CHAPTER 13
Functions of Several Variables

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Section 13.1 Introduction to Functions of Several Variables

1. $x^2z + yz - xy = 10$

$z(x^2 + y) = 10 + xy$

$z = \dfrac{10 + xy}{x^2 + y}$

Yes, z is a function of x and y.

2. $xz^2 + 2xy - y^2 = 4$

No, z is not a function of x and y. For example, $(x, y) = (1, 0)$ corresponds to both $z = \pm 2$.

3. $\dfrac{x^2}{4} + \dfrac{y^2}{9} + z^2 = 1$

No, z is not a function of x and y. For example, $(x, y) = (0, 0)$ corresponds to both $z = \pm 1$.

4. $z + x \ln y - 8 = 0$

$z = 8 - x \ln y$

Yes, z is a function of x and y.

5. $f(x, y) = \dfrac{x}{y}$

(a) $f(3, 2) = \dfrac{3}{2}$

(b) $f(-1, 4) = -\dfrac{1}{4}$

(c) $f(30, 5) = \dfrac{30}{5} = 6$

(d) $f(5, y) = \dfrac{5}{y}$

(e) $f(x, 2) = \dfrac{x}{2}$

(f) $f(5, t) = \dfrac{5}{t}$

6. $f(x, y) = 4 - x^2 - 4y^2$

(a) $f(0, 0) = 4$

(b) $f(0, 1) = 4 - 0 - 4 = 0$

(c) $f(2, 3) = 4 - 4 - 36 = -36$

(d) $f(1, y) = 4 - 1 - 4y^2 = 3 - 4y^2$

(e) $f(x, 0) = 4 - x^2 - 0 = 4 - x^2$

(f) $f(t, 1) = 4 - t^2 - 4 = -t^2$

7. $f(x, y) = xe^y$

(a) $f(5, 0) = 5e^0 = 5$

(b) $f(3, 2) = 3e^2$

(c) $f(2, -1) = 2e^{-1} = \dfrac{2}{e}$

(d) $f(5, y) = 5e^y$

(e) $f(x, 2) = xe^2$

(f) $f(t, t) = te^t$

8. $g(x, y) = \ln|x + y|$

(a) $g(2, 3) = \ln|2 + 3| = \ln 5$

(b) $g(5, 6) = \ln|5 + 6| = \ln 11$

(c) $g(e, 0) = \ln|e + 0| = 1$

(d) $g(0, 1) = \ln|0 + 1| = 0$

(e) $g(2, -3) = \ln|2 - 3| = \ln 1 = 0$

(f) $g(e, e) = \ln|e + e| = \ln 2e$

$= \ln 2 + \ln e = (\ln 2) + 1$

9. $h(x, y, z) = \dfrac{xy}{z}$

(a) $h(2, 3, 9) = \dfrac{2(3)}{9} = \dfrac{2}{3}$

(b) $h(1, 0, 1) = \dfrac{1(0)}{1} = 0$

(c) $h(-2, 3, 4) = \dfrac{(-2)(3)}{4} = -\dfrac{3}{2}$

(d) $h(5, 4, -6) = \dfrac{5(4)}{-6} = -\dfrac{10}{3}$

10. $f(x, y, z) = \sqrt{x + y + z}$

(a) $f(0, 5, 4) = \sqrt{0 + 5 + 4} = 3$

(b) $f(6, 8, -3) = \sqrt{6 + 8 - 3} = \sqrt{11}$

(c) $f(4, 6, 2) = \sqrt{4 + 6 + 2} = \sqrt{12} = 2\sqrt{3}$

(d) $f(10, -4, -3) = \sqrt{10 - 4 - 3} = \sqrt{3}$

11. $f(x, y) = x \sin y$

(a) $f\left(2, \frac{\pi}{4}\right) = 2 \sin \frac{\pi}{4} = \sqrt{2}$

(b) $f(3, 1) = 3 \sin(1)$

(c) $f\left(-3, \frac{\pi}{3}\right) = -3 \sin \frac{\pi}{3} = -3\left(\frac{\sqrt{3}}{2}\right) = \frac{-3\sqrt{3}}{2}$

