dx \\ &= \frac{2}{3}\int_0^{b/2} \left[\left(y - \frac{2L}{3}\right)^3\right]_{2Lx/b}^{L} dx \\ &= \frac{2}{3}\int_0^{b/2} \left[\frac{L}{27} - \left(\frac{2Lx}{b} - \frac{2L}{3}\right)^3\right] dx \\ &= \frac{2}{3}\left[\frac{L^3x}{27} - \frac{b}{8L}\left(\frac{2Lx}{b} - \frac{2L}{3}\right)^4\right]_0^{b/2} = \frac{L^3b}{36} \end{aligned}$$

$$y_a = \frac{2L}{3} - \frac{L^3b/36}{L^2b/6} = \frac{L}{2}$$

58. $\bar{y} = 0, A = \pi a^2, h = L$

$$\begin{aligned} I_{\bar{y}} &= \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} y^2\, dy\, dx \\ &= \int_0^{2\pi} \int_0^a r^3 \sin^2\theta\, dr\, d\theta \\ &= \int_0^{2\pi} \frac{a^4}{4} \sin^2\theta\, d\theta \\ &= \frac{a^4\pi}{4} \end{aligned}$$

$$y_a = -\frac{(a^4\pi/4)}{L\pi a^2} = -\frac{a^2}{4L}$$

Section 14.5 Surface Area

1. $f(x, y) = 2x + 2y$

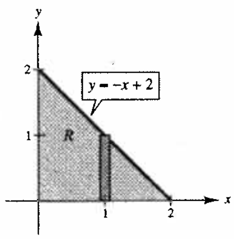
R = triangle with vertices $(0, 0), (2, 0), (0, 2)$

$f_x = 2, f_y = 2$

$\sqrt{1 + (f_x)^2 + (f_y)^2} = 3$

$$S = \int_0^2 \int_0^{2-x} 3\, dy\, dx = 3 \int_0^2 (2 - x)\, dx$$

$$= \left[3\left(2x - \frac{x^2}{2}\right)\right]_0^2 = 6$$

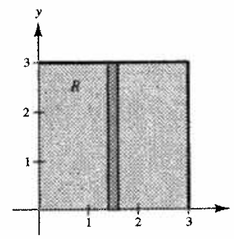


2. $f(x, y) = 15 + 2x - 3y$

$f_x = 2, f_y = -3$

$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{14}$

$$S = \int_0^3 \int_0^3 \sqrt{14}\, dy\, dx = \int_0^3 3\sqrt{14}\, dx = 9\sqrt{14}$$



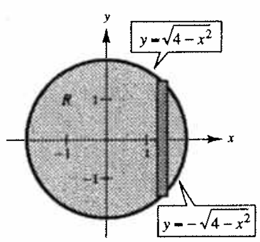
3. $f(x, y) = 8 + 2x + 2y$

$R = \{(x, y): x^2 + y^2 \le 4\}$

$f_x = 2, f_y = 2$

$\sqrt{1 + (f_x)^2 + (f_y)^2} = 3$

$$S = \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 3\, dy\, dx = \int_0^{2\pi} \int_0^2 3r\, dr\, d\theta = 12\pi$$



4. $f(x, y) = 10 + 2x - 3y$

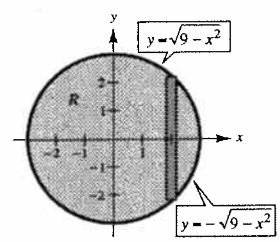
$R = \{(x, y): x^2 + y^2 \le 9\}$

$f_x = 2, f_y = -3$

$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{14}$

$$S = \int_{-3}^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \sqrt{14}\, dy\, dx$$

$$= \int_0^{2\pi} \int_0^3 \sqrt{14}\, r\, dr\, d\theta = 9\sqrt{14}\pi$$



5. $f(x, y) = 9 - x^2$

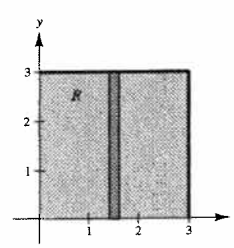
R = square with vertices, $(0, 0), (3, 0), (0, 3), (3, 3)$

$f_x = -2x, f_y = 0$

$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + 4x^2}$

$$S = \int_0^3 \int_0^3 \sqrt{1 + 4x^2}\, dy\, dx = \int_0^3 3\sqrt{1 + 4x^2}\, dx$$

$$= \left[\frac{3}{4}\left(2x\sqrt{1 + 4x^2} + \ln\left|2x + \sqrt{1 + 4x^2}\right|\right)\right]_0^3 = \frac{3}{4}\left(6\sqrt{37} + \ln\left|6 + \sqrt{37}\right|\right)$$



6. $f(x, y) = y^2$

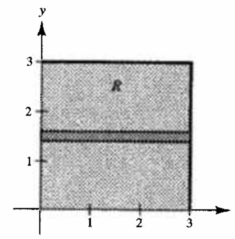
R = square with vertices $(0, 0), (3, 0), (0, 3), (3, 3)$

$f_x = 0,\ f_y = 2y$

$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + 4y^2}$

$$S = \int_0^3 \int_0^3 \sqrt{1 + 4y^2}\,dx\,dy = \int_0^3 3\sqrt{1 + 4y^2}\,dy$$

$$= \left[\frac{3}{4}\left(2y\sqrt{1 + 4y^2} + \ln\left|2y + \sqrt{1 + 4y^2}\right|\right)\right]_0^3 = \frac{3}{4}\left(6\sqrt{37} + \ln\left|6 + \sqrt{37}\right|\right)$$



7. $f(x, y) = 2 + x^{3/2}$

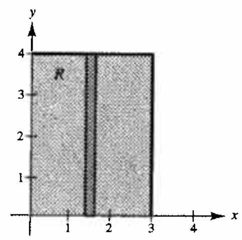
R = rectangle with vertices $(0, 0), (0, 4), (3, 4), (3, 0)$

$f_x = \frac{3}{2}x^{1/2},\ f_y = 0$

$$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + \left(\frac{9}{4}\right)x} = \frac{\sqrt{4 + 9x}}{2}$$

$$S = \int_0^3 \int_0^4 \frac{\sqrt{4 + 9x}}{2}\,dy\,dx = \int_0^3 4\left(\frac{\sqrt{4 + 9x}}{2}\right)dx$$

$$= \left[\frac{4}{27}(4 + 9x)^{3/2}\right]_0^3 = \frac{4}{27}\left(31\sqrt{31} - 8\right)$$



8. $f(x, y) = 2 + \frac{2}{3}y^{3/2}$

$f_x = 0,\ f_y = y^{1/2}$

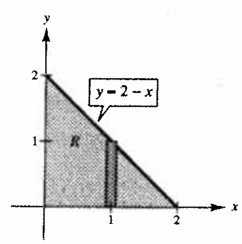
$\sqrt{1 + f_x^2 + f_y^2} = \sqrt{1 + y}$

$$S = \int_0^2 \int_0^{2-y} \sqrt{1 + y}\,dx\,dy = \int_0^2 \sqrt{1 + y}(2 - y)\,dy$$

$$= \left[2(1 + y)^{3/2} - \frac{2}{5}(1 + y)^{5/2}\right]_0^2$$

$$= 2 \cdot 3^{3/2} - \frac{2}{5} \cdot 3^{5/2} - 2 + \frac{2}{5}$$

$$= \frac{12}{5}\sqrt{3} - \frac{8}{5}$$



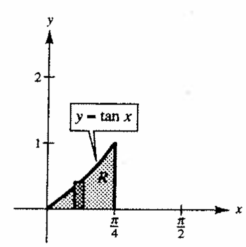
9. $f(x, y) = \ln|\sec x|$

$$R = \left\{(x, y): 0 \le x \le \frac{\pi}{4},\ 0 \le y \le \tan x\right\}$$

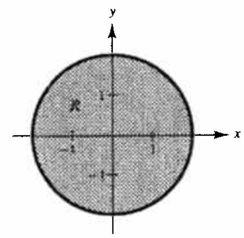
$f_x = \tan x,\ f_y = 0$

$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + \tan^2 x} = \sec x$

$$S = \int_0^{\pi/4} \int_0^{\tan x} \sec x\,dy\,dx = \int_0^{\pi/4} \sec x \tan x\,dx = \left[\sec x\right]_0^{\pi/4} = \sqrt{2} - 1$$



10. $f(x, y) = 9 + x^2 - y^2$



$f_x = 2x,\ f_y = -2y$

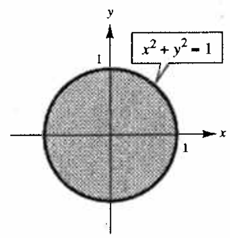
$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + 4x^2 + 4y^2}$

$$S = \int_0^{2\pi}\int_0^2 \sqrt{1 + 4r^2}\, r\, dr\, d\theta$$

$$= \int_0^{2\pi}\left[\frac{1}{12}(1 + 4r^2)^{3/2}\right]_0^2 d\theta$$

$$= \int_0^{2\pi}\frac{1}{12}(17^{3/2} - 1)\, d\theta = \frac{\pi}{6}\left(17\sqrt{17} - 1\right)$$

11. $f(x, y) = \sqrt{x^2 + y^2}$



$R = \{(x, y): 0 \le f(x, y) \le 1\}$

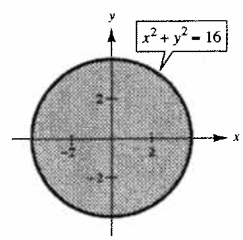
$0 \le \sqrt{x^2 + y^2} \le 1,\ x^2 + y^2 \le 1$

$f_x = \dfrac{x}{\sqrt{x^2 + y^2}},\ f_y = \dfrac{y}{\sqrt{x^2 + y^2}}$

$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + \dfrac{x^2}{x^2 + y^2} + \dfrac{y^2}{x^2 + y^2}} = \sqrt{2}$

$$S = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{2}\, dy\, dx = \int_0^{2\pi}\int_0^1 \sqrt{2}r\, dr\, d\theta = \sqrt{2}\pi$$

12. $f(x, y) = xy$



$R = \{(x, y): x^2 + y^2 \le 16\}$

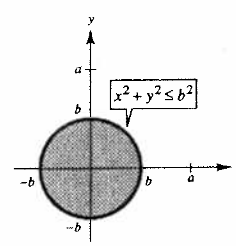
$f_x = y,\ f_y = x$

$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + y^2 + x^2}$

$$S = \int_{-4}^{4}\int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \sqrt{1 + y^2 + x^2}\, dy\, dx$$

$$= \int_0^{2\pi}\int_0^4 \sqrt{1 + r^2}\, r\, dr\, d\theta = \frac{2\pi}{3}\left(17\sqrt{17} - 1\right)$$

13. $f(x, y) = \sqrt{a^2 - x^2 - y^2}$



$R = \{(x, y): x^2 + y^2 \le b^2, 0 < b < a\}$

$f_x = \dfrac{-x}{\sqrt{a^2 - x^2 - y^2}},\ f_y = \dfrac{-y}{\sqrt{a^2 - x^2 - y^2}}$

$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + \dfrac{x^2}{a^2 - x^2 - y^2} + \dfrac{y^2}{a^2 - x^2 - y^2}} = \dfrac{a}{\sqrt{a^2 - x^2 - y^2}}$

$$S = \int_{-b}^{b}\int_{-\sqrt{b^2-x^2}}^{\sqrt{b^2-x^2}} \frac{a}{\sqrt{a^2 - x^2 - y^2}}\, dy\, dx = \int_0^{2\pi}\int_0^b \frac{a}{\sqrt{a^2 - r^2}}\, r\, dr\, d\theta = 2\pi a\left(a - \sqrt{a^2 - b^2}\right)$$

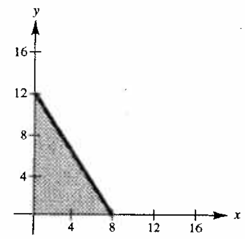
14. See Exercise 13.

$$S = \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \frac{a}{\sqrt{a^2 - x^2 - y^2}}\, dy\, dx = \int_0^{2\pi}\int_0^a \frac{a}{\sqrt{a^2 - r^2}}\, r\, dr\, d\theta = 2\pi a^2$$

15. $z = 24 - 3x - 2y$

$$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{14}$$

$$S = \int_0^8 \int_0^{-(3/2)x+12} \sqrt{14}\, dy\, dx = 48\sqrt{14}$$

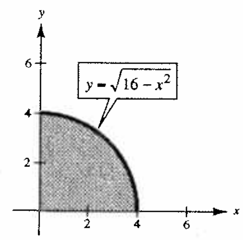


16. $z = 16 - x^2 - y^2$

$$\sqrt{1 + f_y^2 + f_y^2} = \sqrt{1 + 4x^2 + 4y^2}$$

$$S = \int_0^4 \int_0^{\sqrt{16-x^2}} \sqrt{1 + 4(x^2 + y^2)}\, dy\, dx$$

$$= \int_0^{\pi/2} \int_0^4 \sqrt{1 + 4r^2}r\, dr\, d\theta = \frac{\pi}{24}(65\sqrt{65} - 1)$$

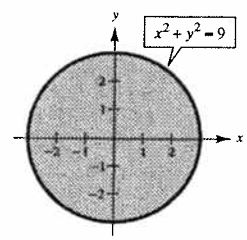


17. $z = \sqrt{25 - x^2 - y^2}$

$$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + \frac{x^2}{25 - x^2 - y^2} + \frac{y^2}{25 - x^2 - y^2}} = \frac{5}{\sqrt{25 - x^2 - y^2}}$$

$$S = 2\int_{-3}^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \frac{5}{\sqrt{25 - (x^2 + y^2)}}\, dy\, dx$$

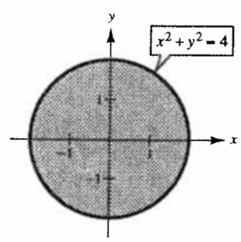
$$= 2\int_0^{2\pi} \int_0^3 \frac{5}{\sqrt{25 - r^2}} r\, dr\, d\theta = 20\pi$$



18. $z = 2\sqrt{x^2 + y^2}$

$$\sqrt{1 + f_x^2 + f_y^2} = \sqrt{1 + \frac{4x^2}{x^2 + y^2} + \frac{4y^2}{x^2 + y^2}} = \sqrt{5}$$

$$S = \int_0^{2\pi} \int_0^2 \sqrt{5}r\, dr\, d\theta = 4\pi\sqrt{5}$$

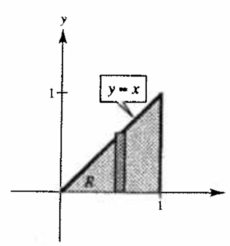


19. $f(x, y) = 2y + x^2$

R = triangle with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$

$$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{5 + 4x^2}$$

$$S = \int_0^1 \int_0^x \sqrt{5 + 4x^2}\, dy\, dx = \frac{1}{12}(27 - 5\sqrt{5})$$

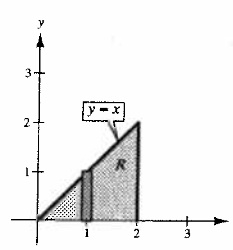


20. $f(x, y) = 2x + y^2$

R = triangle with vertices $(0, 0)$, $(2, 0)$, $(2, 2)$

$$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{5 + 4y^2}$$

$$S = \int_0^2 \int_0^x \sqrt{5 + 4y^2}\, dy\, dx = \frac{5}{4}\ln\left(\frac{8\sqrt{21} + 37}{5}\right) + \frac{\sqrt{21}}{4} + \frac{5\sqrt{5}}{12}$$



21. $f(x, y) = 4 - x^2 - y^2$

$R = \{(x, y): 0 \le f(x, y)\}$

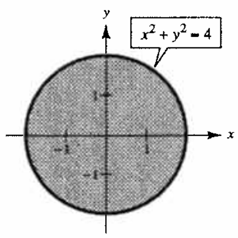
$0 \le 4 - x^2 - y^2,\ x^2 + y^2 \le 4$

$f_x = -2x,\ f_y = -2y$

$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + 4x^2 + 4y^2}$

$$S = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{1 + 4x^2 + 4y^2}\, dy\, dx$$

$$= \int_{0}^{2\pi}\int_{0}^{2} \sqrt{1 + 4r^2}\, r\, dr\, d\theta = \frac{\left(17\sqrt{17} - 1\right)\pi}{6}$$



22. $f(x, y) = x^2 + y^2$

$R = \{(x, y): 0 \le f(x, y) \le 16\}$

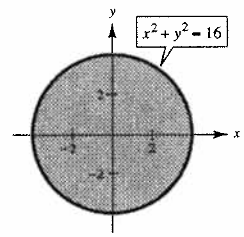
$0 \le x^2 + y^2 \le 16$

$f_x = 2x,\ f_y = 2y$

$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + 4x^2 + 4y^2}$

$$S = \int_{-4}^{4}\int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \sqrt{1 + 4x^2 + 4y^2}\, dy\, dx$$

$$= \int_{0}^{2\pi}\int_{0}^{4} \sqrt{1 + 4r^2}\, dr\, d\theta = \frac{\left(65\sqrt{65} - 1\right)\pi}{6}$$



23. $f(x, y) = 4 - x^2 - y^2$

$R = \{(x, y): 0 \le x \le 1,\ 0 \le y \le 1\}$

$f_x = -2x,\ f_y = -2y$

$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + 4x^2 + 4y^2}$

$$S = \int_{0}^{1}\int_{0}^{1} \sqrt{(1 + 4x^2) + 4y^2}\, dy\, dx \approx 1.8616$$

24. $f(x, y) = \frac{2}{3}x^{3/2} + \cos x$

$R = \{(x, y): 0 \le x \le 1, 0 \le y \le 1\}$

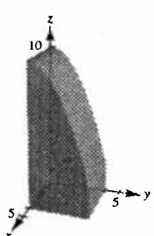
$f_x = x^{1/2} - \sin x,\ f_y = 0$

$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + \left(\sqrt{x} - \sin x\right)^2}$

$$S = \int_{0}^{1}\int_{0}^{1} \sqrt{1 + \left(\sqrt{x} - \sin x\right)^2}\, dy\, dx \approx 1.02185$$

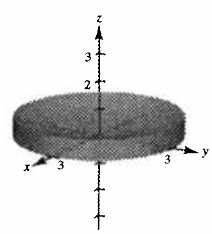
25. Surface area $> (4) \cdot (6) = 24$

Matches (e)



26. Surface area $\approx (9\pi)$

Matches (c)



27. $f(x, y) = e^x$

$R = \{(x, y): 0 \le x \le 1, 0 \le y \le 1\}$

$f_x = e^x,\ f_y = 0$

$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + e^{2x}}$

$$S = \int_{0}^{1}\int_{0}^{1} \sqrt{1 + e^{2x}}\, dy\, dx$$

$$= \int_{0}^{1} \sqrt{1 + e^{2x}} \approx 2.0035$$

28. $f(x, y) = \frac{2}{5}y^{5/2}$

$R = \{(x, y): 0 \le x \le 1, 0 \le y \le 1\}$

$f_x = 0,\ f_y = y^{3/2}$

$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + y^3}$

$$S = \int_{0}^{1}\int_{0}^{1} \sqrt{1 + y^3}\, dx\, dy$$

$$= \int_{0}^{1} \sqrt{1 + y^3}\, dy \approx 1.1114$$

29. $f(x, y) = x^3 - 3xy + y^3$

$R =$ square with vertices $(1, 1), (-1, 1), (-1, -1), (1, -1)$

$f_x = 3x^2 - 3y = 3(x^2 - y), f_y = -3x + 3y^2 = 3(y^2 - x)$

$$S = \int_{-1}^{1}\int_{-1}^{1} \sqrt{1 + 9(x^2 - y)^2 + 9(y^2 - x)^2}\, dy\, dx$$

30. $f(x, y) = x^2 - 3xy - y^2$

$R = \{(x, y): 0 \le x \le 4,\ 0 \le y \le x\}$

$f_x = 2x - 3y,\ f_y = -3x - 2y = -(3x + 2y)$

$$\sqrt{1 + (f_x) + (f_y)^2} = \sqrt{1 + (2x - 3y)^2 + (3x + 2y)^2}$$
$$= \sqrt{1 + 13(x^2 + y^2)}$$

$$S = \int_0^4\int_0^x \sqrt{1 + 13(x^2 + y^2)}\, dy\, dx$$

31. $f(x, y) = e^{-x} \sin y$

$f_x = -e^{-x} \sin y, f_y = e^{-x} \cos y$

$$\sqrt{1 + f_x^2 + f_y^2} = \sqrt{1 + e^{-2x}\sin^2 y + e^{-2x}\cos^2 y} = \sqrt{1 + e^{-2x}}$$

$$S = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{1 + e^{-2x}}\, dy\, dx$$

32. $f(x, y) = \cos(x^2 + y^2)$

$$R = \left\{(x, y):\ x^2 + y^2 \le \frac{\pi}{2}\right\}$$

$f_x = -2x \sin(x^2 + y^2),\ f_y = -2y \sin(x^2 + y^2)$

$$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + 4x^2 \sin^2(x^2 + y^2) + 4y^2\sin^2(x^2 + y^2)} = \sqrt{1 + 4[\sin^2(x^2 + y^2)](x^2 + y^2)}$$

$$S = \int_{-\sqrt{\pi/2}}^{\sqrt{\pi/2}}\int_{-\sqrt{(\pi/2)-x^2}}^{\sqrt{(\pi/2)-x^2}} \sqrt{1 + 4(x^2 + y^2)\sin^2(x^2 + y^2)}\, dy\, dx$$

33. $f(x, y) = e^{xy}$

$R = \{(x, y):\ 0 \le x \le 4,\ 0 \le y \le 10\}$

$f_x = ye^{xy},\ f_y = xe^{xy}$

$$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + y^2e^{2xy} + x^2e^{2xy}} = \sqrt{1 + e^{2xy}(x^2 + y^2)}$$

$$S = \int_0^4\int_0^{10} \sqrt{1 + e^{2xy}(x^2 + y^2)}\, dy\, dx$$

34. $f(x, y) = e^{-x} \sin y$

$R = \{(x, y): 0 \le x \le 4, 0 \le y \le x\}$

$f_x = -e^{-x} \sin y,\ f_y = e^{-x} \cos y$

$$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + e^{-2x}\sin^2 y + e^{-2x}\cos^2 y}$$
$$= \sqrt{1 + e^{-2x}}$$

$$S = \int_0^4\int_0^x \sqrt{1 + e^{-2x}}\, dy\, dx$$

35. See the definition on page 1018.

36. (a) Yes. For example, let R be the square given by

$0 \le x \le 1, 0 \le y \le 1,$

and S the square parallel to R given by

$0 \le x \le 1, 0 \le y \le 1, z = 1.$

(b) Yes. Let R be the region in part (a) and S the surface given by $f(x, y) = xy$.