(d) $f\left(4, \frac{\pi}{2}\right) = 4 \sin \frac{\pi}{2} = 4$

12. $V(r, h) = \pi r^2 h$

(a) $V(3, 10) = \pi(3^2)10 = 90\pi$

(b) $V(5, 2) = \pi(5^2)2 = 50\pi$

(c) $V(4, 8) = \pi(4^2)8 = 128\pi$

(d) $V(6, 4) = \pi(6^2)4 = 144\pi$

13. $g(x, y) = \int_x^y (2t - 3)\, dt = \Big[t^2 - 3t\Big]_x^y = y^2 - 3y - x^2 + 3x$

(a) $g(0, 4) = 16 - 12 = 4$

(b) $g(1, 4) = 16 - 12 - 1 + 3 = 6$

(c) $g\left(\frac{3}{2}, 4\right) = 16 - 12 - \frac{9}{4} + \frac{9}{2} = \frac{25}{4}$

(d) $g\left(0, \frac{3}{2}\right) = \frac{9}{4} - 3\left(\frac{3}{2}\right) = -\frac{9}{4}$

14. $g(x, y) = \int_x^y \frac{1}{t}\, dt = \ln|t|\Big]_x^y = \ln|y| - \ln|x| = \ln\left|\frac{y}{x}\right|$

(a) $g(4, 1) = \ln \frac{1}{4} = -\ln 4$

(b) $g(6, 3) = \ln \frac{3}{6} = -\ln 2$

(c) $g(2, 5) = \ln \frac{5}{2}$

(d) $g\left(\frac{1}{2}, 7\right) = \ln \frac{7}{(1/2)} = \ln 14$

15. $f(x, y) = x^2 - 2y$

(a) $$\frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} = \frac{[(x + \Delta x)^2 - 2y] - (x^2 - 2y)}{\Delta x}$$

$$= \frac{x^2 + 2x(\Delta x) + (\Delta x)^2 - 2y - x^2 + 2y}{\Delta x} = \frac{\Delta x(2x + \Delta x)}{\Delta x} = 2x + \Delta x,\ \Delta x \neq 0$$

(b) $$\frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} = \frac{[x^2 - 2(y + \Delta y)] - (x^2 - 2y)}{\Delta y} = \frac{x^2 - 2y - 2\Delta y - x^2 + 2y}{\Delta y} = \frac{-2\Delta y}{\Delta y} = -2,\ \Delta y \neq 0$$

16. $f(x, y) = 3xy + y^2$

(a) $$\frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} = \frac{[3(x + \Delta x)y + y^2] - (3xy + y^2)}{\Delta x}$$

$$= \frac{3xy + 3(\Delta x)y + y^2 - 3xy - y^2}{\Delta x} = \frac{3(\Delta x)y}{\Delta x} = 3y,\ \Delta x \neq 0$$

(b) $$\frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} = \frac{[3x(y + \Delta y) + (y + \Delta y)^2] - (3xy + y^2)}{\Delta y}$$

$$= \frac{3xy + 3x(\Delta y) + y^2 + 2y(\Delta y) + (\Delta y)^2 - 3xy - y^2}{\Delta y}$$

$$= \frac{\Delta y(3x + 2y + \Delta y)}{\Delta y} = 3x + 2y + \Delta y,\ \Delta y \neq 0$$