(c) No

37. $f(x, y) = \sqrt{1 - x^2}; f_x = \dfrac{-x}{\sqrt{1^2 - x^2}}, f_y = 0$

$$S = \iint_R \sqrt{1 + f_x^2 + f_y^2}\, dA$$
$$= 16\int_0^1\int_0^x \frac{1}{\sqrt{1 - x^2}}\, dy\, dx$$
$$= 16\int_0^1 \frac{x}{\sqrt{1 - x^2}}\, dx = \Big[-16(1 - x^2)^{1/2}\Big]_0^1 = 16$$

38. $f(x, y) = k\sqrt{x^2 + y^2}$

$$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + \frac{k^2x^2}{x^2 + y^2} + \frac{k^2y^2}{x^2 + y^2}} = \sqrt{k^2 + 1}$$

$$S = \int\!\!\int_R \sqrt{1 + (f_x)^2 + (f_y)^2}\, dA = \int\!\!\int_R \sqrt{k^2 + 1}\, dA = \sqrt{k^2 + 1}\int\!\!\int_R dA = A\sqrt{k^2 + 1} = \pi r^2\sqrt{k^2 + 1}$$

39. (a) $V = \int_0^{50}\int_0^{\sqrt{50^2 - x^2}} \left(20 + \frac{xy}{100} - \frac{x + y}{5}\right) dy\, dx$

$$= \int_0^{50} \left[20\sqrt{50^2 - x^2} + \frac{x}{200}(50^2 - x^2) - \frac{x}{5}\sqrt{50^2 - x^2} - \frac{50^2 - x^2}{10}\right] dy$$

$$= \left[10\left(x\sqrt{50 - x^2} + 50^2 \arcsin\frac{x}{50}\right) + \frac{25}{4}x^2 - \frac{x^4}{800} + \frac{1}{15}(50^2 - x^2)^{3/2} - 250x + \frac{x^3}{30}\right]_0^{50}$$

$\approx 30{,}415.74$ ft^3

(b) $z = 20 + \frac{xy}{100}$

$$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + \frac{y^2}{100^2} + \frac{x^2}{100^2}} = \frac{\sqrt{100^2 + x^2 + y^2}}{100}$$

$$S = \frac{1}{100}\int_0^{50}\int_0^{\sqrt{50^2 - x^2}} \sqrt{100^2 + x^2 + y^2}\, dy\, dx$$

$$= \frac{1}{100}\int_0^{\pi/2}\int_0^{50} \sqrt{100^2 + r^2}\, r\, dr\, d\theta \approx 2081.53 \text{ ft}^2$$

40. (a) $z = \frac{-1}{75}y^3 + \frac{4}{25}y^2 - \frac{16}{15}y + 25$

(b) $V \approx 2(50)\int_0^{15}\left(-\frac{1}{75}y^3 + \frac{4}{25}y^2 - \frac{16}{15}y + 25\right) dy$

$= 100(266.25) = 26{,}625$ cubic feet

(c) $f(x, y) = -\frac{1}{75}y^3 + \frac{4}{25}y^2 - \frac{16}{15}y + 25$

$f_x = 0, f_y = -\frac{1}{25}y^2 + \frac{8}{25}y - \frac{16}{15}$

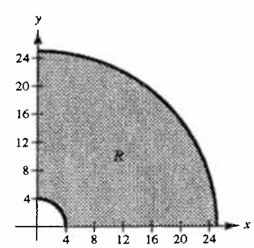
$$S = 2\int_0^{50}\int_0^{15} \sqrt{1 + f_y^2 + f_x^2}\, dy\, dx \approx 3087.58 \text{ sq ft}$$

(d) Arc length ≈ 30.8758

Surface area of roof $\approx 2(50)(30.8758) = 3087.58$ sq ft

41. (a) $V = \int\!\!\int_R f(x, y)$

$= 8\int\!\!\int_R \sqrt{625 - x^2 - y^2}\, dA$ where R is the region in the first quadrant



$$= 8\int_0^{\pi/2}\int_4^{25} \sqrt{625 - r^2}\, r\, dr\, d\theta$$

$$= -4\int_0^{\pi/2}\left[\frac{2}{3}(625 - r^2)^{3/2}\right]_4^{25} d\theta$$

$$= -\frac{8}{3}\left[0 - 609\sqrt{609}\right] \cdot \frac{\pi}{2}$$

$= 812\pi\sqrt{609}$ cm^3

—CONTINUED—

41. —CONTINUED—

(b) $A = \iint_R \sqrt{1 + (f_x)^2 + (f_y)^2}\, dA = 8\iint_R \sqrt{1 + \frac{x^2}{625 - x^2 - y^2} + \frac{y^2}{625 - x^2 - y^2}}\, dA$

$= 8\iint_R \frac{25}{\sqrt{625 - x^2 - y^2}}\, dA = 8\int_0^{\pi/2}\int_4^{25} \frac{25}{\sqrt{625 - r^2}}\, r\, dr\, d\theta$

$= \lim_{b \to 25^-} \left[-200\sqrt{625 - r^2}\right]_4^b \cdot \frac{\pi}{2} = 100\pi\sqrt{609}\ \text{cm}^2$

Section 14.6 Triple Integrals and Applications

1. $\int_0^3\int_0^2\int_0^1 (x + y + z)\, dx\, dy\, dx = \int_0^3\int_0^2 \left[\frac{1}{2}x^2 + xy + xz\right]_0^1 dy\, dx$

$= \int_0^3\int_0^2 \left(\frac{1}{2} + y + z\right) dy\, dz = \int_0^3 \left[\frac{1}{2}y + \frac{1}{2}y^2 + yz\right]_0^2 dz = \left[3z + z^2\right]_0^3 = 18$

2. $\int_{-1}^1\int_{-1}^1\int_{-1}^1 x^2y^2z^2\, dx\, dy\, dz = \frac{1}{3}\int_{-1}^1\int_{-1}^1 \left[x^3y^2z^2\right]_{-1}^1 dy\, dz$

$= \frac{2}{3}\int_{-1}^1\int_{-1}^1 y^2z^2\, dy\, dz = \frac{2}{9}\int_{-1}^1 \left[y^3z^2\right]_{-1}^1 dz = \frac{4}{9}\int_{-1}^1 z^2\, dz = \left[\frac{4}{27}z^3\right]_{-1}^1 = \frac{8}{27}$

3. $\int_0^1\int_0^x\int_0^{xy} x\, dz\, dy\, dx = \int_0^1\int_0^x \left[xz\right]_0^{xy} dy\, dx$

$= \int_0^1\int_0^x x^2y\, dy\, dx = \int_0^1 \left[\frac{x^2y^2}{2}\right]_0^x dx = \int_0^1 \frac{x^4}{2}\, dx = \left[\frac{x^5}{10}\right]_0^1 = \frac{1}{10}$

4. $\int_0^9\int_0^{y/3}\int_0^{\sqrt{y^2-9x^2}} z\, dz\, dx\, dy = \frac{1}{2}\int_0^9\int_0^{y/3} (y^2 - 9x^2)\, dx\, dy$

$= \frac{1}{2}\int_0^9 \left[xy^2 - 3x^3\right]_0^{y/3} dy = \frac{2}{18}\int_0^9 y^3\, dy = \left[\frac{1}{36}y^4\right]_0^9 = \frac{729}{4}$

5. $\int_1^4\int_0^1\int_0^x 2ze^{-x^2}\, dy\, dx\, dz = \int_1^4\int_0^1 \left[(2ze^{-x^2})y\right]_0^x dx\, dz = \int_1^4\int_0^1 2zxe^{-x^2}\, dx\, dz$

$= \int_1^4 \left[-ze^{-x^2}\right]_0^1 dz = \int_1^4 z(1 - e^{-1})\, dz = \left[(1 - e^{-1})\frac{z^2}{2}\right]_1^4 = \frac{15}{2}\left(1 - \frac{1}{e}\right)$

6. $\int_1^4\int_1^{e^2}\int_0^{1/xz} \ln z\, dy\, dz\, dx = \int_1^4\int_1^{e^2} \left[(\ln z)y\right]_0^{1/xz} dz\, dx = \int_1^4\int_1^{e^2} \frac{\ln z}{xz}\, dz\, dx$

$= \int_1^4 \left[\frac{1}{x}\frac{(\ln z)^2}{2}\right]_1^{e^2} dx = \int_1^4 \frac{2}{x}\, dx = \left[2\ln|x|\right]_1^4 = 2\ln 4$

7. $\int_0^4\int_0^{\pi/2}\int_0^{1-x} x\cos y\, dz\, dy\, dx = \int_0^4\int_0^{\pi/2} \left[(x\cos y)z\right]_0^{1-x} dy\, dx = \int_0^4\int_0^{\pi/2} x(1 - x)\cos y\, dy\, dx$

$= \int_0^4 \left[x(1 - x)\sin y\right]_0^{\pi/2} dx = \int_0^4 x(1 - x)\, dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^4 = 8 - \frac{64}{3} = \frac{-40}{3}$

8. $\displaystyle\int_0^{\pi/2}\int_0^{y/2}\int_0^{1/y} \sin y\, dz\, dx\, dy = \int_0^{\pi/2}\int_0^{y/2} \frac{\sin y}{y}\, dx\, dy = \frac{1}{2}\int_0^{\pi/2} \sin y\, dy = \left[-\frac{1}{2}\cos y\right]_0^{\pi/2} = \frac{1}{2}$

9. $\displaystyle\int_0^2\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_0^{x^2} x\, dz\, dy\, dx = \int_0^2\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} x^3\, dy\, dx = \frac{128}{15}$

10. $\displaystyle\int_0^{\sqrt{2}}\int_0^{\sqrt{2-x^2}}\int_{2x^2+y^2}^{4-y^2} y\, dz\, dy\, dx = \int_0^{\sqrt{2}}\int_0^{\sqrt{2-x^2}} (4y - 2x^2y - 2y^3)\, dy\, dx = \frac{16\sqrt{2}}{15}$

11. $\displaystyle\int_0^2\int_0^{\sqrt{4-x^2}}\int_1^4 \frac{x^2 \sin y}{z}\, dz\, dy\, dx = \int_0^2\int_0^{\sqrt{4-x^2}} \Big[x^2 \sin y \ln|z|\Big]_1^4\, dy\, dx$

$\displaystyle= \int_0^2 \Big[x^2 \ln 4(-\cos y)\Big]_0^{\sqrt{4-x^2}} dx = \int_0^2 x^2 \ln 4\left[1 - \cos\sqrt{4-x^2}\right] dx \approx 2.44167$

12. $\displaystyle\int_0^3\int_0^{2-(2y/3)}\int_0^{6-2y-3z} ze^{-x^2y^2}\, dx\, dz\, dy = \int_0^6\int_0^{(6-x)/2}\int_0^{(6-x-2y)/3} ze^{-x^2y^2}\, dz\, dy\, dx$

$\displaystyle= \int_0^6\int_0^{3-(x/2)} \frac{1}{2}\left(\frac{6-x-2y}{3}\right)^2 e^{-x^2y^2}\, dy\, dx \approx 2.118$

13. $\displaystyle\int_0^4\int_0^{4-x}\int_0^{4-x-y} dz\, dy\, dx$

14. $\displaystyle\int_0^3\int_0^{2x}\int_0^{9-x^2} dz\, dy\, dx$

15. $\displaystyle\int_{-3}^3\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_0^{9-x^2-y^2} dz\, dy\, dx$

16. $z = \frac{1}{2}(x^2 + y^2) \implies 2z = x^2 + y^2$

$x^2 + y^2 + z^2 = 2z + z^2 = 80 \implies z^2 + 2z - 80 = 0 \implies (z-8)(z+10) = 0 \implies z = 8 \implies x^2 + y^2 = 2z = 16$

$\displaystyle\int_{-4}^4\int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}}\int_{1/2(x^2+y^2)}^{\sqrt{80-x^2-y^2}} dz\, dy\, dx$

17. $\displaystyle\int_{-2}^2\int_0^{4-y^2}\int_0^x dz\, dx\, dy = \int_{-2}^2\int_0^{4-y^2} x\, dx\, dy$

$\displaystyle= \frac{1}{2}\int_{-2}^2 (4-y^2)^2\, dy = \int_0^2 (16 - 8y^2 + y^4)\, dy = \left[16y - \frac{8}{3}y^3 + \frac{1}{5}y^5\right]_0^2 = \frac{256}{15}$

18. $\displaystyle\int_0^1\int_0^1\int_0^{xy} dz\, dy\, dx = \int_0^1\int_0^1 xy\, dy\, dx = \int_0^1 \frac{x}{2}\, dx = \left[\frac{x^2}{4}\right]_0^1 = \frac{1}{4}$

19. $\displaystyle 8\int_0^a\int_0^{\sqrt{a^2-x^2}}\int_0^{\sqrt{a^2-x^2-y^2}} dz\, dy\, dx = 8\int_0^a\int_0^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2}\, dy\, dx$

$\displaystyle= 4\int_0^a \left[y\sqrt{a^2-x^2-y^2} + (a^2-x^2)\arcsin\left(\frac{y}{\sqrt{a^2-x^2}}\right)\right]_0^{\sqrt{a^2-x^2}} dx$

$\displaystyle= 4\left(\frac{\pi}{2}\right)\int_0^a (a^2-x^2)\, dx = \left[2\pi\left(a^2x - \frac{1}{3}x^3\right)\right]_0^a = \frac{4}{3}\pi a^3$

20. $4\int_0^6\int_0^{\sqrt{36-x^2}}\int_0^{36-x^2-y^2} dz\,dy\,dx = 4\int_0^6\int_0^{\sqrt{36-x^2}}(36-x^2-y^2)dy\,dx = 4\int_0^6\left[36y - x^2y - \frac{y^3}{3}\right]_0^{\sqrt{36-x^2}} dx$

$$= 4\int_0^6\left[36\sqrt{36-x^2} - x^2\sqrt{36-x^2} - \frac{1}{3}(36-x^2)^{3/2}\right]dx$$

$$= 4\left[9x\sqrt{36-x^2} + 324\arcsin\left(\frac{x}{6}\right) + \frac{1}{6}x(36-x^2)^{3/2}\right]_0^6 = 4(162\pi) = 648\pi$$

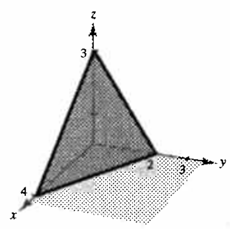
21. $\int_0^2\int_0^{4-x^2}\int_0^{4-x^2} dz\,dy\,dx = \int_0^2 (4-x^2)^2\,dx = \int_0^2 (16-8x^2+x^4)\,dx = \left[16x - \frac{8}{3}x^3 + \frac{1}{5}x^5\right]_0^2 = \frac{256}{15}$

22. $\int_0^2\int_0^{2-x}\int_0^{9-x^2} dz\,dy\,dx = \int_0^2\int_0^{2-x}(9-x^2)\,dy\,dx = \int_0^2 (9-x^2)(2-x)\,dx$

$$= \int_0^2 (18-9x-2x^2+x^3)\,dx = \left[18x - \frac{9}{2}x^2 - \frac{2}{3}x^3 + \frac{1}{4}x^4\right]_0^2 = \frac{50}{3}$$

23. Plane: $3x + 6y + 4z = 12$

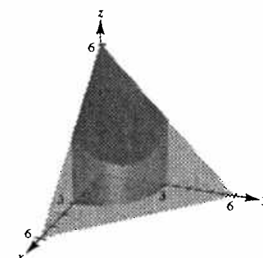
$$\int_0^3\int_0^{(12-4z)/3}\int_0^{(12-4z-3x)/6} dy\,dx\,dz$$



24. Top plane: $x + y + z = 6$

Side cylinder: $x^2 + y^2 = 9$

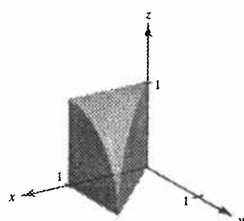
$$\int_0^3\int_0^{\sqrt{9-y^2}}\int_0^{6-x-y} dz\,dx\,dy$$



25. Top cylinder: $y^2 + z^2 = 1$

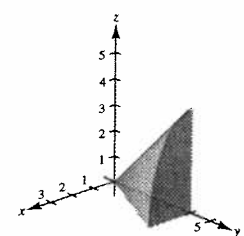
Side plane: $x = y$

$$\int_0^1\int_0^x\int_0^{\sqrt{1-y^2}} dz\,dy\,dx$$



26. Elliptic cone: $4x^2 + z^2 = y^2$

$$\int_0^4\int_z^4\int_0^{\sqrt{y^2-z^2}/2} dx\,dy\,dz$$



27. $Q = \{(x, y, z): 0 \le x \le 1, 0 \le y \le x, 0 \le z \le 3\}$

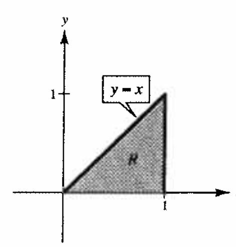
$$\iiint_Q xyz\,dV = \int_0^3\int_0^1\int_y^1 xyz\,dx\,dy\,dz = \int_0^3\int_0^1\int_0^x xyz\,dy\,dx\,dz$$

$$= \int_0^1\int_0^3\int_y^1 xyz\,dx\,dz\,dy$$

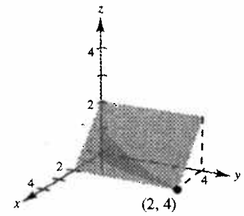
$$= \int_0^1\int_0^3\int_0^x xyz\,dy\,dz\,dx$$

$$= \int_0^1\int_y^1\int_0^3 xyz\,dz\,dx\,dy$$

$$= \int_0^1\int_0^x\int_0^3 xyz\,dz\,dy\,dx\left(=\frac{9}{16}\right)$$

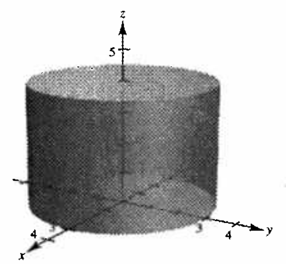


28. $Q = \{(x, y, z): 0 \le x \le 2, x^2 \le y \le 4, 0 \le z \le 2 - x\}$



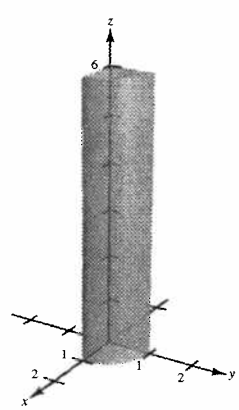
$$\iiint_Q xyz\, dV = \int_0^2 \int_{x^2}^4 \int_0^{2-x} xyz\, dz\, dy\, dx$$
$$= \int_0^4 \int_0^{\sqrt{y}} \int_0^{2-x} xyz\, dz\, dx\, dy$$
$$= \int_0^2 \int_0^{2-x} \int_{x^2}^4 xyz\, dy\, dz\, dx$$
$$= \int_0^2 \int_0^{2-z} \int_{x^2}^4 xyz\, dy\, dx\, dz$$
$$= \int_0^2 \int_0^{(2-z)^2} \int_0^{\sqrt{y}} xyz\, dx\, dy\, dz + \int_0^2 \int_{(2-z)^2}^4 \int_0^{2-z} xyz\, dx\, dy\, dz$$
$$= \int_0^4 \int_0^{2-\sqrt{y}} \int_0^{\sqrt{y}} xyz\, dx\, dz\, dy + \int_0^4 \int_{2-\sqrt{y}}^2 \int_0^{2-z} dx\, dz\, dy \left(= \frac{104}{21}\right)$$

29. $Q = \{(x, y, z): x^2 + y^2 \le 9, 0 \le z \le 4\}$



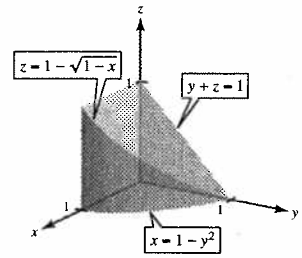
$$\iiint_Q xyz\, dV = \int_0^4 \int_{-3}^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} xyz\, dy\, dx\, dz$$
$$= \int_0^4 \int_{-3}^3 \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} xyz\, dx\, dy\, dz$$
$$= \int_{-3}^3 \int_0^4 \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} xyz\, dx\, dz\, dy$$
$$= \int_{-3}^3 \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} \int_0^4 xyz\, dz\, dx\, dy$$
$$= \int_{-3}^3 \int_0^4 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} xyz\, dy\, dz\, dx$$
$$= \int_{-3}^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_0^4 xyz\, dz\, dy\, dx\ (= 0)$$

30. $Q = \{(x, y, z): 0 \le x \le 1, y \le 1 - x^2, 0 \le z \le 6\}$



$$\iiint_Q xyz\, dV = \int_0^1 \int_0^{1-x^2} \int_0^6 xyz\, dz\, dy\, dx$$
$$= \int_0^1 \int_0^{\sqrt{1-y}} \int_0^6 xyz\, dz\, dx\, dy$$
$$= \int_0^1 \int_0^6 \int_0^{\sqrt{1-y}} xyz\, dx\, dz\, dy$$
$$= \int_0^6 \int_0^1 \int_0^{\sqrt{1-y}} xyz\, dx\, dy\, dz$$
$$= \int_0^1 \int_0^6 \int_0^{1-x^2} xyz\, dy\, dz\, dx$$
$$= \int_0^6 \int_0^1 \int_0^{1-x^2} xyz\, dy\, dx\, dz = \frac{3}{2}$$

31. $Q = \{(x, y, z)\colon\ 0 \le y \le 1,\ 0 \le x \le 1 - y^2,\ 0 \le z \le 1 - y\}$



$$\int_0^1\int_0^{1-y^2}\int_0^{1-y} dz\,dx\,dy = \int_0^1\int_0^{\sqrt{1-x}}\int_0^{1-y} dz\,dy\,dx$$

$$= \int_0^1\int_0^{2z-z^2}\int_0^{1-z} dy\,dx\,dz + \int_0^1\int_{2z-z^2}^1\int_0^{\sqrt{1-x}} dy\,dx\,dz$$

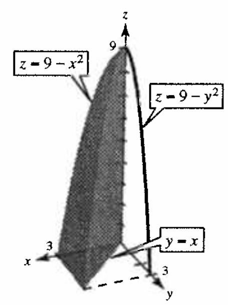
$$= \int_0^1\int_{1-\sqrt{1-x}}^1\int_0^{1-z} dy\,dz\,dx + \int_0^1\int_0^{1-\sqrt{1-x}}\int_0^{\sqrt{1-x}} dy\,dz\,dx$$

$$= \int_0^1\int_0^{1-y}\int_0^{1-y^2} dx\,dz\,dy$$

$$= \int_0^1\int_0^{1-z}\int_0^{1-y^2} dx\,dy\,dz$$

$$= \frac{5}{12}$$

32. $Q = \{(x, y, z)\colon\ 0 \le x \le 3,\ 0 \le y \le x,\ 0 \le z \le 9 - x^2\}$



$$\int_0^3\int_0^x\int_0^{9-x^2} dz\,dy\,dx = \int_0^3\int_y^3\int_0^{9-x^2} dz\,dx\,dy$$

$$= \int_0^3\int_0^{9-x^2}\int_0^x dy\,dz\,dx$$

$$= \int_0^9\int_0^{\sqrt{9-z}}\int_0^x dy\,dx\,dz$$

$$= \int_0^9\int_0^{\sqrt{9-z}}\int_y^{\sqrt{9-z}} dx\,dy\,dz$$

$$= \int_0^3\int_0^{9-y^2}\int_y^{\sqrt{9-z}} dx\,dz\,dy$$

$$= \frac{81}{4}$$

33. $$m = k\int_0^6\int_0^{4-(2x/3)}\int_0^{2-(y/2)-(x/3)} dz\,dy\,dx$$

$$= 8k$$

$$M_{yz} = k\int_0^6\int_0^{4-(2x/3)}\int_0^{2-(y/2)-(x/3)} x\,dz\,dy\,dx$$

$$= 12k$$

$$\bar{x} = \frac{M_{yz}}{m} = \frac{12k}{8k} = \frac{3}{2}$$

34. $$m = k\int_0^5\int_0^{5-x}\int_0^{1/5(15-3x-3y)} y\,dz\,dy\,dz = \frac{125}{8}k$$

$$M_{xz} = k\int_0^5\int_0^{5-x}\int_0^{1/5(15-3x-3y)} y^2\,dz\,dy\,dx = \frac{125}{4}k$$

$$\bar{y} = \frac{M_{xz}}{m} = 2$$

35. $$m = k\int_0^4\int_0^4\int_0^{4-x} x\,dz\,dy\,dx = k\int_0^4\int_0^4 x(4-x)\,dy\,dx$$

$$= 4k\int_0^4 (4x - x^2)\,dx = \frac{128k}{3}$$

$$M_{xy} = k\int_0^4\int_0^4\int_0^{4-x} xz\,dz\,dy\,dx = k\int_0^4\int_0^4 x\frac{(4-x)^2}{2}\,dy\,dx$$

$$= 2k\int_0^4 (16x - 8x^2 + x^3)\,dx = \frac{128k}{3}$$

$$\bar{z} = \frac{M_{xy}}{m} = 1$$

36. $$m = k\int_0^b\int_0^{a[1-(y/b)]}\int_0^{c[1-(y/b)-(x/a)]} dz\,dx\,dy = \frac{kabc}{6}$$

$$M_{xz} = k\int_0^b\int_0^{a[1-(y/b)]}\int_0^{c[1-(y/b)-(x/a)]} y\,dz\,dx\,dy = \frac{kab^2c}{24}$$

$$\bar{y} = \frac{M_{xz}}{m} = \frac{kab^2c/24}{kabc/6} = \frac{b}{4}$$

37. $m = k\int_0^b\int_0^b\int_0^b xy\,dz\,dy\,dx = \frac{kb^5}{4}$

$M_{yz} = k\int_0^b\int_0^b\int_0^b x^2y\,dz\,dy\,dx = \frac{kb^6}{6}$

$M_{xz} = k\int_0^b\int_0^b\int_0^b xy^2\,dz\,dy\,dx = \frac{kb^6}{6}$

$M_{xy} = k\int_0^b\int_0^b\int_0^b xyz\,dz\,dy\,dx = \frac{kb^6}{8}$

$\bar{x} = \frac{M_{yz}}{m} = \frac{kb^6/6}{kb^5/4} = \frac{2b}{3}$

$\bar{y} = \frac{M_{xz}}{m} = \frac{kb^6/6}{kb^5/4} = \frac{2b}{3}$

$\bar{z} = \frac{M_{xy}}{m} = \frac{kb^6/8}{kb^5/4} = \frac{b}{2}$

38. $m = k\int_0^a\int_0^b\int_0^c z\,dz\,dy\,dx = \frac{kabc^2}{2}$

$M_{xy} = k\int_0^a\int_0^b\int_0^c z^2\,dz\,dy\,dx = \frac{kabc^3}{3}$

$M_{yz} = k\int_0^a\int_0^b\int_0^c xz\,dz\,dy\,dx = \frac{ka^2bc^2}{4}$

$M_{xz} = k\int_0^a\int_0^b\int_0^c yz\,dz\,dy\,dx = \frac{kab^2c^2}{4}$

$\bar{x} = \frac{M_{yz}}{m} = \frac{ka^2bc^2/4}{kabc^2/2} = \frac{a}{2}$

$\bar{y} = \frac{M_{xz}}{m} = \frac{kab^2c^2/4}{kabc^2/2} = \frac{b}{2}$

$\bar{z} = \frac{M_{xy}}{m} = \frac{kabc^3/3}{kabc^2/2} = \frac{2c}{3}$

39. $\bar{x}$ will be greater than 2, whereas $\bar{y}$ and $\bar{z}$ will be unchanged.