17. $f(x, y) = \sqrt{4 - x^2 - y^2}$

Domain: $4 - x^2 - y^2 \geq 0$

$x^2 + y^2 \leq 4$

$\{(x, y): x^2 + y^2 \leq 4\}$

Range: $0 \leq z \leq 2$

18. $f(x, y) = \sqrt{4 - x^2 - 4y^2}$

Domain: $4 - x^2 - 4y^2 \geq 0$

$x^2 + 4y^2 \leq 4$

$\frac{x^2}{4} + \frac{y^2}{1} \leq 1$

$\left\{(x, y): \frac{x^2}{4} + \frac{y^2}{1} \leq 1\right\}$

Range: $0 \leq z \leq 2$

19. $f(x, y) = \arcsin(x + y)$

Domain:
$\{(x, y): -1 \leq x + y \leq 1\}$

Range: $-\frac{\pi}{2} \leq z \leq \frac{\pi}{2}$

20. $f(x, y) = \arccos \frac{y}{x}$

Domain: $\left\{(x, y): -1 \leq \frac{y}{x} \leq 1\right\}$

Range: $0 \leq z \leq \pi$

21. $f(x, y) = \ln(4 - x - y)$

Domain: $4 - x - y > 0$

$x + y < 4$

$\{(x, y): y < -x + 4\}$

Range: all real numbers

22. $f(x, y) = \ln(xy - 6)$

Domain: $xy - 6 > 0$

$xy > 6$

$\{(x, y): xy > 6\}$

Range: all real numbers

23. $z = \frac{x + y}{xy}$

Domain: $\{(x, y): x \neq 0 \text{ and } y \neq 0\}$

Range: all real numbers

24. $z = \frac{xy}{x - y}$

Domain: $\{(x, y): x \neq y\}$

Range: all real numbers

25. $f(x, y) = e^{x/y}$

Domain: $\{(x, y): y \neq 0\}$

Range: $z > 0$

26. $f(x, y) = x^2 + y^2$

Domain: $\{(x, y)$: x is any real number, y is any real number$\}$

Range: $z \geq 0$

27. $g(x, y) = \frac{1}{xy}$

Domain: $\{(x, y): x \neq 0 \text{ and } y \neq 0\}$

Range: all real numbers except zero

28. $g(x, y) = x\sqrt{y}$

Domain: $\{(x, y): y \geq 0\}$

Range: all real numbers

29. $f(x, y) = \frac{-4x}{x^2 + y^2 + 1}$

(a) View from the positive x-axis: $(20, 0, 0)$

(b) View where x is negative, y and z are positive: $(-15, 10, 20)$

(c) View from the first octant: $(20, 15, 25)$

(d) View from the line $y = x$ in the xy-plane: $(20, 20, 0)$

30. (a) Domain: $\{(x, y)$: x is any real number, y is any real number$\}$

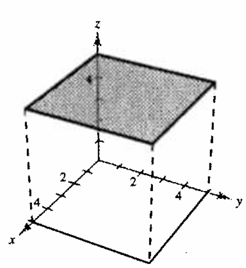
Range: $-2 \leq z \leq 2$

(b) $z = 0$ when $x = 0$ which represents points on the y-axis.

(c) No. When x is positive, z is negative. When x is negative, z is positive. The surface does not pass through the first octant, the octant where y is negative and x and z are positive, the octant where y is positive and x and z are negative, and the octant where x, y and z are all negative.

31. $f(x, y) = 5$

Plane: $z = 5$

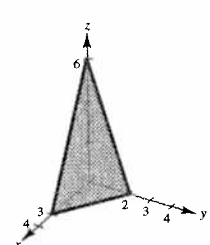


32. $f(x, y) = 6 - 2x - 3y$

Plane

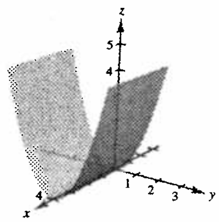
Domain: entire xy-plane

Range: $-\infty < z < \infty$



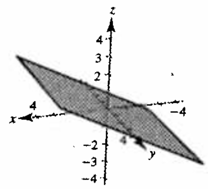
33. $f(x, y) = y^2$

Since the variable x is missing, the surface is a cylinder with rulings parallel to the x-axis. The generating curve is $z = y^2$. The domain is the entire xy-plane and the range is $z \geq 0$.



34. $g(x, y) = \frac{1}{2}x$

Plane: $z = \frac{1}{2}x$

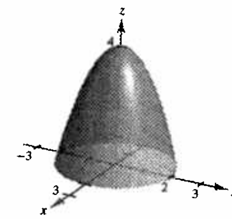


35. $z = 4 - x^2 - y^2$

Paraboloid

Domain: entire xy-plane

Range: $z \leq 4$

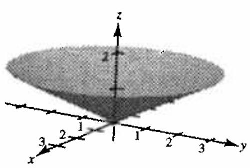


36. $z = \dfrac{1}{2}\sqrt{x^2 + y^2}$

Cone

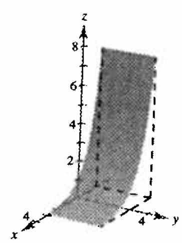
Domain of f: entire xy-plane

Range: $z \geq 0$



37. $f(x, y) = e^{-x}$

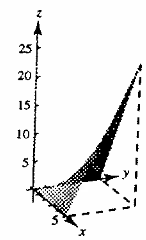
Since the variable y is missing, the surface is a cylinder with rulings parallel to the y-axis. The generating curve is $z = e^{-x}$. The domain is the entire xy-plane and the range is $z > 0$.



38. $f(x, y) = \begin{cases} xy, & x \geq 0,\ y \geq 0 \\ 0, & \text{elsewhere} \end{cases}$

Domain of f: entire xy-plane

Range: $z \geq 0$

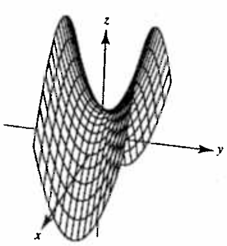


39. $z = y^2 - x^2 + 1$

Hyperbolic paraboloid

Domain: entire xy-plane

Range: $-\infty < z < \infty$

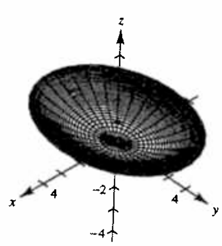


40. $f(x, y) = \frac{1}{12}\sqrt{144 - 16x^2 - 9y^2}$

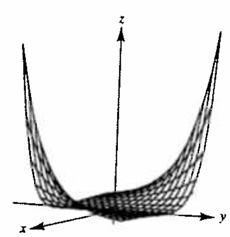
Semi-ellipsoid

Domain: set of all points lying on or inside the ellipse $(x^2/9) + (y^2/16) = 1$

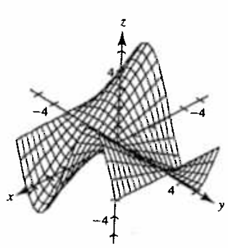
Range: $0 \leq z \leq 1$



41. $f(x, y) = x^2e^{(-xy/2)}$

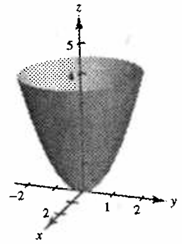


42. $f(x, y) = x \sin y$



43. $f(x, y) = x^2 + y^2$

(a)

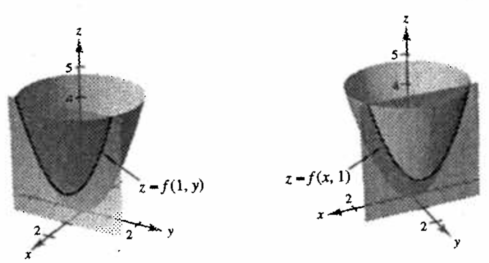


(b) g is a vertical translation of f two units upward.

(c) g is a horizontal translation of f two units to the right. The vertex moves from $(0, 0, 0)$ to $(0, 2, 0)$.

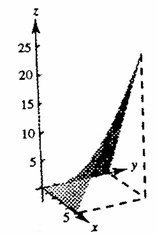
(d) g is a reflection of f in the xy-plane followed by a vertical translation 4 units upward.

(e)



44. $f(x, y) = xy, x \geq 0, y \geq 0$

(a)

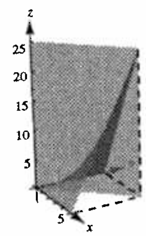


(b) g is a vertical translation of f three units downward.

(c) g is a reflection of f in the xy-plane.

(d) The graph of g is lower than the graph of f. If $z = f(x, y)$ is on the graph of f, then $\frac{1}{2}z$ is on the graph of g.