40. $\bar{z}$ will be greater than 8/5, whereas $\bar{x}$ and $\bar{y}$ will be unchanged.

41. $\bar{y}$ will be greater than 0, whereas $\bar{x}$ and $\bar{z}$ will be unchanged.

42. $\bar{x}$, $\bar{y}$ and $\bar{z}$ will all be greater than their original values.

43. $m = \frac{1}{3}k\pi r^2h$

$\bar{x} = \bar{y} = 0$ by symmetry

$M_{xy} = 4k\int_0^r\int_0^{\sqrt{r^2-x^2}}\int_{h\sqrt{x^2+y^2}/r}^{h} z\,dz\,dy\,dx$

$= \frac{2kh^2}{r^2}\int_0^r\int_0^{\sqrt{r^2-x^2}} (r^2 - x^2 - y^2)\,dy\,dx$

$= \frac{4kh^2}{3r^2}\int_0^r (r^2 - x^2)^{3/2}\,dx$

$= \frac{k\pi r^2h^2}{4}$

$\bar{z} = \frac{M_{xy}}{m} = \frac{k\pi r^2h^2/4}{k\pi r^2h/3} = \frac{3h}{4}$

44. $m = 2k\int_0^2\int_0^{\sqrt{4-x^2}}\int_0^y dz\,dy\,dx$

$= k\int_0^2 (4 - x^2)\,dx = \frac{16k}{3}$

$M_{yz} = k\int_{-2}^2\int_0^{\sqrt{4-x^2}}\int_0^y x\,dz\,dy\,dx = 0$

$M_{xz} = 2k\int_0^2\int_0^{\sqrt{4-x^2}}\int_0^y y\,dz\,dy\,dx = 2k\pi$

$M_{xy} = 2k\int_0^2\int_0^{\sqrt{4-x^2}}\int_0^y z\,dz\,dy\,dx = k\pi$

$\bar{x} = \frac{M_{yz}}{m} = \frac{0}{16k/3} = 0$

$\bar{y} = \frac{M_{xz}}{m} = \frac{2k\pi}{16k/3} = \frac{3\pi}{8}$

$\bar{z} = \frac{M_{xy}}{m} = \frac{k\pi}{16k/3} = \frac{3\pi}{16}$

45. $m = \dfrac{128k\pi}{3}$

$\bar{x} = \bar{y} = 0$ by symmetry

$z = \sqrt{4^2 - x^2 - y^2}$

$$M_{xy} = 4k\int_0^4\int_0^{\sqrt{4^2-x^2}}\int_0^{\sqrt{4^2-x^2-y^2}} z\,dz\,dy\,dx$$

$$= 2k\int_0^4\int_0^{\sqrt{4^2-x^2}} (4^2 - x^2 - y^2)\,dy\,dx = 2k\int_0^4 \left[16y - x^2y - \frac{1}{3}y^3\right]_0^{\sqrt{4^2-x^2}} dx = \frac{4k}{3}\int_0^4 (4^2 - x^2)^{3/2}\,dx$$

$$= \frac{1024k}{3}\int_0^{\pi/2} \cos^4\theta\,d\theta \qquad (\text{let } x = 4\sin\theta)$$

$= 64\pi k$ by Wallis's Formula

$$\bar{z} = \frac{M_{xy}}{m} = \frac{64k\pi}{1}\cdot\frac{3}{128k\pi} = \frac{3}{2}$$

46. $\bar{x} = 0$

$$m = 2k\int_0^2\int_0^1\int_0^{1/(y^2+1)} dz\,dy\,dx = 2k\int_0^2\int_0^1 \frac{1}{y^2+1}\,dy\,dx = 2k\left(\frac{\pi}{4}\right)\int_0^2 dx = k\pi$$

$$M_{xz} = 2k\int_0^2\int_0^1\int_0^{1/(y^2+1)} y\,dz\,dy\,dx = 2k\int_0^2\int_0^1 \frac{y}{y^2+1}\,dy\,dx = k\int_0^2 (\ln 2)\,dx = k\ln 4$$

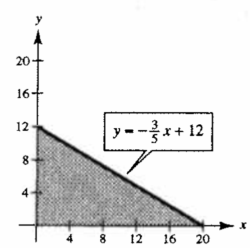
$$M_{xy} = 2k\int_0^2\int_0^1\int_0^{1/(y^2+1)} z\,dz\,dy\,dx$$

$$= k\int_0^2\int_0^1 \frac{1}{(y^2+1)^2}\,dy\,dx = k\int_0^2\left[\frac{y}{2(y^2+1)} + \frac{1}{2}\arctan y\right]_0^1 dx = k\left(\frac{1}{4} + \frac{\pi}{8}\right)\int_0^2 dx = k\left(\frac{1}{2} + \frac{\pi}{4}\right)$$

$$\bar{y} = \frac{M_{xz}}{m} = \frac{k\ln 4}{k\pi} = \frac{\ln 4}{\pi}$$

$$\bar{z} = \frac{M_{xy}}{m} = k\left(\frac{1}{2} + \frac{\pi}{4}\right)/k\pi = \frac{2+\pi}{4\pi}$$

47. $f(x, y) = \dfrac{5}{12}y$



$$m = k\int_0^{20}\int_0^{-(3/5)x+12}\int_0^{(5/12)y} dz\,dy\,dx = 200k$$

$$M_{yz} = k\int_0^{20}\int_0^{-(3/5)x+12}\int_0^{(5/12)y} x\,dz\,dy\,dx = 1000k$$

$$M_{xz} = k\int_0^{20}\int_0^{-(3/5)x+12}\int_0^{(5/12)y} y\,dz\,dy\,dx = 1200k$$

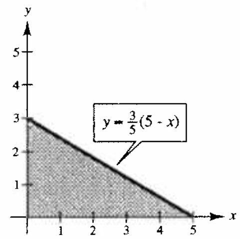
$$M_{xy} = k\int_0^{20}\int_0^{-(3/5)x+12}\int_0^{(5/12)y} z\,dz\,dy\,dx = 250k$$

$$\bar{x} = \frac{M_{yz}}{m} = \frac{1000k}{200k} = 5$$

$$\bar{y} = \frac{M_{xz}}{m} = \frac{1200k}{200k} = 6$$

$$\bar{z} = \frac{M_{xy}}{m} = \frac{250k}{200k} = \frac{5}{4}$$

48. $f(x, y) = \frac{1}{15}(60 - 12x - 20y)$



$$m = k\int_0^5\int_0^{-(3/5)x+3}\int_0^{(1/15)(60-12x-20y)} dz\,dy\,dx = 10k$$

$$M_{yz} = k\int_0^5\int_0^{-(3/5)x+3}\int_0^{(1/15)(60-12x-20y)} x\,dz\,dy\,dx = \frac{25k}{2}$$

$$M_{xz} = k\int_0^5\int_0^{-(3/5)x+3}\int_0^{-(1/15)(60-12x-20y)} y\,dz\,dy\,dx = \frac{15k}{2}$$

$$M_{xy} = k\int_0^5\int_0^{-(3/5)x+3}\int_0^{(1/15)(60-12x-20y)} z\,dz\,dy\,dx = 10k$$

$$\bar{x} = \frac{M_{yz}}{m} = \frac{25k/2}{10k} = \frac{5}{4}$$

$$\bar{y} = \frac{M_{xz}}{m} = \frac{15k/2}{10k} = \frac{3}{4}$$

$$\bar{z} = \frac{M_{xy}}{m} = \frac{10k}{10k} = 1$$

49. (a) $I_x = k\int_0^a\int_0^a\int_0^a (y^2 + z^2)\,dx\,dy\,dz = ka\int_0^a\int_0^a (y^2 + z^2)\,dy\,dz$

$$= ka\int_0^a \left[\frac{1}{3}y^3 + z^2y\right]_0^a dz = ka\int_0^a \left(\frac{1}{3}a^3 + az^2\right) dz = \left[ka\left(\frac{1}{3}a^3z + \frac{1}{3}az^3\right)\right]_0^a = \frac{2ka^5}{3}$$

$I_x = I_y = I_z = \frac{2ka^5}{3}$ by symmetry

(b) $I_x = k\int_0^a\int_0^a\int_0^a (y^2 + z^2)xyz\,dx\,dy\,dz = \frac{ka^2}{2}\int_0^a\int_0^a (y^3z + yz^3)\,dy\,dz$

$$= \frac{ka^2}{2}\int_0^a \left[\frac{y^4z}{4} + \frac{y^2z^3}{2}\right]_0^a dz = \frac{ka^4}{8}\int_0^a (a^2z + 2z^3)\,dz = \left[\frac{ka^4}{8}\left(\frac{a^2z^2}{2} + \frac{2z^4}{4}\right)\right]_0^a = \frac{ka^8}{8}$$

$I_x = I_y = I_z = \frac{ka^8}{8}$ by symmetry

50. (a) $I_{xy} = k\int_{-a/2}^{a/2}\int_{-a/2}^{a/2}\int_{-a/2}^{a/2} z^2\,dz\,dy\,dx = \frac{ka^5}{12}$

$I_{xz} = I_{yz} = \frac{ka^5}{12}$ by symmetry

$$I_x = I_y = I_z = \frac{ka^5}{12} + \frac{ka^5}{12} = \frac{ka^5}{6}$$

(b) $I_{xy} = k\int_{-a/2}^{a/2}\int_{-a/2}^{a/2}\int_{-a/2}^{a/2} z^2(x^2 + y^2)\,dz\,dy\,dx = \frac{a^3k}{12}\int_{-a/2}^{a/2}\int_{-a/2}^{a/2} (x^2 + y^2)\,dy\,dx = \frac{a^7k}{72}$

$$I_{xz} = k\int_{-a/2}^{a/2}\int_{-a/2}^{a/2}\int_{-a/2}^{a/2} y^2(x^2 + y^2)\,dz\,dy\,dx = ka\int_{-a/2}^{a/2}\int_{-a/2}^{a/2} (x^2y^2 + y^4)\,dy\,dx = \frac{7ka^7}{360}$$

$I_{yz} = I_{xz}$ by symmetry

$$I_x = I_{xy} + I_{xz} = \frac{a^7k}{30}$$

$$I_y = I_{xy} + I_{yz} = \frac{a^7k}{30}$$

$$I_z = I_{yz} + I_{xz} = \frac{7ka^7}{180}$$

51. (a) $I_x = k\int_0^4\int_0^4\int_0^{4-x}(y^2+z^2)\,dz\,dy\,dx = k\int_0^4\int_0^4\left[y^2(4-x)+\frac{1}{3}(4-x)^3\right]dy\,dx$

$= k\int_0^4\left[\frac{y^3}{3}(4-x)+\frac{y}{3}(4-x)^3\right]_0^4 dx = k\int_0^4\left[\frac{64}{3}(4-x)+\frac{4}{3}(4-x)^3\right]dx$

$= k\left[-\frac{32}{3}(4-x)^2-\frac{1}{3}(4-x)^4\right]_0^4 = 256k$

$I_y = k\int_0^4\int_0^4\int_0^{4-x}(x^2+z^2)\,dz\,dy\,dx = k\int_0^4\int_0^4\left[x^2(4-x)+\frac{1}{3}(4-x)^3\right]dy\,dx$

$= 4k\int_0^4\left[4x^2-x^3+\frac{1}{3}(4-x)^3\right]dx = 4k\left[\frac{4}{3}x^3-\frac{1}{4}x^4-\frac{1}{12}(4-x)^4\right]_0^4 = \frac{512k}{3}$

$I_z = k\int_0^4\int_0^4\int_0^{4-x}(x^2+y^2)\,dz\,dy\,dx = k\int_0^4\int_0^4(x^2+y^2)(4-x)\,dy\,dx$

$= k\int_0^4\left[\left(x^2y+\frac{y^3}{3}\right)(4-x)\right]_0^4 dx = k\int_0^4\left(4x^2+\frac{64}{3}\right)(4-x)\,dx = 256k$

(b) $I_x = k\int_0^4\int_0^4\int_0^{4-x}y(y^2+z^2)\,dz\,dy\,dx = k\int_0^4\int_0^4\left[y^3(4-x)+\frac{1}{3}y(4-x)^3\right]dy\,dx$

$= k\int_0^4\left[\frac{y^4}{4}(4-x)+\frac{y^2}{6}(4-x)^3\right]_0^4 dx = k\int_0^4\left[64(4-x)+\frac{8}{3}(4-x)^3\right]dx$

$= k\left[-32(4-x)^2-\frac{2}{3}(4-x)^4\right]_0^4 = \frac{2048k}{3}$

$I_y = k\int_0^4\int_0^4\int_0^{4-x}y(x^2+z^2)\,dz\,dy\,dx = k\int_0^4\int_0^4\left[x^2y(4-x)+\frac{1}{3}y(4-x)^3\right]dy\,dx$

$= 8k\int_0^4\left[4x^2-x^3+\frac{1}{3}(4-x)^3\right]dx = 8k\left[\frac{4}{3}x^3-\frac{1}{4}x^4-\frac{1}{12}(4-x)^4\right]_0^4 = \frac{1024k}{3}$

$I_z = k\int_0^4\int_0^4\int_0^{4-x}y(x^2+y^2)\,dz\,dy\,dx = k\int_0^4\int_0^4(x^2y+y^3)(4-x)\,dx$

$= k\int_0^4\left[\left(\frac{x^2y^2}{2}+\frac{y^4}{4}\right)(4-x)\right]_0^4 dx = k\int_0^4(8x^2+64)(4-x)\,dx$

$= 8k\int_0^4(32-8x+4x^2-x^3)\,dx = \left[8k\left(32x-4x^2+\frac{4}{3}x^3-\frac{1}{4}x^4\right)\right]_0^4 = \frac{2048k}{3}$

52. (a) $I_{xy} = k\int_0^4\int_0^2\int_0^{4-y^2}z^3\,dz\,dy\,dx = k\int_0^4\int_0^2\frac{1}{4}(4-y^2)^4\,dy\,dx$

$= \frac{k}{4}\int_0^4\int_0^2(256-256y^2+96y^4-16y^6+y^8)\,dy\,dx$

$= \frac{k}{4}\int_0^4\left[256y-\frac{256y^3}{3}+\frac{96y^5}{5}-\frac{16y^7}{7}+\frac{y^9}{9}\right]_0^2 dx = k\int_0^4\frac{16,384}{945}\,dx = \frac{65,536k}{315}$

$I_{xz} = k\int_0^4\int_0^2\int_0^{4-y^2}y^2z\,dz\,dy\,dx = k\int_0^4\int_0^2\frac{1}{2}y^2(4-y^2)^2\,dy\,dx$

$= k\int_0^4\int_0^2\frac{1}{2}(16y^2-8y^4+y^6)\,dy\,dx = \frac{k}{2}\int_0^4\left[\frac{16y^3}{3}-\frac{8y^5}{5}+\frac{y^7}{7}\right]_0^2 dx = \frac{k}{2}\int_0^4\frac{1024}{105}\,dx = \frac{2048k}{105}$

$I_{yz} = k\int_0^4\int_0^2\int_0^{4-y^2}x^2z\,dz\,dy\,dx = k\int_0^4\int_0^2\frac{1}{2}x^2(4-y^2)^2\,dy\,dx$

$= k\int_0^4\int_0^2\frac{1}{2}x^2(16-8y^2+y^4)\,dy\,dx = \frac{k}{2}\int_0^4\left[x^2\left(16y-\frac{8y^3}{3}+\frac{y^5}{5}\right)\right]_0^2 dx = \frac{k}{2}\int_0^4\frac{256}{15}x^2\,dx = \frac{8192k}{45}$

$I_x = I_{xz}+I_{xy} = \frac{2048k}{9}, I_y = I_{yz}+I_{xy} = \frac{8192k}{21}, I_z = I_{yz}+I_{xz} = \frac{63,488k}{315}$

—CONTINUED—

52. —CONTINUED—

(b) $I_{xy} = \int_0^4 \int_0^2 \int_0^{4-y^2} z^2(4 - z)\,dz\,dy\,dx$

$$= k\int_0^4 \int_0^2 \int_0^{4-y^2} 4z^2\,dz\,dy\,dx - k\int_0^4 \int_0^2 \int_0^{4-y^2} z^3\,dz\,dy\,dx = \frac{32{,}768k}{105} - \frac{65{,}536k}{315} = \frac{32{,}768k}{315}$$

$$I_{xz} = \int_0^4 \int_0^2 \int_0^{4-y^2} y^2(4 - z)\,dz\,dy\,dx$$

$$= k\int_0^4 \int_0^2 \int_0^{4-y^2} 4y^2\,dz\,dy\,dx - k\int_0^4 \int_0^2 \int_0^{4-y^2} y^2 z\,dz\,dy\,dx = \frac{1024k}{15} - \frac{2048k}{105} = \frac{1024k}{21}$$

$$I_{yz} = k\int_0^4 \int_0^2 \int_0^{4-y^2} x^2(4 - z)\,dz\,dy\,dx$$

$$= k\int_0^4 \int_0^2 \int_0^{4-y^2} 4x^2\,dz\,dy\,dx - k\int_0^4 \int_0^2 \int_0^{4-y^2} x^2 z\,dz\,dy\,dx = \frac{4096k}{9} - \frac{8192k}{45} = \frac{4096k}{15}$$

$$I_x = I_{xz} + I_{xy} = \frac{48{,}128k}{315},\quad I_y = I_{yz} + I_{xy} = \frac{118{,}784k}{315},\quad I_z = I_{xz} + I_{yz} = \frac{11{,}264k}{35}$$

53. $I_{xy} = k\int_{-L/2}^{L/2} \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} z^2\,dz\,dx\,dy = k\int_{-L/2}^{L/2} \int_{-a}^{a} \frac{2}{3}(a^2 - x^2)\sqrt{a^2 - x^2}\,dx\,dy$

$$= \frac{2}{3}\int_{-L/2}^{L/2} k\left[\frac{a^2}{2}\left(x\sqrt{a^2 - x^2} + a^2 \arcsin\frac{x}{a}\right) - \frac{1}{8}\left(x(2x^2 - a^2)\sqrt{x^2 - a^2} + a^4 \arcsin\frac{x}{a}\right)\right]_{-a}^{a} dy$$

$$= \frac{2k}{3}\int_{-L/2}^{L/2} 2\left(\frac{a^4\pi}{4} - \frac{a^4\pi}{16}\right) dy = \frac{a^4\pi Lk}{4}$$

Since $m = \pi a^2 Lk$, $I_{xy} = ma^2/4$.

$$I_{xz} = k\int_{-L/2}^{L/2} \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} y^2\,dz\,dx\,dy = 2k\int_{-L/2}^{L/2} \int_{-a}^{a} y^2\sqrt{a^2 - x^2}\,dx\,dy$$

$$= 2k\int_{-L/2}^{L/2} \left[\frac{y^2}{2}\left(x\sqrt{a^2 - x^2} + a^2 \arcsin\frac{x}{a}\right)\right]_{-a}^{a} dy = k\pi a^2\int_{-L/2}^{L/2} y^2\,dy = \frac{2k\pi a^2}{3}\left(\frac{L^3}{8}\right) = \frac{1}{12}mL^2$$

$$I_{yz} = k\int_{-L/2}^{L/2} \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} x^2\,dz\,dx\,dy = 2k\int_{-L/2}^{L/2} \int_{-a}^{a} x^2\sqrt{a^2 - x^2}\,dx\,dy$$

$$= 2k\int_{-L/2}^{L/2} \frac{1}{8}\left[x(2x^2 - a^2)\sqrt{a^2 - x^2} + a^4 \arcsin\frac{x}{a}\right]_{-a}^{a} dy = \frac{ka^4\pi}{4}\int_{-L/2}^{L/2} dy = \frac{ka^4\pi L}{4} = \frac{ma^2}{4}$$

$$I_x = I_{xy} + I_{xz} = \frac{ma^2}{4} + \frac{mL^2}{12} = \frac{m}{12}(3a^2 + L^2)$$

$$I_y = I_{xy} + I_{yz} = \frac{ma^2}{4} + \frac{ma^2}{4} = \frac{ma^2}{2}$$

$$I_z = I_{xz} + I_{yz} = \frac{mL^2}{12} + \frac{ma^2}{4} = \frac{m}{12}(3a^2 + L^2)$$

54. $I_{xy} = \int_{-c/2}^{c/2}\int_{-a/2}^{a/2}\int_{-b/2}^{b/2} z^2\,dz\,dy\,dx = \frac{b^3}{12}\int_{-c/2}^{c/2}\int_{-a/2}^{a/2} dy\,dx = \frac{1}{12}b^2(abc) = \frac{1}{12}mb^2$

$I_{xz} = \int_{-c/2}^{c/2}\int_{-a/2}^{a/2}\int_{-b/2}^{b/2} y^2\,dz\,dy\,dx = b\int_{-c/2}^{c/2}\int_{-a/2}^{a/2} y^2\,dy\,dx = \frac{ba^3}{12}\int_{-c/2}^{c/2} dx = \frac{ba^3c}{12} = \frac{1}{12}a^2(abc) = \frac{1}{12}ma^2$

$I_{yz} = \int_{-c/2}^{c/2}\int_{-a/2}^{a/2}\int_{-b/2}^{b/2} x^2\,dz\,dy\,dx = ab\int_{-c/2}^{c/2} x^2\,dx = \frac{abc^3}{12} = \frac{1}{12}c^2(abc) = \frac{1}{12}mc^2$

$I_x = I_{xy} + I_{xz} = \frac{1}{12}m(a^2 + b^2)$

$I_y = I_{xy} + I_{yz} = \frac{1}{12}m(b^2 + c^2)$

$I_z = I_{xz} + I_{yz} = \frac{1}{12}m(a^2 + c^2)$

55. $\int_{-1}^{1}\int_{-1}^{1}\int_{0}^{1-x} (x^2 + y^2)\sqrt{x^2 + y^2 + z^2}\,dz\,dy\,dx$

56. $\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{0}^{4-x^2-y^2} kx^2(x^2 + y^2)\,dz\,dy\,dx$

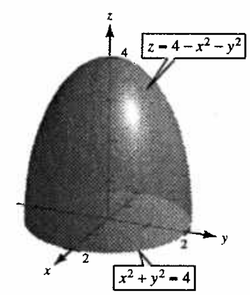
57. $\rho = kz$

(a) $m = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{0}^{4-x^2-y^2} (kz)\,dz\,dy\,dx \left(= \frac{32k\pi}{3}\right)$

(b) $\bar{x} = \bar{y} = 0$ by symmetry

$\bar{z} = \frac{M_{xy}}{m} = \frac{1}{m}\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{0}^{4-x^2-y^2} kz^2\,dz\,dy\,dx\ (= 2)$

$I_z = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{0}^{4-x^2-y^2} (x^2 + y^2)kz\,dz\,dy\,dx \left(= \frac{32k\pi}{3}\right)$



58. $\rho = kxy$

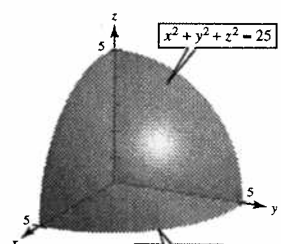
(a) $m = \int_{0}^{5}\int_{0}^{\sqrt{25-x^2}}\int_{0}^{\sqrt{25-x^2-y^2}} kxy\,dz\,dy\,dx \left(= \frac{625}{3}k\right)$

(b) $\bar{x} = \frac{M_{yz}}{m} = \frac{1}{m}\int_{0}^{5}\int_{0}^{\sqrt{25-x^2}}\int_{0}^{\sqrt{25-x^2-y^2}} x(kxy)\,dz\,dy\,dx \left(= \frac{25\pi}{32}\right)$

$\bar{y} = \bar{x}$ by symmetry

$\bar{z} = \frac{M_{xy}}{m} = \frac{1}{m}\int_{0}^{5}\int_{0}^{\sqrt{25-x^2}}\int_{0}^{\sqrt{25-x^2-y^2}} z(kxy)\,dz\,dy\,dx \left(= \frac{25}{16}\right)$

(c) $I_z = \int_{0}^{5}\int_{0}^{\sqrt{25-x^2}}\int_{0}^{\sqrt{25-x^2-y^2}} (x^2 + y^2)kxy\,dz\,dy\,dx \left(= \frac{62500}{21}k\right)$



59. See the definition, page 1024.

See Theorem 14.4, page 1025.

60. 6

61. (a) The annular solid on the right has the greater density.

(b) The annular solid on the right has the greater movement of inertia.

(c) The solid on the left will reach the bottom first. The solid on the right has a greater resistance to rotational motion.

62. Because the density increases as you move away from the axis of symmetry, the moment of inertia will increase.

63. $V = 1$ (unit cube)

$$\text{Average value} = \frac{1}{V}\iiint_Q f(x, y, z)\, dV$$
$$= \int_0^1\int_0^1\int_0^1 (z^2 + 4)\, dx\, dy\, dz$$
$$= \int_0^1\int_0^1 (z^2 + 4)\, dy\, dz$$
$$= \int_0^1 (z^2 + 4)\, dz$$
$$= \left[\frac{z^3}{3} + 4z\right]_0^1$$
$$= \frac{1}{3} + 4 = \frac{13}{3}$$

64. $V = 27$ (cube with sides of length 3)

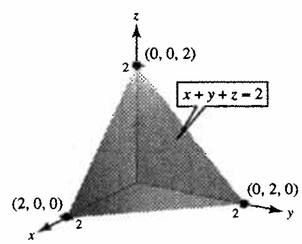
$$\text{Average value} = \frac{1}{V}\iiint_Q f(x, y, z)\, dV$$
$$= \frac{1}{27}\int_0^3\int_0^3\int_0^3 xyz\, dx\, dy\, dz$$
$$= \frac{1}{27}\int_0^3\int_0^3 \frac{9}{2}yz\, dy\, dz$$
$$= \frac{1}{27}\int_0^3 \frac{81}{4}z\, dz$$
$$= \frac{1}{27}\frac{729}{8} = \frac{27}{8}$$

65. $V = \frac{1}{3}\text{base} \times \text{height}$

$$= \frac{1}{3}\left(\frac{1}{2}(2)(2)\right)(2) = \frac{4}{3}$$

$f(x, y, z) = x + y + z$

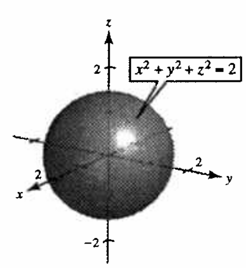
Plane: $x + y + z = 2$



$$\text{Average value} = \frac{1}{V}\iiint_Q f(x, y, z)\, dV$$
$$= \frac{3}{4}\int_0^2\int_0^{2-x}\int_0^{2-x-y} (x + y + z)\, dz\, dy\, dx$$
$$= \frac{3}{4}\int_0^2\int_0^{2-x} \frac{1}{2}(2 - x - y)(x + y + 2)\, dy\, dx$$
$$= \frac{3}{4}\int_0^2 \frac{1}{6}(x + 4)(x - 2)^2\, dx$$
$$= \frac{3}{4}(2) = \frac{3}{2}$$

66. $V = \frac{4}{3}\pi\left(\sqrt{2}\right)^3 = \frac{8}{3}\pi\sqrt{2}$

$f(x, y, z) = x + y$



$$\text{Average value} = \frac{1}{V}\iiint_Q f(x, y, z)\, dV$$
$$= \frac{3}{8\pi\sqrt{2}}\int_{-\sqrt{2}}^{\sqrt{2}}\int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}}\int_{-\sqrt{2-x^2-y^2}}^{\sqrt{2-x^2-y^2}} (x + y)\, dz\, dy\, dx$$
$$= 0 \text{ (by symmetry)}$$

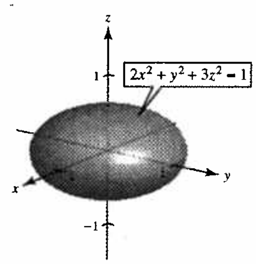
67. $1 - 2x^2 - y^2 - 3z^2 \ge 0$

$2x^2 + y^2 + 3z^2 \le 1$

$Q = \{(x, y, z)\colon 2x^2 + y^2 + 3z^2 \le 1\}$ ellipsoid

$$\int_{-1/\sqrt{2}}^{1/\sqrt{2}} \int_{-\sqrt{1-2x^2}}^{\sqrt{1-2x^2}} \int_{-\sqrt{(1-2x^2-y^2)/3}}^{\sqrt{(1-2x^2-y^2)/3}} (1 - 2x^2 - y^2 - 3z^2)\, dz\, dy\, dx \approx 0.684$$

Exact value: $\dfrac{4\sqrt{6}\pi}{45}$



68. $1 - x^2 - y^2 - z^2 \ge 0$

$x^2 + y^2 + z^2 \le 1$

$Q = \{(x, y, z)\colon x^2 + y^2 + z^2 \le 1\}$ sphere

$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} (1 - x^2 - y^2 - z^2)\, dz\, dy\, dx \approx 1.6755$$

Exact value: $\dfrac{8\pi}{15}$

69. $$\frac{14}{15} = \int_0^1 \int_0^{3-a-y^2} \int_a^{4-x-y^2} dz\, dx\, dy$$

$$= \int_0^1 \int_0^{3-a-y^2} (4 - x - y^2 - a)\, dx\, dy$$

$$= \int_0^1 \left[(4 - y^2 - a)x - \frac{x^2}{2} \right]_0^{3-a-y^2} dy$$

$$= \int_0^1 \left[(4 - y^2 - a)(3 - a - y^2) - \frac{(3 - a - y^2)}{2} \right] dy$$

$$= \frac{94}{15} - \frac{11a}{3} + \frac{1}{2}a^2$$

Hence, $3a^2 - 22a + 32 = 0$

$(a - 2)(3a - 16) = 0$

$a = 2, \dfrac{16}{3}.$

70. $x^2 + \dfrac{y^2}{b^2} + \dfrac{z^2}{9} = 1$

By symmetry, the volume in the first octant is

$\dfrac{1}{8}(16\pi) = 2\pi.$

$$2\pi = \int_0^1 \int_0^{b\sqrt{1-x^2}} \int_0^{3\sqrt{1-x^2-y^2/b^2}} 1\, dz\, dy\, dz$$

By trial and error, $b = 4$.

$\left[\textbf{Note:} \text{ Volume at ellipsoid } \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1 \text{ is } \dfrac{4}{3}\pi abc.\right]$

71. Let $y_k = 1 - x_k$.

$$\frac{\pi}{2n}(x_1 + \cdots + x_n) = \frac{\pi}{2n}(n - y_1 - y_2 - \cdots - y_n)$$

$$= \frac{\pi}{2} - \frac{\pi}{2n}(y_1 + \cdots + y_n)$$

Hence,

$$I_1 = \int_0^1 \int_0^1 \cdots \int_0^1 \cos^2\left\{\frac{\pi}{2n}(x_1 + \ldots + x_n)\right\} dx_1\, dx_2 \cdots dx_n$$

$$= \int_1^0 \int_1^0 \cdots \int_1^0 \sin^2\left\{\frac{\pi}{2n}(y_1 + \ldots + y_n)\right\}(-dy_1)(-dy_2) \cdots (-dy_n)$$

$$= \int_0^1 \int_0^1 \cdots \int_0^1 \sin^2\left\{\frac{\pi}{2n}(x_1 + \ldots + x_n)\right\} dx_1\, dx_2 \cdots dx_n = I_2$$

$I_1 + I_2 = 1 \implies I_1 = \dfrac{1}{2}.$

Finally, $\lim\limits_{n\to\infty} I_1 = \dfrac{1}{2}.$

Section 14.7 Triple Integrals in Cylindrical and Spherical Coordinates

1. $$\int_0^4\int_0^{\pi/2}\int_0^2 r\cos\theta\,dr\,d\theta\,dz = \int_0^4\int_0^{\pi/2}\left[\frac{r^2}{2}\cos\theta\right]_0^2 d\theta\,dz$$
$$= \int_0^4\int_0^{\pi/2} 2\cos\theta\,d\theta\,dz = \int_0^4\left[2\sin\theta\right]_0^{\pi/2} dz = \int_0^4 2\,dz = 8$$

2. $$\int_0^{\pi/4}\int_0^2\int_0^{2-r} rz\,dz\,dr\,d\theta = \int_0^{\pi/4}\int_0^2\left[\frac{rz^2}{2}\right]_0^{2-r} dr\,d\theta$$
$$= \frac{1}{2}\int_0^{\pi/4}\int_0^2 (4r - 4r^2 + r^3)\,dr\,d\theta = \frac{1}{2}\int_0^{\pi/4}\left[2r^2 - \frac{4r^3}{3} + \frac{r^4}{4}\right]_0^2 d\theta = \frac{2}{3}\int_0^{\pi/4} d\theta = \frac{\pi}{6}$$

3. $$\int_0^{\pi/2}\int_0^{2\cos^2\theta}\int_0^{4-r^2} r\sin\theta\,dz\,dr\,d\theta = \int_0^{\pi/2}\int_0^{2\cos^2\theta} r(4-r^2)\sin\theta\,dr\,d\theta = \int_0^{\pi/2}\left[\left(2r^2 - \frac{r^4}{4}\right)\sin\theta\right]_0^{2\cos^2\theta} d\theta$$
$$= \int_0^{\pi/2}[8\cos^4\theta - 4\cos^8\theta]\sin\theta\,d\theta = \left[-\frac{8\cos^5\theta}{5} + \frac{4\cos^9\theta}{9}\right]_0^{\pi/2} = \frac{52}{45}$$

4. $$\int_0^{\pi/2}\int_0^{\pi}\int_0^2 e^{-\rho^3}\rho^2\,d\rho\,d\theta\,d\phi = \int_0^{\pi/2}\int_0^{\pi}\left[-\frac{1}{3}e^{-\rho^3}\right]_0^2 d\theta\,d\phi = \int_0^{\pi/2}\int_0^{\pi}\frac{1}{3}(1 - e^{-8})\,d\theta\,d\phi = \frac{\pi^2}{6}(1 - e^{-8})$$

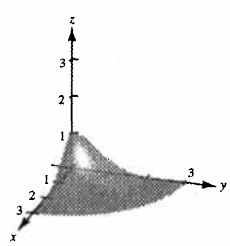
5. $$\int_0^{2\pi}\int_0^{\pi/4}\int_0^{\cos\phi} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \frac{1}{3}\int_0^{2\pi}\int_0^{\pi/4}\cos^3\phi\sin\phi\,d\phi\,d\theta = -\frac{1}{12}\int_0^{2\pi}\left[\cos^4\phi\right]_0^{\pi/4} d\theta = \frac{\pi}{8}$$

6. $$\int_0^{\pi/4}\int_0^{\pi/4}\int_0^{\cos\theta} \rho^2\sin\phi\cos\phi\,d\rho\,d\theta\,d\phi = \frac{1}{3}\int_0^{\pi/4}\int_0^{\pi/4}\cos^3\theta\sin\phi\cos\phi\,d\theta\,d\phi$$
$$= \frac{1}{3}\int_0^{\pi/4}\int_0^{\pi/4}\sin\phi\cos\phi[\cos\theta(1 - \sin^2\theta)]\,d\theta\,d\phi$$
$$= \frac{1}{3}\int_0^{\pi/4}\sin\phi\cos\phi\left[\sin\theta - \frac{\sin^3\theta}{3}\right]_0^{\pi/4} d\phi$$
$$= \frac{5\sqrt{2}}{36}\int_0^{\pi/4}\sin\phi\cos\phi\,d\phi = \left[\frac{5\sqrt{2}}{36}\frac{\sin^2\phi}{2}\right]_0^{\pi/4} = \frac{5\sqrt{2}}{144}$$

7. $$\int_0^4\int_0^z\int_0^{\pi/2} re^r\,d\theta\,dr\,dz = \pi(e^4 + 3)$$

8. $$\int_0^{\pi/2}\int_0^{\pi}\int_0^{\sin\theta}(2\cos\phi)\rho^2\,d\rho\,d\theta\,d\phi = \frac{8}{9}$$

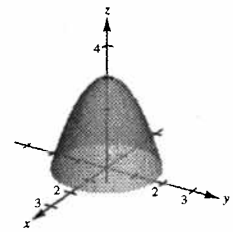
9. $$\int_0^{\pi/2}\int_0^3\int_0^{e^{-r^2}} r\,dz\,dr\,d\theta = \int_0^{\pi/2}\int_0^3 re^{-r^2}\,dr\,d\theta$$
$$= \int_0^{\pi/2}\left[-\frac{1}{2}e^{-r^2}\right]_0^3 d\theta$$
$$= \int_0^{\pi/2}\frac{1}{2}(1 - e^{-9})\,d\theta$$
$$= \frac{\pi}{4}(1 - e^{-9})$$



10. $\displaystyle\int_0^{2\pi}\int_0^{\sqrt{3}}\int_0^{3-r^2} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^{\sqrt{3}} r(3-r^2)\,dr\,d\theta$

$\displaystyle= \int_0^{2\pi}\left[\left(\frac{3r^2}{2}-\frac{r^4}{4}\right)\right]_0^{\sqrt{3}}\,d\theta$

$\displaystyle= \int_0^{2\pi}\frac{9}{4}\,d\theta = \frac{9\pi}{2}$

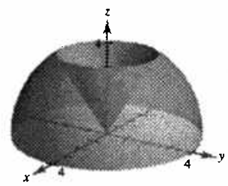


11. $\displaystyle\int_0^{2\pi}\int_{\pi/6}^{\pi/2}\int_0^4 \rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \frac{64}{3}\int_0^{2\pi}\int_{\pi/6}^{\pi/2}\sin\phi\,d\phi\,d\theta$

$\displaystyle= \frac{64}{3}\int_0^{2\pi}\Big[-\cos\phi\Big]_{\pi/6}^{\pi/2}\,d\theta$

$\displaystyle= \frac{32\sqrt{3}}{3}\int_0^{2\pi}d\theta$

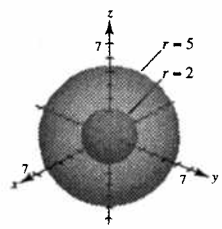
$\displaystyle= \frac{64\sqrt{3}\pi}{3}$



12. $\displaystyle\int_0^{2\pi}\int_0^{\pi}\int_2^5 \rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \frac{117}{3}\int_0^{2\pi}\int_0^{\pi}\sin\phi\,d\phi\,d\theta$

$\displaystyle= \frac{117}{3}\int_0^{2\pi}\Big[-\cos\phi\Big]_0^{\pi}\,d\theta$

$\displaystyle= \frac{468\pi}{3} = 156\pi$



13. $\displaystyle\int_0^{2\pi}\int_0^2\int_{r^2}^4 r^2\cos\theta\,dz\,dr\,d\theta = 0$

$\displaystyle\int_0^{2\pi}\int_0^{\arctan(1/2)}\int_0^{4\sec\phi}\rho^3\sin^2\phi\cos\theta\,d\rho\,d\phi\,d\theta + \int_0^{2\pi}\int_{\arctan(1/2)}^{\pi/2}\int_0^{\cot\phi\csc\phi}\rho^3\sin^2\phi\cos\theta\,d\rho\,d\phi\,d\theta = 0$

14. $\displaystyle\int_0^{\pi/2}\int_0^2\int_0^{\sqrt{16-r^2}} r^2\,dz\,dr\,d\theta = \frac{8\pi^2}{3} - 2\pi\sqrt{3}$

$\displaystyle\int_0^{\pi/2}\int_0^{\pi/6}\int_0^4\rho^3\sin^2\phi\,d\rho\,d\phi\,d\theta + \int_0^{\pi/2}\int_{\pi/6}^{\pi/2}\int_4^{2\csc\phi}\rho^3\sin^2\phi\,d\rho\,d\phi\,d\theta = \frac{8\pi^2}{3} - 2\pi\sqrt{3}$

15. $\displaystyle\int_0^{2\pi}\int_0^a\int_a^{a+\sqrt{a^2-r^2}} r^2\cos\theta\,dz\,dr\,d\theta = 0$

$\displaystyle\int_0^{\pi/4}\int_0^{2\pi}\int_{a\sec\phi}^{2a\cos\phi}\rho^3\sin^2\phi\cos\theta\,d\rho\,d\theta\,d\phi = 0$

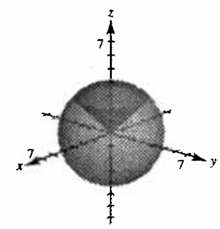
16. $\displaystyle\int_0^{\pi/2}\int_0^1\int_0^{\sqrt{1-r^2}} r\sqrt{r^2+z^2}\,dz\,dr\,d\theta = \frac{\pi}{8}$

$\displaystyle\int_0^{\pi/2}\int_0^{\pi/2}\int_0^1\rho^3\sin\phi\,d\rho\,d\phi\,d\theta = \frac{\pi}{8}$

17. $\displaystyle V = 4\int_0^{\pi/2}\int_0^{a\cos\theta}\int_0^{\sqrt{a^2-r^2}} r\,dz\,dr\,d\theta = 4\int_0^{\pi/2}\int_0^{a\cos\theta} r\sqrt{a^2-r^2}\,dr\,d\theta$

$\displaystyle= \frac{4}{3}a^3\int_0^{\pi/2}(1-\sin^3\theta)\,d\theta = \frac{4}{3}a^3\left[\theta + \frac{1}{3}\cos\theta(\sin^2\theta+2)\right]_0^{\pi/2} = \frac{4}{3}a^3\left(\frac{\pi}{2}-\frac{2}{3}\right) = \frac{2a^3}{9}(3\pi-4)$

18. $V = \frac{2}{3}\pi(4)^3 + 4\left[\int_0^{\pi/2}\int_0^{2\sqrt{2}}\int_0^r r\,dz\,dr\,d\theta + \int_0^{\pi/2}\int_{2\sqrt{2}}^4\int_0^{\sqrt{16-r^2}} r\,dz\,dr\,d\theta\right]$



(Volume of lower hemisphere) + 4(Volume in the first octant)

$$V = \frac{128\pi}{3} + 4\left[\int_0^{\pi/2}\int_0^{2\sqrt{2}} r^2\,dr\,d\theta + \int_0^{\pi/2}\int_{2\sqrt{2}}^4 r\sqrt{16-r^2}\,dr\,d\theta\right]$$

$$= \frac{128\pi}{3} + 4\left[\frac{8\sqrt{2}\pi}{3} + \int_0^{\pi/2}\left[-\frac{1}{3}(16-r^2)^{3/2}\right]_{2\sqrt{2}}^4 d\theta\right]$$

$$= \frac{128\pi}{3} + 4\left[\frac{8\sqrt{2}\pi}{3} + \frac{8\sqrt{2}\pi}{3}\right]$$

$$= \frac{128\pi}{3} + \frac{64\sqrt{2}\pi}{3} = \frac{64\pi}{3}(2+\sqrt{2})$$

19. $V = 2\int_0^{\pi}\int_0^{a\cos\theta}\int_0^{\sqrt{a^2-r^2}} r\,dz\,dr\,d\theta$

$$= 2\int_0^{\pi}\int_0^{a\cos\theta} r\sqrt{a^2-r^2}\,dr\,d\theta$$

$$= 2\int_0^{\pi}\left[-\frac{1}{3}(a^2-r^2)^{3/2}\right]_0^{a\cos\theta} d\theta$$

$$= \frac{2a^3}{3}\int_0^{\pi}(1-\sin^3\theta)\,d\theta$$

$$= \frac{2a^3}{3}\left[\theta + \cos\theta - \frac{\cos^3\theta}{3}\right]_0^{\pi}$$

$$= \frac{2a^3}{9}(3\pi - 4)$$

20. $V = \int_0^{2\pi}\int_0^{\sqrt{2}}\int_r^{\sqrt{4-r^2}} r\,dz\,dr\,d\theta$

$$= \int_0^{2\pi}\int_0^{\sqrt{2}}\left(r\sqrt{4-r^2} - r^2\right)dr\,d\theta$$

$$= \int_0^{2\pi}\left[-\frac{1}{3}(4-r^2)^{3/2} - \frac{r^3}{3}\right]_0^{\sqrt{2}} d\theta$$

$$= \frac{8\pi}{3}(2-\sqrt{2})$$

21. $m = \int_0^{2\pi}\int_0^2\int_0^{9-r\cos\theta-2r\sin\theta}(kr)r\,dz\,dr\,d\theta$

$$= \int_0^{2\pi}\int_0^2 kr^2(9 - r\cos\theta - 2r\sin\theta)\,dr\,d\theta$$

$$= \int_0^{2\pi} k\left[3r^3 - \frac{r^4}{4}\cos\theta - \frac{r^4}{2}\sin\theta\right]_0^2 d\theta$$

$$= \int_0^{2\pi} k[24 - 4\cos\theta - 8\sin\theta]\,d\theta$$

$$= k\left[24\theta - 4\sin\theta + 8\cos\theta\right]_0^{2\pi}$$

$$= k[48\pi + 8 - 8] = 48k\pi$$

22. $\int_0^{\pi/2}\int_0^2\int_0^{12e^{-r^2}} k\,r\,dz\,dr\,d\theta = \int_0^{\pi/2}\int_0^2 12ke^{-r^2}\,r\,dr\,d\theta$

$$= \int_0^{\pi/2}\left[-6ke^{-r^2}\right]_0^2$$

$$= \int_0^{\pi/2}(-6ke^{-4} + 6k)\,d\theta$$

$$= 3k\pi(1 - e^{-4})$$

23. $z = h - \frac{h}{r_0}\sqrt{x^2 + y^2} = \frac{h}{r_0}(r_0 - r)$

$$V = 4\int_0^{\pi/2}\int_0^{r_0}\int_0^{h(r_0-r)/r_0} r\,dz\,dr\,d\theta$$
$$= \frac{4h}{r_0}\int_0^{\pi/2}\int_0^{r_0}(r_0r - r^2)\,dr\,d\theta$$
$$= \frac{4h}{r_0}\int_0^{\pi/2}\frac{r_0^3}{6}\,d\theta$$
$$= \frac{4h}{r_0}\left(\frac{r_0^3}{6}\right)\left(\frac{\pi}{2}\right) = \frac{1}{3}\pi r_0^2 h$$