(e)



45. $z = e^{1-x^2-y^2}$

Level curves:

$$c = e^{1-x^2-y^2}$$

$$\ln c = 1 - x^2 - y^2$$

$$x^2 + y^2 = 1 - \ln c$$

Circles centered at $(0, 0)$

Matches (c)

46. $z = e^{1-x^2+y^2}$

Level curves:

$$c = e^{1-x^2+y^2}$$

$$\ln c = 1 - x^2 + y^2$$

$$x^2 - y^2 = 1 - \ln c$$

Hyperbolas centered at $(0, 0)$

Matches (d)

47. $z = \ln|y - x^2|$

Level curves:

$$c = \ln|y - x^2|$$

$$\pm e^c = y - x^2$$

$$y = x^2 \pm e^c$$

Parabolas

Matches (b)

48. $z = \cos\left(\frac{x + 2y^2}{4}\right)$

Level curves:

$$c = \cos\left(\frac{x^2 + 2y^2}{4}\right)$$

$$\cos^{-1} c = \frac{x^2 + 2y^2}{4}$$

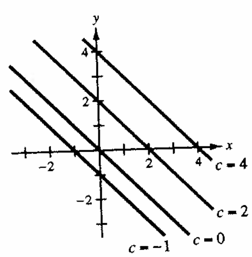
$$x^2 + 2y^2 = 4 \cos^{-1} c$$

Ellipses

Matches (a)

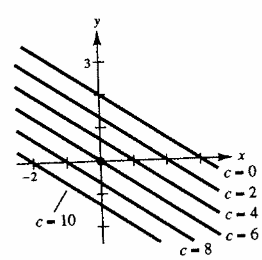
49. $z = x + y$

Level curves are parallel lines of the form $x + y = c$.



50. $f(x, y) = 6 - 2x - 3y$

The level curves are of the form $6 - 2x - 3y = c$ or $2x + 3y = 6 - c$. Thus, the level curves are straight lines with a slope of $-\frac{2}{3}$.



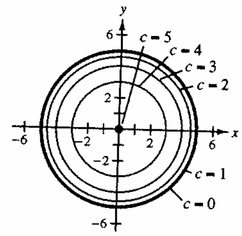
51. $f(x, y) = \sqrt{25 - x^2 - y^2}$

The level curves are of the form

$$c = \sqrt{25 - x^2 - y^2},$$

$$x^2 + y^2 = 25 - c^2.$$

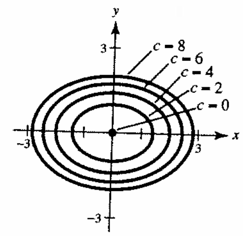
Thus, the level curves are circles of radius 5 or less, centered at the origin.



52. $f(x, y) = x^2 + 2y^2$

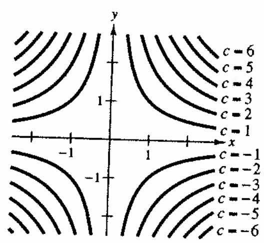
The level curves are ellipses of the form

$$x^2 + 2y^2 = c \text{ (except } x^2 + 2y^2 = 0 \text{ is the point } (0, 0)).$$



53. $f(x, y) = xy$

The level curves are hyperbolas of the form $xy = c$.

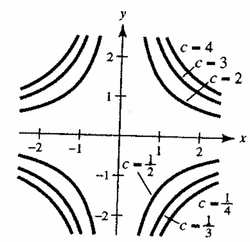


54. $f(x, y) = e^{xy/2}$

The level curves are of the form

$$e^{xy/2} = c, \text{ or } \ln c = \frac{xy}{2}.$$

Thus, the level curves are hyperbolas.



55. $f(x, y) = \dfrac{x}{x^2 + y^2}$

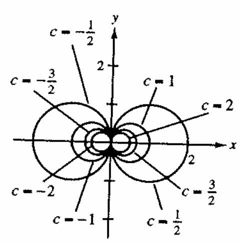
The level curves are of the form

$$c = \frac{x}{x^2 + y^2}$$

$$x^2 - \frac{x}{c} + y^2 = 0$$

$$\left(x - \frac{1}{2c}\right)^2 + y^2 = \left(\frac{1}{2c}\right)^2.$$

Thus, the level curves are circles passing through the origin and centered at $(\pm 1/2c, 0)$.