24. $\bar{x} = \bar{y} = 0$ by symmetry

$m = \frac{1}{3}\pi r_0^2 hk$ from Exercise 23

$$M_{xy} = 4k\int_0^{\pi/2}\int_0^{r_0}\int_0^{h(r_0-r)/r_0} zr\,dz\,dr\,d\theta$$
$$= \frac{2kh^2}{r_0^2}\int_0^{\pi/2}\int_0^{r_0}(r_0^2 r - 2r_0 r^2 + r^3)\,dr\,d\theta$$
$$= \frac{2kh^2}{r_0^2}\left(\frac{r_0^4}{12}\right)\left(\frac{\pi}{2}\right) = \frac{kr_0^2 h^2\pi}{12}$$
$$\bar{z} = \frac{M_{xy}}{m} = \frac{kr_0^2h^2\pi}{12}\left(\frac{3}{\pi r_0^2 hk}\right) = \frac{h}{4}$$

25. $\rho = k\sqrt{x^2 + y^2} = kr$

$\bar{x} = \bar{y} = 0$ by symmetry

$$m = 4k\int_0^{\pi/2}\int_0^{r_0}\int_0^{h(r_0-r)/r_0} r^2\,dz\,dr\,d\theta$$
$$= \frac{1}{6}k\pi r_0^3 h$$
$$M_{xy} = 4k\int_0^{\pi/2}\int_0^{r_0}\int_0^{h(r_0-r)/r_0} r^2 z\,dz\,dr\,d\theta$$
$$= \frac{1}{30}k\pi r_0^3 h^2$$
$$\bar{z} = \frac{M_{xy}}{m} = \frac{k\pi r_0^3h^2/30}{k\pi r_0^3 h/6} = \frac{h}{5}$$

26. $\rho = kz$

$\bar{x} = \bar{y} = 0$ by symmetry

$$m = 4k\int_0^{\pi/2}\int_0^{r_0}\int_0^{h(r_0-r)/r_0} zr\,dz\,dr\,d\theta$$
$$= \frac{1}{12}k\pi r_0^2 h^2$$
$$M_{xy} = 4k\int_0^{\pi/2}\int_0^{r_0}\int_0^{h(r_0-r)/r_0} z^2 r\,dz\,dr\,d\theta$$
$$= \frac{1}{30}k\pi r_0^2 h^3$$
$$\bar{z} = \frac{M_{xy}}{m} = \frac{k\pi r_0^2h^3/30}{k\pi r_0^2 h^2/12} = \frac{2h}{5}$$

27. $I_z = 4k\int_0^{\pi/2}\int_0^{r_0}\int_0^{h(r_0-r)/r_0} r^3\,dz\,dr\,d\theta$

$$= \frac{4kh}{r_0}\int_0^{\pi/2}\int_0^{r_0}(r_0r^3 - r^4)\,dr\,d\theta$$
$$= \frac{4kh}{r_0}\left(\frac{r_0^5}{20}\right)\left(\frac{\pi}{2}\right)$$
$$= \frac{1}{10}k\pi r_0^4 h$$

Since the mass of the core is $m = kV = k\left(\frac{1}{3}\pi r_0^2 h\right)$ from Exercise 23, we have $k = 3m/\pi r_0^2 h$. Thus,

$$I_z = \frac{1}{10}k\pi r_0^4 h$$
$$= \frac{1}{10}\left(\frac{3m}{\pi r_0^2 h}\right)\pi r_0^4 h$$
$$= \frac{3}{10}mr_0^2.$$

28. $I_z = \iiint_Q (x^2 + y^2)\rho(x, y, z)\,dV$

$$= 4k\int_0^{\pi/2}\int_0^{r_0}\int_0^{h(r_0-r)/r_0} r^4\,dz\,dr\,d\theta$$
$$= 4kh\int_0^{\pi/2}\int_0^{r_0}\frac{r_0 - r}{r_0}r^4\,dr\,d\theta$$
$$= 4kh\int_0^{\pi/2}\left[\frac{r^5}{5} - \frac{r^6}{6r_0}\right]_0^{r_0}d\theta$$
$$= 4kh\int_0^{\pi/2}\left[\frac{r_0^5}{5} - \frac{r_0^5}{6}\right]d\theta$$
$$= 4kh\int_0^{\pi/2}\frac{1}{30}r_0^5\,d\theta$$
$$= 4kh\frac{1}{30}r_0^5\frac{\pi}{2}$$
$$= \frac{1}{15}r_0^5\pi kh$$

29. $m = k(\pi b^2 h - \pi a^2 h) = k\pi h(b^2 - a^2)$

$$I_z = 4k\int_0^{\pi/2}\int_a^b\int_0^h r^3\,dz\,dr\,d\theta$$

$$= 4kh\int_0^{\pi/2}\int_a^b r^3\,dr\,d\theta$$

$$= kh\int_0^{\pi/2}(b^4 - a^4)\,d\theta$$

$$= \frac{k\pi(b^4 - a^4)h}{2}$$

$$= \frac{k\pi(b^2 - a^2)(b^2 + a^2)h}{2}$$

$$= \frac{1}{2}m(a^2 + b^2)$$

30. $m = k\pi a^2 h$

$$I_z = 2k\int_0^{\pi/2}\int_0^{2a\sin\theta}\int_0^h r^3\,dz\,dr\,d\theta$$

$$= \frac{3}{2}k\pi a^4 h$$

$$= \frac{3}{2}ma^2$$

31. $V = \displaystyle\int_0^{2\pi}\int_0^{\pi}\int_0^{4\sin\phi}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = 16\pi^2$

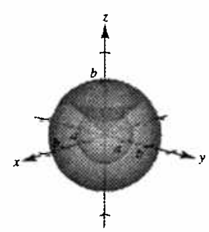
32. $V = 8\displaystyle\int_0^{\pi/4}\int_0^{\pi/2}\int_a^b \rho^2\sin\phi\,d\rho\,d\theta\,d\phi$ (includes upper and lower cones)

$$= \frac{8}{3}(b^3 - a^3)\int_0^{\pi/4}\int_0^{\pi/2}\sin\phi\,d\theta\,d\phi$$

$$= \frac{4\pi}{3}(b^3 - a^3)\int_0^{\pi/4}\sin\phi\,d\phi$$

$$= \left[\frac{4\pi}{3}(b^3 - a^3)(-\cos\phi)\right]_0^{\pi/4}$$

$$= \left(1 - \frac{\sqrt{2}}{2}\right)\frac{4\pi}{3}(b^3 - a^3) = \frac{2\pi}{3}(2 - \sqrt{2})(b^3 - a^3)$$



33. $m = 8k\displaystyle\int_0^{\pi/2}\int_0^{\pi/2}\int_0^a \rho^3\sin\phi\,d\rho\,d\theta\,d\phi$

$$= 2ka^4\int_0^{\pi/2}\int_0^{\pi/2}\sin\phi\,d\theta\,d\phi$$

$$= k\pi a^4\int_0^{\pi/2}\sin\phi\,d\phi$$

$$= \left[k\pi a^4(-\cos\phi)\right]_0^{\pi/2}$$

$$= k\pi a^4$$

34. $m = 8k\displaystyle\int_0^{\pi/2}\int_0^{\pi/2}\int_0^a \rho^3\sin^2\phi\,d\rho\,d\theta\,d\phi$

$$= 2ka^4\int_0^{\pi/2}\int_0^{\pi/2}\sin^2\phi\,d\theta\,d\phi$$

$$= k\pi a^4\int_0^{\pi/2}\sin^2\phi\,d\phi$$

$$= \left[k\pi a^4\left(\frac{1}{2}\phi - \frac{1}{4}\sin 2\phi\right)\right]_0^{\pi/2}$$

$$= k\pi a^4\frac{\pi}{4} = \frac{1}{4}k\pi^2 a^4$$

35. $m = \frac{2}{3}k\pi r^3$

$\bar{x} = \bar{y} = 0$ by symmetry

$$M_{xy} = 4k\int_0^{\pi/2}\int_0^{\pi/2}\int_0^r \rho^3 \cos\phi \sin\phi \, d\rho \, d\theta \, d\phi$$
$$= \frac{1}{2}kr^4\int_0^{\pi/2}\int_0^{\pi/2} \sin 2\phi \, d\theta \, d\phi$$
$$= \frac{kr^4\pi}{4}\int_0^{\pi/2} \sin 2\phi \, d\phi$$
$$= \left[-\frac{1}{8}k\pi r^4 \cos 2\phi\right]_0^{\pi/2} = \frac{1}{4}k\pi r^4$$
$$\bar{z} = \frac{M_{xy}}{m} = \frac{k\pi r^4/4}{2k\pi r^3/3} = \frac{3r}{8}$$

36. $\bar{x} = \bar{y} = 0$ by symmetry

$$m = k\left(\frac{2}{3}\pi R^3 - \frac{2}{3}\pi r^3\right) = \frac{2}{3}k\pi(R^3 - r^3)$$
$$M_{xy} = 4k\int_0^{\pi/2}\int_0^{\pi/2}\int_r^R \rho^3 \cos\phi \sin\phi \, d\rho \, d\theta \, d\phi$$
$$= \frac{1}{2}k(R^4 - r^4)\int_0^{\pi/2}\int_0^{\pi/2} \sin 2\phi \, d\theta \, d\phi$$
$$= \frac{1}{4}k\pi(R^4 - r^4)\int_0^{\pi/2} \sin 2\phi \, d\phi$$
$$= \left[-\frac{1}{8}k\pi(R^4 - r^4)\cos 2\phi\right]_0^{\pi/2} = \frac{1}{4}k\pi(R^4 - r^4)$$
$$\bar{z} = \frac{M_{xy}}{m} = \frac{k\pi(R^4 - r^4)/4}{2k\pi(R^3 - r^3)/3} = \frac{3(R^4 - r^4)}{8(R^3 - r^3)}$$

37. $$I_z = 4k\int_{\pi/4}^{\pi/2}\int_0^{\pi/2}\int_0^{\cos\phi} \rho^4 \sin^3\phi \, d\rho \, d\theta \, d\phi$$
$$= \frac{4}{5}k\int_{\pi/4}^{\pi/2}\int_0^{\pi/2} \cos^5\phi \sin^3\phi \, d\theta \, d\phi$$
$$= \frac{2}{5}k\pi\int_{\pi/4}^{\pi/2} \cos^5\phi(1 - \cos^2\phi)\sin\phi \, d\phi$$
$$= \left[\frac{2}{5}k\pi\left(-\frac{1}{6}\cos^6\phi + \frac{1}{8}\cos^8\phi\right)\right]_{\pi/4}^{\pi/2}$$
$$= \frac{k\pi}{192}$$

38. $$I_z = 4k\int_0^{\pi/2}\int_0^{\pi/2}\int_r^R \rho^4 \sin^3\phi \, d\rho \, d\theta \, d\phi$$
$$= \frac{4k}{5}(R^5 - r^5)\int_0^{\pi/2}\int_0^{\pi/2} \sin^3\phi \, d\theta \, d\phi$$
$$= \frac{2k\pi}{5}(R^5 - r^5)\int_0^{\pi/2} \sin\phi(1 - \cos^2\phi) \, d\phi$$
$$= \left[\frac{2k\pi}{5}(R^5 - r^5)\left(-\cos\phi + \frac{\cos^3\phi}{3}\right)\right]_0^{\pi/2}$$
$$= \frac{4k\pi}{15}(R^5 - r^5)$$

39. $x = r\cos\theta$ $\qquad x^2 + y^2 = r^2$

$y = r\sin\theta$ $\qquad \tan\theta = \frac{y}{x}$

$z = z$ $\qquad z = z$

40. $x = \rho\sin\phi\cos\theta$ $\qquad \rho^2 = x^2 + y^2 + z^2$

$y = \rho\sin\phi\sin\theta$ $\qquad \tan\theta = \frac{y}{x}$

$z = \rho\cos\phi$ $\qquad \cos\phi = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$

41. $$\int_{\theta_1}^{\theta_2}\int_{g_1(\theta)}^{g_2(\theta)}\int_{h_1(r\cos\theta,\, r\sin\theta)}^{h_2(r\cos\theta,\, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)r \, dz \, dr \, d\theta$$

42. $$\int_{\theta_1}^{\theta_2}\int_{\phi_1}^{\phi_2}\int_{\rho_1}^{\rho_2} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

43. (a) $r = r_0$: right circular cylinder about z-axis

$\theta = \theta_0$: plane parallel to z-axis

$z = z_0$: plane parallel to xy-plane

(b) $\rho = \rho_0$: sphere of radius ρ_0

$\theta = \theta_0$: plane parallel to z-axis

$\phi = \phi_0$: cone

44. (a) You are integrating over a cylindrical wedge.

(b) You are integrating over a spherical block.

45. $16\int_0^a\int_0^{\sqrt{a^2-x^2}}\int_0^{\sqrt{a^2-x^2-y^2}}\int_0^{\sqrt{a^2-x^2-y^2-z^2}} dw\,dz\,dy\,dx$

$$= 16\int_0^a\int_0^{\sqrt{a^2-x^2}}\int_0^{\sqrt{a^2-x^2-y^2}} \sqrt{a^2-x^2-y^2-z^2}\,dz\,dy\,dx$$

$$= 16\int_0^{\pi/2}\int_0^a\int_0^{\sqrt{a^2-r^2}} \sqrt{(a^2-r^2)-z^2}\,dz(r\,dr\,d\theta)$$

$$= 16\int_0^{\pi/2}\int_0^a \frac{1}{2}\left[z\sqrt{(a^2-r^2)-z^2} + (a^2-r^2)\arcsin\frac{z}{\sqrt{a^2-r^2}}\right]_0^{\sqrt{a^2-r^2}} r\,dr\,d\theta$$

$$= 8\int_0^{\pi/2}\int_0^a \frac{\pi}{2}(a^2-r^2)r\,dr\,d\theta$$

$$= 4\pi\int_0^{\pi/2}\left[\frac{a^2r^2}{2} - \frac{r^4}{4}\right]_0^a d\theta$$

$$= a^4\pi\int_0^{\pi/2} d\theta = \frac{a^4\pi^2}{2}$$

46. $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sqrt{x^2+y^2+z^2}\,e^{-(x^2+y^2+z^2)}\,dx\,dy\,dz$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \rho e^{-\rho^2}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$= \lim_{k\to\infty}\int_0^{2\pi}\int_0^{\pi}\int_0^{k} \rho^3 e^{-\rho^2}\sin\phi\,d\rho\,d\phi\,d\theta$$

$$= \lim_{k\infty}\left(\int_0^{\pi}\sin\phi\,d\phi\right)\left(\int_0^{2\pi}d\theta\right)\left(\int_0^{k}\rho^3e^{-\rho^2}\,d\rho\right)$$

$$= (2)(2\pi)\lim_{k\to\infty}\left[\frac{-(\rho^2+1)e^{-\rho^2}}{2}\right]_0^k$$

$$= 4\pi\left[\frac{1}{2}\right] = 2\pi$$

Section 14.8 Change of Variables: Jacobians

1. $x = -\frac{1}{2}(u - v)$

$y = \frac{1}{2}(u + v)$

$$\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} = \left(-\frac{1}{2}\right)\left(\frac{1}{2}\right) - \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = -\frac{1}{2}$$

2. $x = au + bv$

$y = cu + dv$

$$\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} = ad - cb$$

3. $x = u - v^2$

$y = u + v$

$$\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} = (1)(1) - (1)(-2v) = 1 + 2v$$

4. $x = uv - 2u$

$y = uv$

$$\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} = (v - 2)u - vu = -2u$$

5. $x = u\cos\theta - v\sin\theta$

$y = u\sin\theta + v\cos\theta$

$$\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} = \cos^2\theta + \sin^2\theta = 1$$

6. $x = u + a$

$y = v + a$

$$\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} = (1)(1) - (0)(0) = 1$$

7. $x = e^u \sin v$

$y = e^u \cos v$

$$\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} = (e^u\sin v)(-e^u\sin v) - (e^u\cos v)(e^u\cos v) = -e^{2u}$$

8. $x = \dfrac{u}{v}$

$y = u + v$

$$\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} = \left(\frac{1}{v}\right)(1) - (1)\left(-\frac{u}{v^2}\right) = \frac{1}{v} + \frac{u}{v^2} = \frac{u+v}{v^2}$$

9. $x = 3u + 2v$

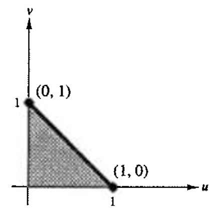
$y = 3v$

$v = \dfrac{y}{3}$

$u = \dfrac{x - 2v}{3} = \dfrac{x - 2(y/3)}{3}$

$= \dfrac{x}{3} - \dfrac{2y}{9}$

(x, y)	(u, v)
(0, 0)	(0, 0)
(3, 0)	(1, 0)
(2, 3)	(0, 1)



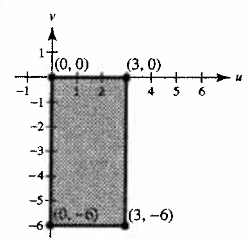
10. $x = \dfrac{1}{3}(4u - v)$

$y = \dfrac{1}{3}(u - v)$

$u = x - y$

$v = x - 4y$

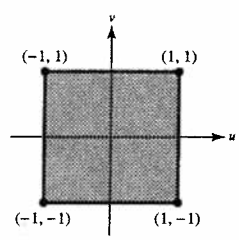
(x, y)	(u, v)
(0, 0)	(0, 0)
(4, 1)	(3, 0)
(2, 2)	(0, −6)
(6, 3)	(3, −6)



11. $x = \dfrac{1}{2}(u + v)$

$y = \dfrac{1}{2}(u - v)$

$$\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} = \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right) - \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = -\frac{1}{2}$$



$$\int\!\!\int_R 4(x^2 + y^2)\,dA = \int_{-1}^{1}\int_{-1}^{1} 4\left[\frac{1}{4}(u+v)^2 + \frac{1}{4}(u-v)^2\right]\left(\frac{1}{2}\right)dv\,du$$

$$= \int_{-1}^{1}\int_{-1}^{1}(u^2 + v^2)\,dv\,du = \int_{-1}^{1} 2\left(u^2 + \frac{1}{3}\right)du = \left[2\left(\frac{u^3}{3} + \frac{u}{3}\right)\right]_{-1}^{1} = \frac{8}{3}$$

12. $x = \frac{1}{2}(u + v), \qquad u = x - y$

$y = -\frac{1}{2}(u - v), \qquad v = x + y$

$$\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} = \frac{1}{2}\left(\frac{1}{2}\right) - \left(-\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{2}$$

(x, y)	(u, v)
$(0, 1)$	$(-1, 1)$
$(2, 1)$	$(1, 3)$
$(1, 2)$	$(-1, 3)$
$(1, 0)$	$(1, 1)$

$$\int_R\int 60xy\,dA$$

$$= \int_{-1}^{1}\int_{1}^{3} 60\left(\frac{1}{2}(u + v)\right)\left(-\frac{1}{2}(u - v)\right)\left(\frac{1}{2}\right) dv\,du$$

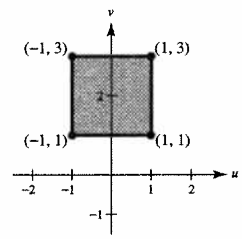
$$= \int_{-1}^{1}\int_{1}^{3} -\frac{15}{2}(v^2 - u^2)\,dv\,du$$

$$= \int_{-1}^{1}\left[-\frac{15}{2}\left(\frac{v^3}{3} - u^2v\right)\right]_{1}^{3} du$$

$$= \int_{-1}^{1}\frac{15}{2}\left(2u^2 - \frac{26}{3}\right) du$$

$$= \left[\frac{15}{2}\left(\frac{2}{3}u^3 - \frac{26}{3}u\right)\right]_{-1}^{1}$$

$$= 15\left(\frac{2}{3} - \frac{26}{3}\right) = -120$$

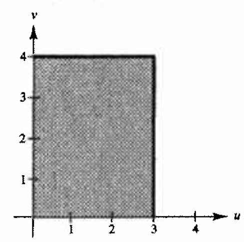


13. $x = u + v$

$y = u$

$$\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} = (1)(0) - (1)(1) = -1$$

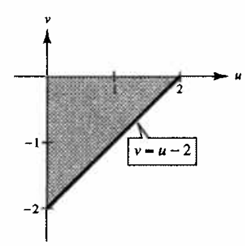
$$\int_R\int y(x - y)\,dA = \int_0^3\int_0^4 uv(1)\,dv\,du = \int_0^3 8u\,du = 36$$



14. $x = \frac{1}{2}(u + v)$

$y = \frac{1}{2}(u - v)$

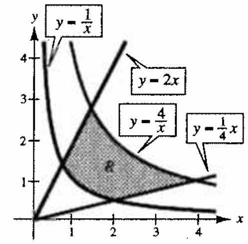
$$\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} = -\frac{1}{2}$$



$$\int_R\int 4(x + y)e^{x-y}\,dA = \int_0^2\int_{u-2}^{0} 4ue^v\left(\frac{1}{2}\right) dv\,du$$

$$= \int_0^2 2u(1 - e^{u-2})\,du = 2\left[\frac{u^2}{2} - ue^{u-2} + e^{u-2}\right]_0^2 = 2(1 - e^{-2})$$

15. $\displaystyle\int_R\int e^{-xy/2}\,dA$

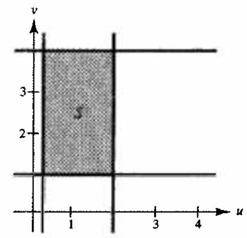


R: $y = \dfrac{x}{4}, y = 2x, y = \dfrac{1}{x}, y = \dfrac{4}{x}$

$x = \sqrt{v/u}, y = \sqrt{uv} \implies u = \dfrac{y}{x}, \; v = xy$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} -\frac{1}{2}\frac{v^{1/2}}{u^{3/2}} & \frac{1}{2}\frac{1}{u^{1/2}v^{1/2}} \\ \frac{1}{2}\frac{v^{1/2}}{u^{1/2}} & \frac{1}{2}\frac{u^{1/2}}{v^{1/2}} \end{vmatrix} = -\frac{1}{4}\left(\frac{1}{u} + \frac{1}{u}\right) = -\frac{1}{2u}$$

Transformed Region:



$y = \dfrac{1}{x} \implies yx = 1 \implies v = 1$

$y = \dfrac{4}{x} \implies ux = 4 \implies v = 4$

$y = 2x \implies \dfrac{y}{x} = 2 \implies u = 2$

$y = \dfrac{x}{4} \implies \dfrac{y}{x} = \dfrac{1}{4} \implies u = \dfrac{1}{4}$

$$\int_R\int e^{-xy/2}\,dA = \int_{1/4}^{2}\int_1^4 e^{-v/2}\left(\frac{1}{2u}\right)dv\,du = -\int_{1/4}^{2}\left[\frac{e^{-v/2}}{u}\right]_1^4 du = -\int_{1/4}^{2}(e^{-2} - e^{-1/2})\frac{1}{u}\,du$$

$$= -\Big[(e^{-2} - e^{-1/2})\ln u\Big]_{1/4}^{2} = -(e^{-2} - e^{-1/2})\left(\ln 2 - \ln\frac{1}{4}\right) = (e^{-1/2} - e^{-2})\ln 8 \approx 0.9798$$

16. $x = \dfrac{u}{v}$

$y = v$

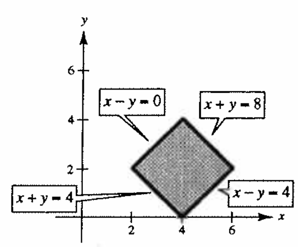
$\dfrac{\partial x}{\partial u}\dfrac{\partial y}{\partial v} - \dfrac{\partial y}{\partial u}\dfrac{\partial x}{\partial v} = \dfrac{1}{v}$

$$\int_R\int y\sin xy\,dA = \int_1^4\int_1^4 v(\sin u)\frac{1}{v}\,dv\,du = \int_1^4 3\sin u\,du = \Big[-3\cos u\Big]_1^4 = 3(\cos 1 - \cos 4) \approx 3.5818$$

17. $u = x + y = 4, \quad v = x - y = 0$

$u = x + y = 8, \quad v = x - y = 4$

$x = \dfrac{1}{2}(u + v) \quad y = \dfrac{1}{2}(u - v)$



$\dfrac{\partial(x,y)}{\partial(u,v)} = -\dfrac{1}{2}$

$$\int_R\int (x + y)e^{x-y}\,dA = \int_4^8\int_0^4 ue^v\left(\frac{1}{2}\right)dv\,du$$

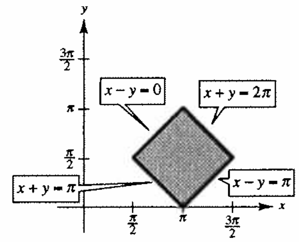
$$= \frac{1}{2}\int_4^8 u(e^4 - 1)\,du = \left[\frac{1}{4}u^2(e^4 - 1)\right]_4^8 = 12(e^4 - 1)$$

18. $u = x + y = \pi, \quad v = x - y = 0$

$u = x + y = 2\pi, \quad v = x - y = \pi$

$x = \frac{1}{2}(u + v), \quad y = \frac{1}{2}(u - v)$

$\frac{\partial(x, y)}{\partial(u, v)} = -\frac{1}{2}$



$$\iint_R (x + y)^2 \sin^2(x - y)\, dA = \int_0^{\pi}\int_{\pi}^{2\pi} u^2 \sin^2 v\left(\frac{1}{2}\right) du\, dv$$

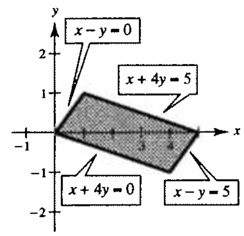
$$= \int_0^{\pi}\left[\frac{1}{2}\left(\frac{u^3}{3}\right)\frac{1 - \cos 2v}{2}\right]_{\pi}^{2\pi} dv = \left[\frac{7\pi^3}{12}\left(v - \frac{1}{2}\sin 2v\right)\right]_0^{\pi} = \frac{7\pi^4}{12}$$

19. $u = x + 4y = 0, \quad v = x - y = 0$

$u = x + 4y = 5, \quad v = x - y = 5$

$x = \frac{1}{5}(u + 4v), \quad y = \frac{1}{5}(u - v)$

$\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} = \left(\frac{1}{5}\right)\left(-\frac{1}{5}\right) - \left(\frac{1}{5}\right)\left(\frac{4}{5}\right) = -\frac{1}{5}$



$$\iint_R \sqrt{(x - y)(x + 4y)}\, dA = \int_0^5\int_0^5 \sqrt{uv}\left(\frac{1}{5}\right) du\, dv$$

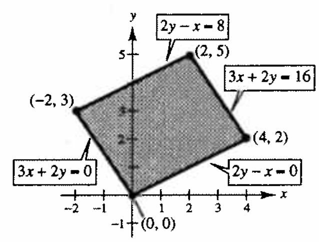
$$= \int_0^5\left[\frac{1}{5}\left(\frac{2}{3}\right)u^{3/2}\sqrt{v}\right]_0^5 dv = \left[\frac{2\sqrt{5}}{3}\left(\frac{2}{3}\right)v^{3/2}\right]_0^5 = \frac{100}{9}$$

20. $u = 3x + 2y = 0, \quad v = 2y - x = 0$

$u = 3x + 2y = 16, \quad v = 2y - x = 8$

$x = \frac{1}{4}(u - v), \quad y = \frac{1}{8}(u + 3v)$

$\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} = \frac{1}{4}\left(\frac{3}{8}\right) - \frac{1}{8}\left(-\frac{1}{4}\right) = \frac{1}{8}$



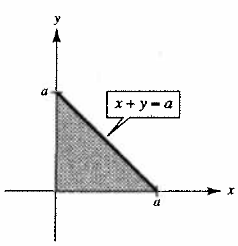
$$\iint_R (3x + 2y)(2y - x)^{3/2}\, dA = \int_0^8\int_0^{16} uv^{3/2}\left(\frac{1}{8}\right) du\, dv$$

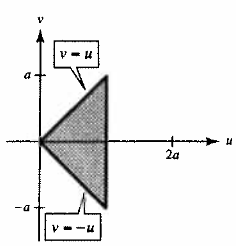
$$= \int_0^8 16v^{3/2}\, dv = \left(\frac{2}{5}\right)16v^{5/2}\Big]_0^8 = \frac{4096}{5}\sqrt{2}$$

21. $u = x + y, v = x - y, x = \frac{1}{2}(u + v), y = \frac{1}{2}(u - v)$

$\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} = -\frac{1}{2}$

$$\iint_R \sqrt{x + y}\, dA = \int_0^a\int_{-u}^{u} \sqrt{u}\left(\frac{1}{2}\right) dv\, du = \int_0^a u\sqrt{u}\, du = \left[\frac{2}{5}u^{5/2}\right]_0^a = \frac{2}{5}a^{5/2}$$



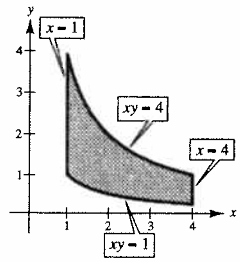


22. $u = x = 1, \quad v = xy = 1$

$u = x = 4, \quad v = xy = 4$

$x = u, \quad y = \dfrac{u}{v}$

$\dfrac{\partial x}{\partial u}\dfrac{\partial y}{\partial v} - \dfrac{\partial y}{\partial u}\dfrac{\partial x}{\partial v} = \dfrac{1}{u}$



$$\iint_R \frac{xy}{1 + x^2y^2}\,dA = \int_1^4\int_1^4 \frac{v}{1 + v^2}\left(\frac{1}{u}\right) dv\,du$$

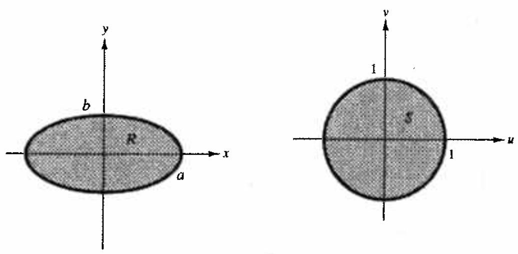
$$= \int_1^4 \left[\frac{1}{2}\ln(1 + v^2)\right]_1^4 \frac{1}{u}\,du = \left[\frac{1}{2}[\ln 17 - \ln 2]\ln u\right]_1^4 = \frac{1}{2}\left(\ln\frac{17}{2}\right)(\ln 4)$$

23. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1, x = au, y = bv$

$\dfrac{(au)^2}{a^2} + \dfrac{(bv)^2}{b^2} = 1$

$u^2 + v^2 = 1$

(a) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \qquad u^2 + v^2 = 1$



(b) $\dfrac{\partial(x, y)}{\partial(u, v)} = \dfrac{\partial x}{\partial u}\dfrac{\partial y}{\partial v} - \dfrac{\partial y}{\partial u}\dfrac{\partial x}{\partial v}$

$= (a)(b) - (0)(0) = ab$

(c) $A = \displaystyle\iint_S ab\,dS$

$= ab(\pi(1)^2) = \pi ab$

24. (a) $f(x, y) = 16 - x^2 - y^2$

$R: \dfrac{x^2}{16} + \dfrac{y^2}{9} \le 1$

$V = \displaystyle\iint_R f(x, y)\,dA$

Let $x = 4u$ and $y = 3v$.

$$\iint_R (16 - x^2 - y^2)\,dA = \int_{-1}^{1}\int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} (16 - 16u^2 - 9v^2)\,12\,dv\,du \quad (\text{Let } u = r\cos\theta,\ v = r\sin\theta.)$$

$$= \int_0^{2\pi}\int_0^1 (16 - 16r^2\cos^2\theta - 9r^2\sin^2\theta)\,12r\,dr\,d\theta$$

$$= 12\int_0^{2\pi}\left[8r^2 - 4r^4\cos^2\theta - \frac{9}{4}r^4\sin^2\theta\right]_0^1 d\theta = 12\int_0^{2\pi}\left[8 - 4\cos^2\theta - \frac{9}{4}\sin^2\theta\right] d\theta$$

$$= 12\int_0^{2\pi}\left[8 - 4\left(\frac{1 + \cos 2\theta}{2}\right) - \frac{9}{4}\left(\frac{1 - \cos 2\theta}{2}\right)\right] d\theta = 12\int_0^{2\pi}\left[\frac{39}{8} - \frac{7}{8}\cos 2\theta\right] d\theta$$

$$= 12\left[\frac{39}{8}\theta - \frac{7}{16}\sin 2\theta\right]_0^{2\pi} = 12\left[\frac{39\pi}{4}\right] = 117\pi$$

—CONTINUED—

24. —CONTINUED—

(b) $f(x, y) = A\cos\left[\frac{\pi}{2}\sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2}}\right]$

$R: \frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1$

Let $x = au$ and $y = bv$.

$$\int_R\int f(x, y)\,dA = \int_{-1}^{1}\int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} A\cos\left[\frac{\pi}{2}\sqrt{u^2 + v^2}\right] ab\,dv\,du$$

Let $u = r\cos\theta$, $v = r\sin\theta$.

$$Aab\int_0^{2\pi}\int_0^1 \cos\left[\frac{\pi}{2}r\right] r\,dr\,d\theta = Aab\left[\frac{2r}{\pi}\sin\left(\frac{\pi r}{2}\right) + \frac{4}{\pi^2}\cos\left(\frac{\pi r}{2}\right)\right]_0^1 (2\pi)$$

$$= 2\pi Aab\left[\left(\frac{2}{\pi} + 0\right) - \left(0 + \frac{4}{\pi^2}\right)\right] = \frac{4(\pi - 2)Aab}{\pi}$$

25. Jacobian $= \frac{\partial(x, y)}{\partial(u, v)} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}$

26. See Theorem 14.5.

27. $x = u(1 - v),\ y = uv(1 - w),\ z = uvw$

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} 1 - v & -u & 0 \\ v(1 - w) & u(1 - w) & -uv \\ vw & uw & uv \end{vmatrix} = (1 - v)[u^2v(1 - w) + u^2vw] + u[uv^2(1 - w) + uv^2w]$$

$$= (1 - v)(u^2v) + u(uv^2)$$

$$= u^2v$$

28. $x = 4u - v,\ y = 4v - w,\ z = u + w$

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} 4 & -1 & 0 \\ 0 & 4 & -1 \\ 1 & 0 & 1 \end{vmatrix} = 17$$

29. $x = \rho\sin\phi\cos\theta,\ y = \rho\sin\phi\sin\theta,\ z = \rho\cos\phi$

$$\frac{\partial(x, y, z)}{\partial(\rho, \theta, \phi)} = \begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \\ \cos\phi & 0 & -\rho\sin\phi \end{vmatrix}$$

$$= \cos\phi[-\rho^2\sin\phi\cos\phi\sin^2\theta - \rho^2\sin\phi\cos\phi\cos^2\theta] - \rho\sin\phi[\rho\sin^2\phi\cos^2\theta + \rho\sin^2\phi\sin^2\theta]$$

$$= \cos\phi[-\rho^2\sin\phi\cos\phi(\sin^2\theta + \cos^2\theta)] - \rho\sin\phi[\rho\sin^2\phi(\cos^2\theta + \sin^2\theta)]$$

$$= -\rho^2\sin\phi\cos^2\phi - \rho^2\sin^3\phi$$

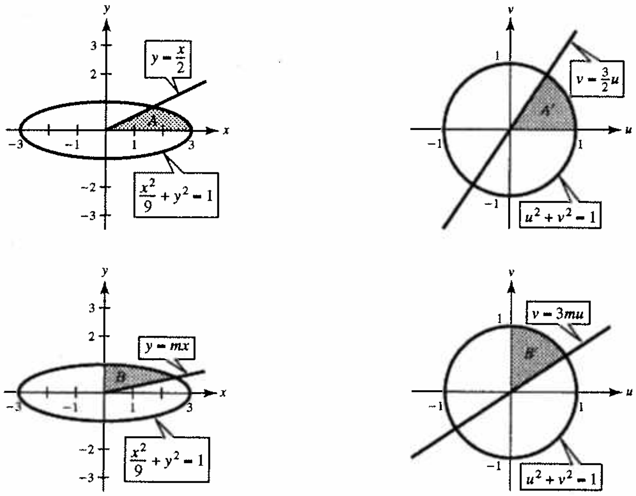
$$= -\rho^2\sin\phi(\cos^2\phi + \sin^2\phi)$$

$$= -\rho^2\sin\phi$$

30. $x = r\cos\theta,\ y = r\sin\theta,\ z = z$

$$\frac{\partial(x, y, z)}{\partial(r, \theta, z)} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1[r\cos^2\theta + r\sin^2\theta] = r$$

31. Let $u = \dfrac{x}{3}, v = y \Rightarrow \dfrac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} 3 & 0 \\ 0 & 1 \end{vmatrix} = 3, y = \dfrac{x}{2} \Rightarrow v = \dfrac{3u}{2}.$



Region A is transformed to region A', and region B is transformed to region B'.

$A' = B' \Rightarrow \dfrac{2}{3} = 3m \Rightarrow m = \dfrac{2}{9}$

Note: You could also calculate the integrals directly.

Review Exercises for Chapter 14

1. $\displaystyle\int_1^{x^2} x \ln y \, dy = \Big[xy(-1 + \ln y)\Big]_1^{x^2} = x^3(-1 + \ln x^2) + x = x - x^3 + x^3 \ln x^2$

2. $\displaystyle\int_y^{2y} (x^2 + y^2)\, dx = \left[\frac{x^3}{3} + xy^2\right]_y^{2y} = \frac{10y^3}{3}$

3. $\displaystyle\int_0^1\int_0^{1+x} (3x + 2y)\, dy\, dx = \int_0^1 \Big[3xy + y^2\Big]_0^{1+x} dx = \int_0^1 (4x^2 + 5x + 1)\, dx = \left[\frac{4}{3}x^3 + \frac{5}{2}x^2 + x\right]_0^1 = \frac{29}{6}$

4. $\displaystyle\int_0^2\int_{x^2}^{2x} (x^2 + 2y)\, dy\, dx = \int_0^2 \Big[x^2y + y^2\Big]_{x^2}^{2x} dx = \int_0^2 (4x^2 + 2x^3 - 2x^4)\, dx = \left[\frac{4}{3}x^3 + \frac{1}{2}x^4 - \frac{2}{5}x^5\right]_0^2 = \frac{88}{15}$

5. $\displaystyle\int_0^3\int_0^{\sqrt{9-x^2}} 4x\, dy\, dx = \int_0^3 4x\sqrt{9 - x^2}\, dx = \left[-\frac{4}{3}(9 - x^2)^{3/2}\right]_0^3 = 36$

6. $\displaystyle\int_0^{\sqrt{3}}\int_{2-\sqrt{4-y^2}}^{2+\sqrt{4-y^2}} dx\, dy = 2\int_0^{\sqrt{3}} \sqrt{4 - y^2}\, dy = \left[y\sqrt{4 - y^2} + 4 \arcsin\frac{y}{2}\right]_0^{\sqrt{3}} = \sqrt{3} + \frac{4\pi}{3}$

7. $\displaystyle\int_0^3\int_0^{(3-x)/3} dy\, dx = \int_0^1\int_0^{3-3y} dx\, dy$

$\displaystyle A = \int_0^1\int_0^{3-3y} dx\, dy = \int_0^1 (3 - 3y)\, dy = \left[3y - \frac{3}{2}y^2\right]_0^1 = \frac{3}{2}$

8. $$\int_0^2\int_0^x dy\,dx + \int_2^3\int_0^{6-2x} dy\,dx = \int_0^2\int_y^{(6-y)/2} dx\,dy$$

$$A = \int_0^2\int_y^{(6-y)/2} dx\,dy = \frac{1}{2}\int_0^2 (6-3y)\,dy = \left[\frac{1}{2}\left(6y - \frac{3}{2}y^2\right)\right]_0^2 = 3$$

9. $$\int_{-5}^{3}\int_{-\sqrt{25-x^2}}^{\sqrt{25-x^2}} dy\,dx = \int_{-5}^{-4}\int_{-\sqrt{25-y^2}}^{\sqrt{25-y^2}} dx\,dy + \int_{-4}^{4}\int_{-\sqrt{25-y^2}}^{3} dx\,dy + \int_{4}^{5}\int_{-\sqrt{25-y^2}}^{\sqrt{25-y^2}} dx\,dy$$

$$A = 2\int_{-5}^{3}\int_0^{\sqrt{25-x^2}} dy\,dx = 2\int_{-5}^{3}\sqrt{25-x^2}\,dx = \left[x\sqrt{25-x^2} + 25\arcsin\frac{x}{5}\right]_{-5}^{3} = \frac{25\pi}{2} + 12 + 25\arcsin\frac{3}{5} \approx 67.36$$

10. $$\int_0^4\int_{x^2-2x}^{6x-x^2} dy\,dx = \int_{-1}^{0}\int_{1-\sqrt{1+y}}^{1+\sqrt{1+y}} dy\,dx + \int_0^8\int_{3-\sqrt{9-y}}^{1+\sqrt{1+y}} dx\,dy + \int_8^9\int_{3-\sqrt{9-y}}^{3+\sqrt{9-y}} dx\,dy$$

$$A = \int_0^4\int_{x^2-2x}^{6x-x^2} dy\,dx = \int_0^4 (8x - 2x^2)\,dx = \left[4x^2 - \frac{2}{3}x^3\right]_0^4 = \frac{64}{3}$$

11. $$A = 4\int_0^1\int_0^{x\sqrt{1-x^2}} dy\,dx = 4\int_0^1 x\sqrt{1-x^2}\,dx = \left[-\frac{4}{3}(1-x^2)^{3/2}\right]_0^1 = \frac{4}{3}$$

$$A = 4\int_0^{1/2}\int_{\sqrt{(1-\sqrt{1-4y^2})/2}}^{\sqrt{(1+\sqrt{1-4y^2})/2}} dx\,dy$$

12. $$A = \int_0^2\int_0^{y^2+1} dx\,dy = \int_0^1\int_0^2 dy\,dx + \int_1^5\int_{\sqrt{x-1}}^{2} dy\,dx = \frac{14}{3}$$

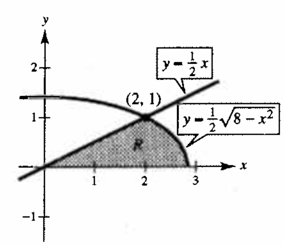
13. $$A = \int_2^5\int_{x-3}^{\sqrt{x-1}} dy\,dx + 2\int_1^2\int_0^{\sqrt{x-1}} dy\,dx = \int_{-1}^{2}\int_{y^2+1}^{y+3} dx\,dy = \frac{9}{2}$$

14. $$A = \int_0^3\int_{-y}^{2y-y^2} dx\,dy = \int_{-3}^{0}\int_{-x}^{1+\sqrt{1-x}} dy\,dx + \int_0^1\int_{1-\sqrt{1-x}}^{1+\sqrt{1-x}} dy\,dx = \frac{9}{2}$$

15. Both integrations are over the common region R shown in the figure. Analytically,

$$\int_0^1\int_{2y}^{2\sqrt{2-y^2}} (x+y)\,dx\,dy = \frac{4}{3} + \frac{4}{3}\sqrt{2}$$

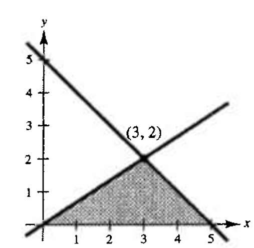
$$\int_0^2\int_0^{x/2} (x+y)\,dy\,dx + \int_2^{2\sqrt{2}}\int_0^{\sqrt{8-x^2}/2} (x+y)\,dy\,dx = \frac{5}{3} + \left(\frac{4}{3}\sqrt{2} - \frac{1}{3}\right) = \frac{4}{3} + \frac{4}{3}\sqrt{2}$$



16. Both integrations are over the common region R shown in the figure. Analytically,

$$\int_0^2\int_{3y/2}^{5-y} e^{x+y}\,dx\,dy = \frac{2}{5} + \frac{8}{5}e^5$$

$$\int_0^3\int_0^{2x/3} e^{x+y}\,dy\,dx + \int_3^5\int_0^{5-x} e^{x+y}\,dy\,dx = \left(\frac{3}{5}e^5 - e^3 + \frac{2}{5}\right) + (e^5 + e^3) = \frac{8}{5}e^5 + \frac{2}{5}$$



17. $V = \int_0^4 \int_0^{x^2+4} (x^2 - y + 4)\, dy\, dx$

$= \int_0^4 \left[x^2y - \frac{1}{2}y^2 + 4y\right]_0^{x^2+4} dx$

$= \int_0^4 \left(\frac{1}{2}x^4 + 4x^2 + 8\right) dx$

$= \left[\frac{1}{10}x^5 + \frac{4}{3}x^3 + 8x\right]_0^4 = \frac{3296}{15}$

18. $V = \int_0^3 \int_0^x (x + y)\, dy\, dx$

$= \int_0^3 \left[xy + \frac{1}{2}y^2\right]_0^x dx$

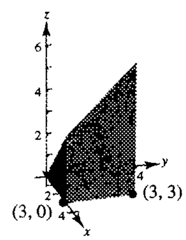
$= \frac{3}{2}\int_0^3 x^2\, dx$

$= \left[\frac{1}{2}x^3\right]_0^3 = \frac{27}{2}$

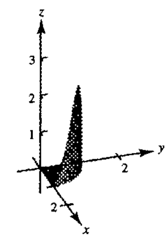
19. Volume ≈ (base)(height)

$\approx \frac{9}{2}(3) = \frac{27}{2}$

Matches (c)



20. Matches (c)



21. $\int_0^\infty \int_0^\infty kxye^{-(x+y)}\, dy\, dx = \int_0^\infty \left[-kxe^{-(x+y)}(y + 1)\right]_0^\infty dx$

$= \int_0^\infty kxe^{-x}\, dx$

$= \left[-k(x + 1)e^{-x}\right]_0^\infty = k$

Therefore, $k = 1$.