56. $f(x, y) = \ln(x - y)$

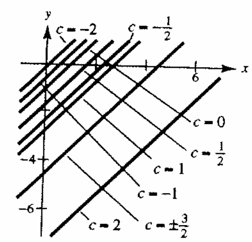
The level curves are of the form

$$c = \ln(x - y)$$

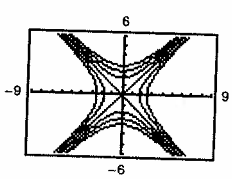
$$e^c = x - y$$

$$y = x - e^c.$$

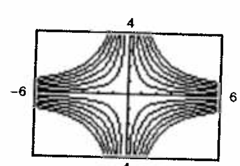
Thus, the level curves are parallel lines of slope 1 passing through the fourth quadrant.



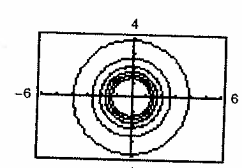
57. $f(x, y) = x^2 - y^2 + 2$



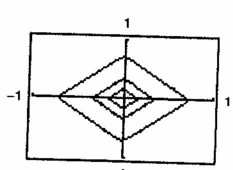
58. $f(x, y) = |xy|$



59. $g(x, y) = \dfrac{8}{1 + x^2 + y^2}$



60. $h(x, y) = 3\sin(|x| + |y|)$



61. See Definition, page 884.

62. The graph of a function of two variables is the set of all points (x, y, z) for which $z = f(x, y)$ and (x, y) is in the domain of f. The graph can be interpreted as a surface in space. Level curves are the scalar fields $f(x, y) = c$, for c, a constant.

63. No, The following graphs are not hemispheres.

$z = e^{-(x^2+y^2)}$

$z = x^2 + y^2$

64. $f(x, y) = \dfrac{x}{y}$

The level curves are the lines

$$c = \frac{x}{y} \text{ or } y = \frac{1}{c}x.$$

These lines all pass through the origin.

65. The surface is sloped like a saddle. The graph is not unique. Any vertical translation would have the same level curves.
One possible function is

$$f(x, y) = |xy|.$$

66. The surface could be an ellipsoid centered at (0, 1, 0).
One possible function is

$$f(x, y) = x^2 + \frac{(y-1)^2}{4} - 1.$$

67. $V(I, R) = 1000\left[\dfrac{1 + 0.10(1 - R)}{1 + I}\right]^{10}$

	Inflation Rate		
Tax Rate	0	0.03	0.05
0	2593.74	1929.99	1592.33
0.28	2004.23	1491.34	1230.42
0.35	1877.14	1396.77	1152.40

68. $A(r, t) = 1000e^{rt}$

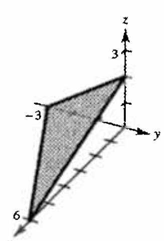
	Number of Years			
Rate	5	10	15	20
0.02	1105.17	1121.40	1349.86	1491.82
0.04	1221.40	1491.82	1822.12	2225.54
0.06	1349.86	1822.12	2459.60	3320.12
0.08	1491.82	2225.54	3320.12	4953.03

69. $f(x, y, z) = x - 2y + 3z$

$c = 6$

$6 = x - 2y + 3z$

Plane

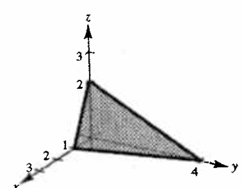


70. $f(x, y, z) = 4x + y + 2z$

$c = 4$

$4 = 4x + y + 2z$

Plane

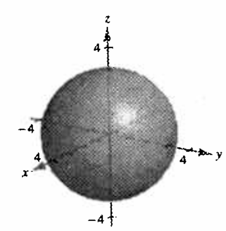


71. $f(x, y, z) = x^2 + y^2 + z^2$

$c = 9$

$9 = x^2 + y^2 + z^2$

Sphere



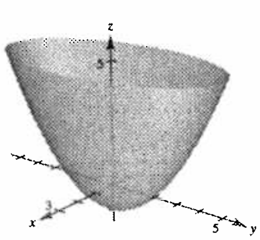
72. $f(x, y, z) = x^2 + \frac{1}{4}y^2 - z$