$P = \int_0^1 \int_0^1 xye^{-(x+y)}\, dy\, dx \approx 0.070$

22. $\int_0^1 \int_0^x kxy\, dy\, dx = \int_0^1 \left[\frac{kxy^2}{2}\right]_0^x dx$

$= \int_0^1 \frac{kx^3}{2} dx$

$= \left[\frac{kx^4}{8}\right]_0^1 = \frac{k}{8}$

Since $k/8 = 1$, we have $k = 8$.

23. True

24. False, $\int_0^1 \int_0^1 x\, dy\, dx \neq \int_1^2 \int_1^2 x\, dy\, dx$

25. True

26. True, $\int_0^1 \int_0^1 \frac{1}{1 + x^2 + y^2} dx\, dy < \int_0^1 \int_0^1 \frac{1}{1 + x^2} dx\, dy = \frac{\pi}{4}$

27. $\int_0^h \int_0^x \sqrt{x^2 + y^2}\, dy\, dx = \int_0^{\pi/4} \int_0^{h \sec\theta} r^2\, dr\, d\theta$

$= \frac{h^3}{3}\int_0^{\pi/4} \sec^3\theta\, d\theta = \frac{h^3}{6}\left[\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right]_0^{\pi/4} = \frac{h^3}{6}\left[\sqrt{2} + \ln(\sqrt{2} + 1)\right]$

28. $\int_0^4 \int_0^{\sqrt{16-y^2}} (x^2 + y^2)\, dx\, dy = \int_0^{\pi/2} \int_0^4 r^3\, dr\, d\theta$

$= \int_0^{\pi/2} \left[\frac{r^4}{4}\right]_0^4 d\theta$

$= \int_0^{\pi/2} 64\, d\theta = 32\pi$

29. $V = 4\int_0^h \int_0^{\pi/2} \int_1^{\sqrt{1+z^2}} r\, dr\, d\theta\, dz$

$= 2\int_0^h \int_0^{\pi/2} (1 + z^2 - 1)\, d\theta\, dz$

$= \pi \int_0^h z^2\, dz$

$= \left[\pi\left(\frac{1}{3}z^3\right)\right]_0^h = \frac{\pi h^3}{3}$

30. $V = 8\int_0^{\pi/2}\int_b^R \sqrt{R^2 - r^2}\,r\,dr\,d\theta$

$= -\frac{8}{3}\int_0^{\pi/2}\left[(R^2 - r^2)^{3/2}\right]_b^R d\theta$

$= \frac{8}{3}(R^2 - b^2)^{3/2}\int_0^{\pi/2} d\theta$

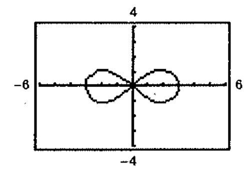
$= \frac{4}{3}\pi(R^2 - b^2)^{3/2}$

31. (a) $(x^2 + y^2)^2 = 9(x^2 - y^2)$

$(r^2)^2 = 9(r^2\cos^2\theta - r^2\sin^2\theta)$

$r^2 = 9(\cos^2\theta - \sin^2\theta) = 9\cos 2\theta$

$r = 3\sqrt{\cos 2\theta}$



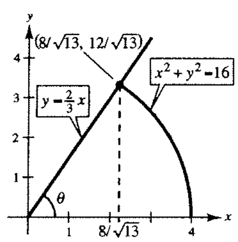
(b) $A = 4\int_0^{\pi/4}\int_0^{3\sqrt{\cos 2\theta}} r\,dr\,d\theta = 9$

(c) $V = 4\int_0^{\pi/4}\int_0^{3\sqrt{\cos 2\theta}} \sqrt{9 - r^2}\,r\,dr\,d\theta \approx 20.392$

32. $\tan\theta = \frac{12\sqrt{13}}{8\sqrt{13}} = \frac{3}{2} \implies \theta \approx 0.9828$

The polar region is given by $0 \le r \le 4$ and $0 \le \theta \le 0.9828$. Hence,

$\int_0^{\arctan(3/2)}\int_0^4 (r\cos\theta)(r\sin\theta)r\,dr\,d\theta = \frac{288}{13}.$



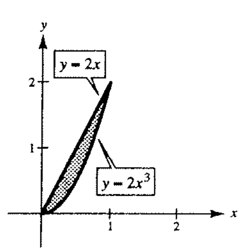
33. (a) $m = k\int_0^1\int_{2x^3}^{2x} xy\,dy\,dx = \frac{k}{4}$

$M_x = k\int_0^1\int_{2x^3}^{2x} xy^2\,dy\,dx = \frac{16k}{55}$

$M_y = k\int_0^1\int_{2x^3}^{2x} x^2y\,dy\,dx = \frac{8k}{45}$

$\bar{x} = \frac{M_y}{m} = \frac{32}{45}$

$\bar{y} = \frac{M_x}{m} = \frac{64}{55}$



(b) $m = k\int_0^1\int_{2x^3}^{2x} (x^2 + y^2)\,dy\,dx = \frac{17k}{30}$

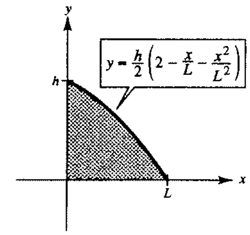
$M_x = k\int_0^1\int_{2x^3}^{2x} y(x^2 + y^2)\,dy\,dx = \frac{392k}{585}$

$M_y = k\int_0^1\int_{2x^3}^{2x} x(x^2 + y^2)\,dy\,dx = \frac{156k}{385}$

$\bar{x} = \frac{M_y}{m} = \frac{936}{1309}$

$\bar{y} = \frac{M_x}{m} = \frac{784}{663}$

34. $m = k\int_0^L \int_0^{(h/2)[2-(x/L)-(x^2/L^2)]} dy\,dx = \frac{kh}{2}\int_0^L \left(2 - \frac{x}{L} - \frac{x^2}{L^2}\right)dx = \frac{7khL}{12}$



$$M_x = k\int_0^L \int_0^{(h/2)[2-(x/L)-(x^2/L^2)]} y\,dy\,dx$$

$$= \frac{kh^2}{8}\int_0^L \left(2 - \frac{x}{L} - \frac{x^2}{L^2}\right)^2 dx$$

$$= \frac{kh^2}{8}\int_0^L \left[4 - \frac{4x}{L} - \frac{3x^2}{L^2} + \frac{2x^3}{L^3} + \frac{x^4}{L^4}\right] dx$$

$$= \frac{kh^2}{8}\left[4x - \frac{2x^2}{L} - \frac{x^3}{L^2} + \frac{x^4}{2L^3} + \frac{x^5}{5L^4}\right]_0^L = \frac{kh^2}{8} \cdot \frac{17L}{10} = \frac{17kh^2L}{80}$$

$$M_y = k\int_0^L \int_0^{(h/2)[2-(x/L)-(x^2/L^2)]} x\,dy\,dx$$

$$= \frac{kh}{2}\int_0^L \left(2x - \frac{x^2}{L} - \frac{x^3}{L^2}\right) dx = \frac{kh}{2}\left[x^2 - \frac{x^3}{3L} - \frac{x^4}{4L^2}\right]_0^L = \frac{kh}{2} \cdot \frac{5L^2}{12} = \frac{5khL^2}{24}$$

$$\bar{x} = \frac{M_y}{m} = \frac{5khL^2}{24} \cdot \frac{12}{7khL} = \frac{5L}{14}$$

$$\bar{y} = \frac{M_x}{m} = \frac{17kh^2L}{80} \cdot \frac{12}{7khL} = \frac{51h}{140}$$

35. $I_x = \int_R\int y^2\rho(x, y)\,dA = \int_0^a \int_0^b kxy^2\,dy\,dx = \frac{1}{6}kb^3a^2$

$$I_y = \int_R\int x^2\rho(x, y)\,dA = \int_0^a \int_0^b kx^3\,dy\,dx = \frac{1}{4}kba^4$$

$$I_0 = I_x + I_y = \frac{1}{6}kb^3a^2 + \frac{1}{4}kba^4 = \frac{ka^2b}{12}(2b^2 + 3a^2)$$

$$m = \int_R\int \rho(x, y)\,dA = \int_0^a \int_0^b kx\,dy\,dx = \frac{1}{2}kba^2$$

$$\bar{\bar{x}} = \sqrt{\frac{I_y}{m}} = \sqrt{\frac{(1/4)kba^4}{(1/2)kba^2}} = \sqrt{\frac{a^2}{2}} = \frac{a\sqrt{2}}{2}$$

$$\bar{\bar{y}} = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{(1/6)kb^3a^2}{(1/2)kba^2}} = \sqrt{\frac{b^2}{3}} = \frac{b\sqrt{3}}{3}$$

36. $I_x = \int_R\int y^2\rho(x, y)\,dA = \int_0^2 \int_0^{4-x^2} ky^3\,dy\,dx = \frac{16{,}384}{315}k$

$$I_y = \int_R\int x^2\rho(x, y)\,dA = \int_0^2 \int_0^{4-x^2} kx^2y\,dy\,dx = \frac{512}{105}k$$

$$I_0 = I_x + I_y = \frac{16{,}384k}{315} + \frac{512k}{105} = \frac{17{,}920}{315}k = \frac{512}{9}k$$

$$m = \int_R\int \rho(x, y)\,dA = \int_0^2 \int_0^{4-x^2} ky\,dy\,dx = \frac{128}{15}k$$

$$\bar{\bar{x}} = \sqrt{\frac{I_y}{m}} = \sqrt{\frac{512k/105}{128k/15}} = \sqrt{\frac{4}{7}}$$

$$\bar{\bar{y}} = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{16{,}384k/315}{128k/15}} = \sqrt{\frac{128}{21}}$$

37. $S = \int_R\int \sqrt{1 + (f_x)^2 + (f_y)^2}\,dA$

$$= 4\int_0^4 \int_0^{\sqrt{16-x^2}} \sqrt{1 + 4x^2 + 4y^2}\,dy\,dx$$

$$= 4\int_0^{\pi/2} \int_0^4 \sqrt{1 + 4r^2}\,r\,dr\,d\theta$$

$$= \left[\frac{1}{3}(65^{3/2} - 1)\theta\right]_0^{\pi/2} = \frac{\pi}{6}\left(65\sqrt{65} - 1\right)$$

38. $f(x, y) = 16 - x - y^2$

$R = \{(x, y): 0 \le x \le 2,\ 0 \le y \le x\}$

$f_x = -1, f_y = -2y$

$\sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{2 + 4y^2}$

$$S = \int_0^2 \int_y^2 \sqrt{2 + 4y^2}\, dx\, dy = \int_0^2 \left[2\sqrt{2 + 4y^2} - y\sqrt{2 + 4y^2}\right] dy$$

$$= \left[\frac{1}{2}\left(2y\sqrt{2 + 4y^2} + 2\ln\left|2y + \sqrt{2 + 4y^2}\right|\right) - \frac{1}{12}(2 + 4y^2)^{3/2}\right]_0^2$$

$$= \left[\frac{1}{2}\left(4\sqrt{18} + 2\ln\left|4 + \sqrt{18}\right|\right) - \frac{1}{12}\left(18\sqrt{18}\right)\right] - \left[\ln\sqrt{2} - \frac{2\sqrt{2}}{12}\right]$$

$$= 6\sqrt{2} + \ln\left|4 + 3\sqrt{2}\right| - \frac{9\sqrt{2}}{2} - \ln\sqrt{2} + \frac{\sqrt{2}}{6} = \frac{5\sqrt{2}}{3} + \ln\left|2\sqrt{2} + 3\right|$$

39. $f(x, y) = 9 - y^2$

$f_x = 0,\ f_y = -2y$

$$S = \iint_R \sqrt{1 + f_x^2 + f_y^2}\, dA$$

$$= \int_0^3 \int_{-y}^{y} \sqrt{1 + 4y^2}\, dx\, dy$$

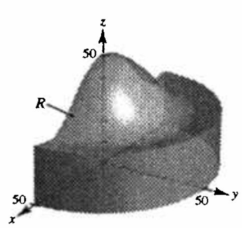
$$= \int_0^3 \left[\sqrt{1 + 4y^2}\, x\right]_{-y}^{y} dy$$

$$= \int_0^3 2y\sqrt{1 + 4y^2}\, dy = \frac{1}{4}\frac{2}{3}(1 + 4y^2)^{3/2}\Big]_0^3 = \frac{1}{6}[(37)^{3/2} - 1]$$

40. (a) Graph of

$$f(x, y) = z$$
$$= 25\left[1 + e^{-(x^2+y^2)/1000}\cos^2\left(\frac{x^2 + y^2}{1000}\right)\right]$$

over region R



(b) Surface area $= \iint_R \sqrt{1 + f_x(x, y)^2 + f_y(x, y)^2}\, dA$

Using a symbolic computer program, you obtain surface area ≈ 4540 sq. ft.

41. $$\int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{x^2+y^2}^{9} \sqrt{x^2 + y^2}\, dz\, dy\, dx = \int_0^{2\pi} \int_0^3 \int_{r^2}^{9} r^2\, dz\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^3 (9r^2 - r^4)\, dr\, d\theta = \int_0^{2\pi} \left[3r^3 - \frac{r^5}{5}\right]_0^3 d\theta = \frac{162}{5}\int_0^{2\pi} d\theta = \frac{324\pi}{5}$$

42. $\displaystyle\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{0}^{(x^2+y^2)/2}(x^2+y^2)\,dz\,dy\,dx = \int_0^{2\pi}\int_0^2\int_0^{r^2/2} r^3\,dz\,dr\,d\theta = \frac{1}{2}\int_0^{2\pi}\int_0^2 r^5\,dr\,d\theta = \frac{16}{3}\int_0^{2\pi}d\theta = \frac{32\pi}{3}$

43. $\displaystyle\int_0^a\int_0^b\int_0^c (x^2+y^2+z^2)\,dx\,dy\,dz = \int_0^a\int_0^b\left(\frac{1}{3}c^3+cy^2+cz^2\right)dy\,dz$

$$= \int_0^a\left(\frac{1}{3}bc^3+\frac{1}{3}b^3c+bcz^2\right)dz = \frac{1}{3}abc^3+\frac{1}{3}ab^3c+\frac{1}{3}a^3bc = \frac{1}{3}abc(a^2+b^2+c^2)$$

44. $\displaystyle\int_0^5\int_0^{\sqrt{25-x^2}}\int_0^{\sqrt{25-x^2-y^2}}\frac{1}{1+x^2+y^2+z^2}\,dz\,dy\,dx = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^5\frac{\rho^2}{1+\rho^2}\sin\phi\,d\rho\,d\phi\,d\theta$

$$= \int_0^{\pi/2}\int_0^{\pi/2}\Big[\rho-\arctan\rho\Big]_0^5\sin\phi\,d\phi\,d\theta$$

$$= \int_0^{\pi/2}\Big[(5-\arctan 5)(-\cos\phi)\Big]_0^{\pi/2}d\theta = \frac{\pi}{2}(5-\arctan 5)$$

45. $\displaystyle\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}}(x^2+y^2)\,dz\,dy\,dx = \int_0^{2\pi}\int_0^1\int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} r^3\,dz\,dr\,d\theta = \frac{8\pi}{15}$

46. $\displaystyle\int_0^2\int_0^{\sqrt{4-x^2}}\int_0^{\sqrt{4-x^2-y^2}} xyz\,dz\,dy\,dx = \frac{4}{3}$

47. $\displaystyle V = 4\int_0^{\pi/2}\int_0^{2\cos\theta}\int_0^{\sqrt{4-r^2}} r\,dz\,dr\,d\theta = 4\int_0^{\pi/2}\int_0^{2\cos\theta} r\sqrt{4-r^2}\,dr\,d\theta$

$$= -\int_0^{\pi/2}\left[\frac{4}{3}(4-r^2)^{3/2}\right]_0^{2\cos\theta}d\theta = \frac{32}{3}\int_0^{\pi/2}(1-\sin^3\theta)\,d\theta$$

$$= \frac{32}{3}\left[\theta+\cos\theta-\frac{1}{3}\cos^3\theta\right]_0^{\pi/2} = \frac{32}{3}\left(\frac{\pi}{2}-\frac{2}{3}\right)$$

48. $\displaystyle V = 2\int_0^{\pi/2}\int_0^{2\sin\theta}\int_0^{16-r^2} r\,dz\,dr\,d\theta = 2\int_0^{\pi/2}\int_0^{2\sin\theta} r(16-r^2)\,dr\,d\theta$

$$= 2\int_0^{\pi/2}(32\sin^2\theta-4\sin^4\theta)\,d\theta = 8\int_0^{\pi/2}(8\sin^2\theta-\sin^4\theta)\,d\theta$$

$$= 8\left[4\theta-2\sin 2\theta+\frac{1}{4}\sin^3\theta\cos\theta-\frac{3}{4}\left(\frac{1}{2}\theta-\frac{1}{4}\sin 2\theta\right)\right]_0^{\pi/2} = \frac{29\pi}{2}$$

49. $\displaystyle m = 4k\int_{\pi/4}^{\pi/2}\int_0^{\pi/2}\int_0^{\cos\phi}\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$

$$= \frac{4}{3}k\int_{\pi/4}^{\pi/2}\int_0^{\pi/2}\cos^3\phi\sin\phi\,d\theta\,d\phi = \frac{2}{3}k\pi\int_{\pi/4}^{\pi/2}\cos^3\phi\sin\phi\,d\phi = \left[-\frac{2}{3}k\pi\left(\frac{1}{4}\cos^4\phi\right)\right]_{\pi/4}^{\pi/2} = \frac{k\pi}{24}.$$

$$M_{xy} = 4k\int_{\pi/4}^{\pi/2}\int_0^{\pi/2}\int_0^{\cos\phi}\rho^3\cos\phi\sin\phi\,d\rho\,d\theta\,d\phi$$

$$= k\int_{\pi/4}^{\pi/2}\int_0^{\pi/2}\cos^5\phi\sin\phi\,d\theta\,d\phi = \frac{1}{2}k\pi\int_{\pi/4}^{\pi/2}\cos^5\phi\sin\phi\,d\phi = \left[-\frac{1}{12}k\pi\cos^6\phi\right]_{\pi/4}^{\pi/2} = \frac{k\pi}{96}$$

$$\bar{z} = \frac{M_{xy}}{m} = \frac{k\pi/96}{k\pi/24} = \frac{1}{4}$$

$\bar{x} = \bar{y} = 0$ by symmetry

50. $m = 2k\int_0^{\pi/2}\int_0^a\int_0^{cr\sin\theta} r\,dz\,dr\,d\theta = 2kc\int_0^{\pi/2}\int_0^a r^2\sin\theta\,dr\,d\theta = \frac{2}{3}kca^3\int_0^{\pi/2}\sin\theta\,d\theta = \frac{2}{3}kca^3$

$$M_{xz} = 2k\int_0^{\pi/2}\int_0^a\int_0^{cr\sin\theta} r^2\sin\theta\,dz\,dr\,d\theta = 2kc\int_0^{\pi/2}\int_0^a r^3\sin^2\theta\,dr\,d\theta = \frac{1}{2}kca^4\int_0^{\pi/2}\sin^2\theta\,d\theta = \frac{1}{8}\pi kca^4$$

$$M_{xy} = 2k\int_0^{\pi/2}\int_0^a\int_0^{cr\sin\theta} rz\,dz\,dr\,d\theta = kc^2\int_0^{\pi/2}\int_0^a r^3\sin^2\theta\,dr\,d\theta = \frac{1}{4}kc^2a^4\int_0^{\pi/2}\sin^2\theta\,d\theta = \frac{1}{16}\pi kc^2a^4$$

$$\bar{x} = 0$$

$$\bar{y} = \frac{M_{xz}}{m} = \frac{\pi kca^4/8}{2kca^3/3} = \frac{3\pi a}{16}$$

$$\bar{z} = \frac{M_{xy}}{m} = \frac{\pi kc^2a^4/16}{2kca^3/3} = \frac{3\pi ca}{32}$$

51. $m = k\int_0^{\pi/2}\int_0^{\pi/2}\int_0^a \rho^2\sin\phi\,d\rho\,d\theta\,d\phi = \frac{k\pi a^3}{6}$

$$M_{xy} = k\int_0^{\pi/2}\int_0^{\pi/2}\int_0^a (\rho\cos\phi)\rho^2\sin\phi\,d\rho\,d\theta\,d\phi = \frac{k\pi a^4}{16}$$

$$\bar{x} = \bar{y} = \bar{z} = \frac{M_{xy}}{m} = \frac{k\pi a^4}{16}\left(\frac{6}{k\pi a^3}\right) = \frac{3a}{8}$$

52. $m = \frac{500\pi}{3} - \int_0^3\int_0^{2\pi}\int^{\sqrt{25-r^2}} r\,dz\,d\theta\,dr = \frac{500\pi}{3} - \int_0^3\int_0^{2\pi}\left(r\sqrt{25-r^2} - 4r\right)d\theta\,dr$

$$= \frac{500\pi}{3} - 2\pi\left[-\frac{1}{3}(25-r^2)^{3/2} - 2r^2\right]_0^3 = \frac{500\pi}{3} - 2\pi\left[-\frac{64}{3} - 18 + \frac{125}{3}\right] = \frac{500\pi}{3} - \frac{14\pi}{3} = 162\pi$$

$\bar{x} = \bar{y} = 0$ by symmetry

$$M_{xy} = \int_0^{2\pi}\int_0^3\int_{-\sqrt{25-r^2}}^4 zr\,dz\,dr\,d\theta + \int_0^{2\pi}\int_3^5\int_{-\sqrt{25-r^2}}^{\sqrt{25-r^2}} zr\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^3\left[8 - \frac{1}{2}(25-r^2)\right]r\,dr\,d\theta + 0$$

$$= \int_0^{2\pi}\int_0^3\left[\frac{1}{2}r^3 - \frac{9}{2}r\right]dr\,d\theta = \int_0^{2\pi}\left[\frac{1}{8}r^4 - \frac{9}{4}r^2\right]_0^3 d\theta = \left[-\frac{81}{8}\theta\right]_0^{2\pi} = -\frac{81}{4}\pi$$

$$\bar{z} = \frac{M_{xy}}{m} = -\frac{81\pi}{4}\frac{1}{162\pi} = -\frac{1}{8}$$

53. $I_z = 4k\int_0^{\pi/2}\int_3^4\int_0^{16-r^2} r^3\,dz\,dr\,d\theta$

$$= 4k\int_0^{\pi/2}\int_3^4 (16r^3 - r^5)\,dr\,d\theta = \frac{833\pi k}{3}$$

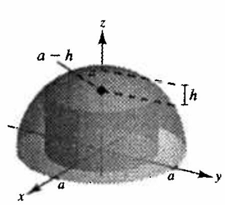
54. $I_z = k\int_0^{\pi}\int_0^{2\pi}\int_0^a \rho^2\sin^2\phi(\rho)\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$

$$= \frac{4k\pi a^6}{9}$$

55. $z = f(x, y) = \sqrt{a^2 - x^2 - y^2}$

$= \sqrt{a^2 - r^2}$

$0 \le r \le \sqrt{2ah - h^2}$



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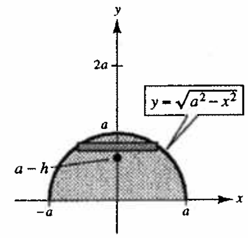
55. —CONTINUED—

(a) Disc Method



$$V = \pi\int_{a-h}^{a} (a^2 - y^2)\,dy$$

$$= \pi\left[a^2y - \frac{y^3}{3}\right]_{a-h}^{a} = \pi\left[\left(a^3 - \frac{a^3}{3}\right) - \left(a^2(a-h) - \frac{(a-h)^3}{3}\right)\right]$$

$$= \pi\left[a^3 - \frac{a^3}{3} - a^3 + a^2h + \frac{a^3}{3} - a^2h + ah^2 - \frac{h^3}{3}\right] = \pi\left[ah^2 - \frac{h^3}{3}\right] = \frac{1}{3}\pi h^2[3a - h]$$

Equivalently, use spherical coordinates.

$$V = \int_0^{2\pi}\int_0^{\cos^{-1}(a-h/a)}\int_{(a-h)\sec\phi}^{a} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

(b) $$M_{xy} = \int_0^{2\pi}\int_0^{\cos^{-1}(a-h/a)}\int_{(a-h)\sec\phi}^{a} (\rho\cos\phi)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$= \frac{1}{4}h^2\pi(2a-h)^2$$

$$\bar{z} = \frac{M_{xy}}{V} = \frac{\frac{1}{4}h^2\pi(2a-h)^2}{\frac{1}{3}h^2\pi(3a-h)} = \frac{3}{4}\frac{(2a-h)^2}{3a-h}$$

Centroid: $\left(0, 0, \dfrac{3(2a-h)^2}{4(3a-h)}\right)$

(c) If $h = a$, $\bar{z} = \dfrac{3(a)^2}{4(2a)} = \dfrac{3}{8}a.$

Centroid of hemisphere: $\left(0, 0, \dfrac{3}{8}a\right)$

(d) $$\lim_{h\to 0}\bar{z} = \lim_{h\to 0}\frac{3(2a-h)^2}{4(3a-h)} = \frac{3(4a^2)}{12a} = a$$

(e) $x^2 + y^2 = \rho^2\sin^2\phi$

$$I_z = \int_0^{2\pi}\int_0^{\cos^{-1}(a-h/a)}\int_{(a-h)\sec\phi}^{a} (\rho^2\sin^2\phi)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \frac{h^3}{30}(20a^2 - 15ah + 3h^2)\pi$$

(f) If $h = a$, $I_z = \dfrac{a^3\pi}{30}(20a^2 - 15a^2 + 3a^2) = \dfrac{4}{15}a^5\pi.$

56. $x^2 + y^2 + \dfrac{z^2}{a^2} = 1$

$$I_z = \iiint_Q (x^2 + y^2)\,dV$$

$$= \int_{-a}^{a}\int_{-\sqrt{1-z^2-a^2}}^{\sqrt{1-z^2-a^2}}\int_{-\sqrt{1-y^2-z^2-a^2}}^{\sqrt{1-y^2-z^2-a^2}} (x^2 + y^2)\,dx\,dy\,dz$$

$$= \frac{8}{15}\pi a$$

57. $$\int_0^{2\pi}\int_0^{\pi}\int_0^{6\sin\phi} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

Since $\rho = 6\sin\phi$ represents (in the yz-plane) a circle of radius 3 centered at $(0, 3, 0)$, the integral represents the volume of the torus formed by revolving $(0 < \theta < 2\pi)$ this circle about the z-axis.

58. $$\int_0^{\pi}\int_0^{2}\int_0^{1+r^2} r\,dz\,dr\,d\theta$$

Since $z = 1 + r^2$ represents a paraboloid with vertex $(0, 0, 1)$, this integral represents the volume of the solid below the paraboloid and above the semi-circle $y = \sqrt{4 - x^2}$ in the xy-plane.

59. $\frac{\partial(x, y)}{\partial(u, v)} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}$

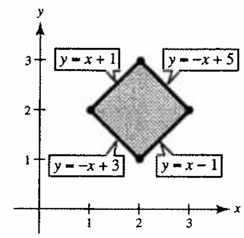
$= 1(-3) - 2(3) = -9$

60. $\frac{\partial(x, y)}{\partial(u, v)} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}$

$= (2u)(-2v) - (2u)(2v) = -8uv$

61. $\frac{\partial(x, y)}{\partial(u, v)} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} = \frac{1}{2}\left(-\frac{1}{2}\right) - \frac{1}{2}\left(\frac{1}{2}\right) = -\frac{1}{2}$

$x = \frac{1}{2}(u + v), y = \frac{1}{2}(u - v) \implies u = x + y, v = x - y$



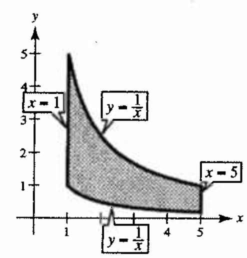
Boundaries in xy-plane	Boundaries in uv-plane
$x + y = 3$	$u = 3$
$x + y = 5$	$u = 5$
$x - y = -1$	$v = -1$
$x - y = 1$	$v = 1$

$$\int\int_R \ln(x + y)\,dA = \int_3^5\int_{-1}^1 \ln\left(\frac{1}{2}(u + v) + \frac{1}{2}(u - v)\right)\left(\frac{1}{2}\right) dv\,du = \int_3^5\int_{-1}^1 \frac{1}{2}\ln u\,dv\,du = \int_3^5 \ln u\,du = \Big[u\ln u - u\Big]_3^5$$

$$= (5\ln 5 - 5) - (3\ln u - 3) = 5\ln 5 - 3\ln 3 - 2 \approx 2.751$$

62. $\frac{\partial(x, y)}{\partial(u, v)} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} = 1\left(\frac{1}{u}\right) - 0 = \frac{1}{u}$

$x = u, y = \frac{v}{u} \implies u = x, v = xy$



Boundary in xy-plane	Boundary in uv-plane
$x = 1$	$u = 1$
$x = 5$	$u = 5$
$xy = 1$	$v = 1$
$xy = 5$	$v = 5$

$$\int\int_R \frac{x}{1 + x^2y^2}\,dA = \int_1^5\int_1^5 \frac{u}{1 + u^2(v/u)^2}\left(\frac{1}{u}\right) du\,dv = \int_1^5\int_1^5 \frac{1}{1 + v^2}\,du\,dv = \int_1^5 \frac{4}{1 + v^2}\,dv$$

$$= 4\arctan v\Big]_1^5 = 4\arctan 5 - \pi$$

Problem Solving for Chapter 14

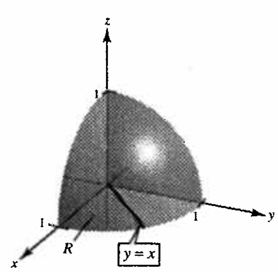
1. (a) $V = 16\int\int_R \sqrt{1 - x^2}\,dA$

$= 16\int_0^{\pi/4}\int_0^1 \sqrt{1 - r^2\cos^2\theta}\,r\,dr\,d\theta$

$= -\frac{16}{3}\int_0^{\pi/4} \frac{1}{\cos^2\theta}\left[(1 - \cos^2\theta)^{3/2} - 1\right] d\theta$

$= -\frac{16}{3}\Big[\sec\theta + \cos\theta - \tan\theta\Big]_0^{\pi/4}$

$= 8(2 - \sqrt{2}) \approx 4.6863$



(b) Programs will vary.

2. $z = \frac{1}{c}(d - ax - by)$ Plane

$$f_x = -\frac{a}{c},\ f_y = -\frac{b}{c}$$

$$\sqrt{1 + f_x^2 + f_y^2} = \sqrt{1 + \frac{a^2}{c^2} + \frac{b^2}{c^2}}$$

$$S = \int\int_R \sqrt{1 + \frac{a^2}{c^2} + \frac{b^2}{c^2}}\, dA = \frac{\sqrt{a^2 + b^2 + c^2}}{c} \int\int_R dA = \frac{\sqrt{a^2 + b^2 + c^2}}{c} A(R)$$

3. (a) $\displaystyle\int \frac{du}{a^2 + u^2} = \frac{1}{a}\arctan\frac{u}{a} + c$. Let $a^2 = 2 - u^2$, $u = v$.

Then $\displaystyle\int \frac{1}{(2 - u^2) + v^2}\, dv = \frac{1}{\sqrt{2 - u^2}} \arctan\frac{v}{\sqrt{2 - u^2}} + C.$

(b) $$I_1 = \int_0^{\sqrt{2}/2} \left[\frac{2}{\sqrt{2 - u^2}} \arctan\frac{v}{\sqrt{2 - u^2}}\right]_{-u}^{u} du$$

$$= \int_0^{\sqrt{2}/2} \frac{2}{\sqrt{2 - u^2}}\left(\arctan\frac{u}{\sqrt{2 - u^2}} - \arctan\frac{-u}{\sqrt{2 - u^2}}\right) du$$

$$= \int_0^{\sqrt{2}/2} \frac{4}{\sqrt{2 - u^2}} \arctan\frac{u}{\sqrt{2 - u^2}}\, du$$

Let $u = \sqrt{2}\sin\theta$, $du = \sqrt{2}\cos\theta\, d\theta$, $2 - u^2 = 2 - 2\sin^2\theta = 2\cos^2\theta$.

$$I_1 = 4\int_0^{\pi/6} \frac{1}{\sqrt{2}\cos\theta} \arctan\left(\frac{\sqrt{2}\sin\theta}{\sqrt{2}\cos\theta}\right) \cdot \sqrt{2}\cos\theta\, d\theta$$

$$= 4\int_0^{\pi/6} \arctan(\tan\theta) d\theta = \frac{4\theta^2}{2}\bigg]_0^{\pi/6} = 2\left(\frac{\pi}{6}\right)^2 = \frac{\pi^2}{18}$$

(c) $$I_2 = \int_{\sqrt{2}/2}^{\sqrt{2}} \left[\frac{2}{\sqrt{2 - u^2}} \arctan\frac{v}{\sqrt{2 - u^2}}\right]_{u-\sqrt{2}}^{-u+\sqrt{2}} du$$

$$= \int_{\sqrt{2}/2}^{\sqrt{2}} \frac{2}{\sqrt{2 - u^2}}\left[\arctan\left(\frac{-u + \sqrt{2}}{\sqrt{2 - u^2}}\right) - \arctan\left(\frac{u - \sqrt{2}}{\sqrt{2 - u^2}}\right)\right] du$$

$$= \int_{\sqrt{2}/2}^{\sqrt{2}} \frac{4}{\sqrt{2 - u^2}} \arctan\left(\frac{\sqrt{2} - u}{\sqrt{2 - u^2}}\right) du$$

Let $u = \sqrt{2}\sin\theta$.

$$I_2 = 4\int_{\pi/6}^{\pi/2} \frac{1}{\sqrt{2}\cos\theta} \arctan\left(\frac{\sqrt{2} - \sqrt{2}\sin\theta}{\sqrt{2}\cos\theta}\right) \cdot \sqrt{2}\cos\theta\, d\theta$$

$$= 4\int_{\pi/6}^{\pi/2} \arctan\left(\frac{1 - \sin\theta}{\cos\theta}\right) d\theta$$

(d) $$\tan\left(\frac{1}{2}\left(\frac{\pi}{2} - \theta\right)\right) = \sqrt{\frac{1 - \cos((\pi/2) - \theta)}{1 + \cos((\pi/2) - \theta)}} = \sqrt{\frac{1 - \sin\theta}{1 + \sin\theta}}$$

$$= \sqrt{\frac{(1 - \sin\theta)^2}{(1 + \sin\theta)(1 - \sin\theta)}} = \sqrt{\frac{(1 - \sin\theta)^2}{\cos^2\theta}} = \frac{1 - \sin\theta}{\cos\theta}$$

—CONTINUED—

3. —CONTINUED—

(e) $$I_2 = 4\int_{\pi/6}^{\pi/2} \arctan\left(\frac{1-\sin\theta}{\cos\theta}\right) d\theta = 4\int_{\pi/6}^{\pi/2} \arctan\left(\tan\left(\frac{1}{2}\left(\frac{\pi}{2}-\theta\right)\right)\right) d\theta$$

$$= 4\int_{\pi/6}^{\pi/2} \frac{1}{2}\left(\frac{\pi}{2}-\theta\right) d\theta = 2\int_{\pi/6}^{\pi/2}\left(\frac{\pi}{2}-\theta\right) d\theta$$

$$= 2\left[\frac{\pi}{2}\theta - \frac{\theta^2}{2}\right]_{\pi/6}^{\pi/2} = 2\left[\left(\frac{\pi^2}{4}-\frac{\pi^2}{8}\right) - \left(\frac{\pi^2}{12}-\frac{\pi^2}{72}\right)\right]$$

$$= 2\left[\frac{18-9-6+1}{72}\pi^2\right] = \frac{4}{36}\pi^2 = \frac{\pi^2}{9}$$

(f) $$\frac{1}{1-xy} = 1 + (xy) + (xy)^2 + \cdots \qquad |xy| < 1$$

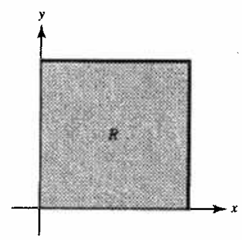
$$\int_0^1\int_0^1 \frac{1}{1-xy}\,dx\,dy = \int_0^1\int_0^1 [1 + (xy) + (xy)^2 + \cdots]\,dx\,dy$$

$$= \int_0^1\int_0^1 \sum_{K=0}^{\infty}(xy)^K\,dx\,dy = \sum_{K=0}^{\infty}\int_0^1 \left.\frac{x^{K+1}y^K}{K+1}\right|_0^1 dy$$

$$= \sum_{K=0}^{\infty}\int_0^1 \frac{y^K}{K+1}\,dy = \sum_{K=0}^{\infty}\left.\frac{y^{K+1}}{(K+1)^2}\right|_0^1$$

$$= \sum_{K=0}^{\infty}\frac{1}{(K+1)^2} = \sum_{n=1}^{\infty}\frac{1}{n^2}$$

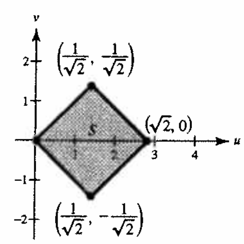
(g) $$u = \frac{x+y}{\sqrt{2}},\ v = \frac{y-x}{\sqrt{2}}$$



$$u - v = \frac{2x}{\sqrt{2}} \Longrightarrow x = \frac{u-v}{\sqrt{2}}$$

$$u + v = \frac{2y}{\sqrt{2}} \Longrightarrow y = \frac{u+v}{\sqrt{2}}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{vmatrix} = 1$$



<u>R</u>		<u>S</u>
$(0, 0)$	$\leftrightarrow$	$(0, 0)$
$(1, 0)$	$\leftrightarrow$	$\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$
$(0, 1)$	$\leftrightarrow$	$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$
$(1, 1)$	$\leftrightarrow$	$(\sqrt{2}, 0)$

$$\int_0^1\int_0^1 \frac{1}{1-xy}\,dx\,dy = \int_0^{\sqrt{2}/2}\int_{-u}^{u} \frac{1}{1-\frac{u^2}{2}+\frac{v^2}{2}}\,dv\,du + \int_{\sqrt{2}/2}^{\sqrt{2}}\int_{u-\sqrt{2}}^{-u+\sqrt{2}} \frac{1}{1-\frac{u^2}{2}+\frac{v^2}{2}}\,dv\,du$$

$$= I_1 + I_2 = \frac{\pi^2}{18} + \frac{\pi^2}{9} = \frac{\pi^2}{6}$$

4. A: $\displaystyle\int_0^{2\pi}\int_4^5\left(\frac{r}{16}-\frac{r^2}{160}\right)r\,dr\,d\theta=\frac{1333\pi}{960}\approx 4.36\text{ ft}^3$

$\displaystyle B=\int_0^{2\pi}\int_9^{10}\left(\frac{r}{16}-\frac{r^2}{160}\right)r\,dr\,d\theta=\frac{523\pi}{960}\approx 1.71\text{ ft}^3$

The distribution is not uniform. Less water in region of greater area.

In one hour, the entire lawn receives

$$\int_0^{2\pi}\int_0^{10}\left(\frac{r}{16}-\frac{r^2}{160}\right)r\,dr\,d\theta=\frac{125\pi}{12}\approx 32.72\text{ ft}^3.$$

5.

Boundary in xy-plane	Boundary in uv-plane
$y=\sqrt{x}$	$u=1$
$y=\sqrt{2x}$	$u=2$
$y=\frac{1}{3}x^2$	$v=3$
$y=\frac{1}{4}x^2$	$v=4$

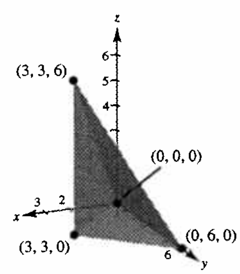
$$\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\frac{1}{3}\left(\frac{v}{u}\right)^{2/3} & \frac{2}{3}\left(\frac{u}{v}\right)^{1/3}\\ \frac{2}{3}\left(\frac{v}{u}\right)^{1/3} & \frac{1}{3}\left(\frac{u}{v}\right)^{2/3}\end{vmatrix}=-\frac{1}{3}$$

$$A=\int_R\int 1\,dA=\int_S\int 1\left|\frac{\partial(x,y)}{\partial(u,v)}\right|dA=\frac{1}{3}$$

6. (a) $\displaystyle V=\int_0^{2\pi}\int_0^2\int_2^{\sqrt{8-r^2}} r\,dz\,dr\,d\theta=\frac{8\pi}{3}\left(4\sqrt{2}-5\right)$

(b) $\displaystyle V=\int_0^{2\pi}\int_0^{\pi/4}\int_{2\sec\phi}^{2\sqrt{2}}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta=\frac{8\pi}{3}\left(4\sqrt{2}-5\right)$

7. $\displaystyle V=\int_0^3\int_0^{2x}\int_x^{6-x}dy\,dz\,dx=18$



8. $$\int_0^1\int_0^1 x^n y^n\,dx\,dy=\int_0^1\left[\frac{x^{n+1}}{n+1}y^n\right]_0^1dy$$
$$=\int_0^1\frac{1}{n+1}y^n\,dy$$
$$=\left[\frac{y^{n+1}}{(n+1)^2}\right]_0^1$$
$$=\frac{1}{(n+1)^2}$$

$$\lim_{n\to\infty}\int_0^1\int_0^1 x^n y^n\,dx\,dy=\lim_{n\to\infty}\frac{1}{(n+1)^2}=0$$

9. From Exercise 55, Section 14.3,

$$\int_{-\infty}^{\infty}e^{-x^2/2}\,dx=\sqrt{2\pi}$$

Thus, $\displaystyle\int_0^{\infty}e^{-x^2/2}\,dx=\frac{\sqrt{2\pi}}{2}$ and $\displaystyle\int_0^{\infty}e^{-x^2}\,dx=\frac{\sqrt{\pi}}{2}$

$$\int_0^{\infty}x^2e^{-x^2}\,dx=\left[-\frac{1}{2}xe^{-x^2}\right]_0^{\infty}+\frac{1}{2}\int_0^{\infty}e^{-x^2}\,dx$$
$$=\frac{1}{2}\frac{\sqrt{\pi}}{2}=\frac{\sqrt{\pi}}{4}$$

10. Let $v=\ln\left(\frac{1}{x}\right)$, $dv=-\frac{dx}{x}$.

$e^v=\frac{1}{x}$, $x=e^{-v}$, $dx=-e^{-v}\,dv$

$$\int_0^1\sqrt{\ln(1/x)}\,dx=\int_{\infty}^0\sqrt{v}\left(-e^{-v}\right)dv=\int_0^{\infty}\sqrt{v}\,e^{-v}\,dv$$

Let $u=\sqrt{v}$, $u^2=v$, $2u\,du=dv$.

$$\int_0^1\sqrt{\ln(1/x)}\,dx=\int_0^{\infty}u\,e^{-u^2}(2u\,du)=2\int_0^{\infty}u^2e^{-u^2}\,du=2\left(\frac{\sqrt{\pi}}{4}\right)=\frac{\sqrt{\pi}}{2}$$ (See Problem Solving #9.)

11. $f(x, y) = \begin{cases} ke^{-(x+y)/a} & x \ge 0, y \ge 0 \\ 0 & \text{elsewhere} \end{cases}$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\, dA = \int_{0}^{\infty}\int_{0}^{\infty} ke^{-(x+y)/a}\, dx\, dy$$
$$= k\int_{0}^{\infty} e^{-x/a}\, dx \cdot \int_{0}^{\infty} e^{-y/a}\, dy$$

These two integrals are equal to

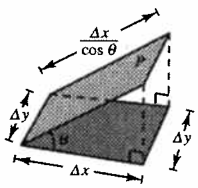
$$\int_{0}^{\infty} e^{-x/a}\, dx = \lim_{b\to\infty}\left[(-a)e^{-x/a}\right]_0^b = a.$$

Hence, assuming $a, k > 0$, you obtain

$$1 = ka^2 \quad\text{or}\quad a = \frac{1}{\sqrt{k}}.$$

12. Essay

13. $A = l \cdot w = \left(\dfrac{\Delta x}{\cos\theta}\right)\Delta y = \sec\theta\, \Delta x\, \Delta y$



Area in xy-plane: $\Delta x\, \Delta y$

14. The greater the angle between the given plane and the xy-plane, the greater the surface area. Hence:

$$z_2 < z_1 < z_4 < z_3$$

15. Converting to polar coordinates,

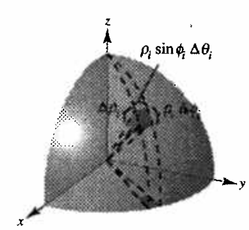
$$\int_{0}^{\infty}\int_{0}^{\infty} \frac{1}{(1 + x^2 + y^2)^2}\, dx\, dy = \int_{0}^{\infty}\int_{0}^{\pi/2} \frac{1}{(1 + r^2)^2}\, r\, d\theta\, dr$$
$$= \int_{0}^{\infty} \frac{r}{(1 + r^2)^2}\frac{\pi}{2}\, dr$$
$$= \lim_{t\to\infty}\int_{0}^{t} \frac{\pi}{4}(1 + r^2)^{-2}(2r\, dr)$$
$$= \lim_{t\to\infty}\left[\frac{\pi}{4}\cdot\frac{-1}{1 + r^2}\right]_0^t$$
$$= \frac{\pi}{4}$$

16. $$\int_{0}^{1}\int_{0}^{1} \frac{x - y}{(x + y)^3}\, dx\, dy = -\frac{1}{2}$$
$$\int_{0}^{1}\int_{0}^{1} \frac{(x - y)}{(x + y)^3}\, dy\, dx = \frac{1}{2}$$

The results are not the same. Fubini's Theorem is not valid because f is not continuous on the region $0 \le x \le 1$, $0 \le y \le 1$.

17. The volume of this spherical block can be determined as follows. One side is length $\Delta\rho$. Another side is $\rho\, \Delta\phi$.Finally, the third side is given by the length of an arc of angle $\Delta\theta$ in a circle of radius $\rho \sin\phi$. Thus:

$$\Delta V \approx (\Delta\rho)(\rho\, \Delta\phi)(\Delta\theta\, \rho \sin\phi)$$
$$= \rho^2 \sin\phi\, \Delta\rho\, \Delta\phi\, \Delta\theta$$



CHAPTER 15
Vector Analysis