$c = 1$

$1 = x^2 + \frac{1}{4}y^2 - z$

Elliptic paraboloid

Vertex: $(0, 0, -1)$

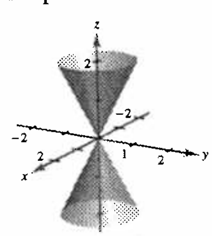


73. $f(x, y, z) = 4x^2 + 4y^2 - z^2$

$c = 0$

$0 = 4x^2 + 4y^2 - z^2$

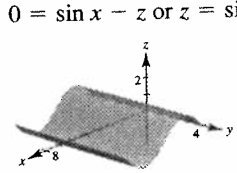
Elliptic cone



74. $f(x, y, z) = \sin x - z$

$c = 0$

$0 = \sin x - z$ or $z = \sin x$



75. $N(d, L) = \left(\frac{d-4}{4}\right)^2 L$

(a) $N(22, 12) = \left(\frac{22-4}{4}\right)^2(12) = 243$ board-feet

(b) $N(30, 12) = \left(\frac{30-4}{4}\right)^2(12) = 507$ board-feet

76. $W(x, y) = \frac{1}{x-y},\ y < x$

(a) $W(15, 10) = \frac{1}{15-10} = \frac{1}{5}$ hr $= 12$ min

(b) $W(12, 9) = \frac{1}{12-9} = \frac{1}{3}$ hr $= 20$ min

(c) $W(12, 6) = \frac{1}{12-6} = \frac{1}{6}$ hr $= 10$ min

(d) $W(4, 2) = \frac{1}{4-2} = \frac{1}{2}$ hr $= 30$ min

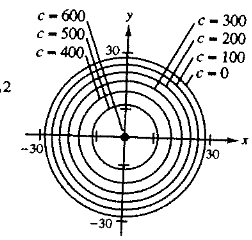
77. $T = 600 - 0.75x^2 - 0.75y^2$

The level curves are of the form

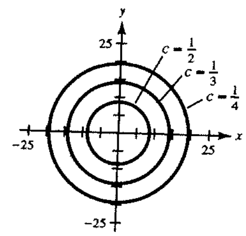
$$c = 600 - 0.75x^2 - 0.75y^2$$

$$x^2 + y^2 = \frac{600-c}{0.75}.$$

The level curves are circles centered at the origin.



78. $V(x, y) = \frac{5}{\sqrt{25 + x^2 + y^2}}$



79. $f(x, y) = 100x^{0.6}y^{0.4}$

$f(2x, 2y) = 100(2x)^{0.6}(2y)^{0.4}$

$= 100(2)^{0.6}x^{0.6}(2)^{0.4}y^{0.4} = 100(2)^{0.6}(2)^{0.4}x^{0.6}y^{0.4} = 2[100x^{0.6}y^{0.4}] = 2f(x, y)$

80. $z = Cx^a y^{1-a}$

$\ln z = \ln C + a \ln x + (1 - a) \ln y$

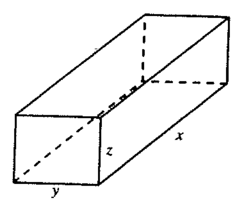
$\ln z - \ln y = \ln C + a \ln x - a \ln y$

$\ln \frac{z}{y} = \ln C + a \ln \frac{x}{y}$

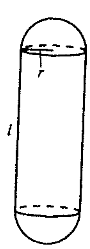
81. $C = 0.75xy + 2(0.40)xz + 2(0.40)yz$

base + front & back + two ends

$= 0.75xy + 0.80(xz + yz)$



82. $V = \pi r^2 l + \frac{4}{3}\pi r^3 = \frac{\pi r^2}{3}(3l + 4r)$



83. $PV = kT,\ 20(2600) = k(300)$

(a) $k = \frac{20(2600)}{300} = \frac{520}{3}$

(b) $P = \frac{kT}{V} = \frac{520}{3}\left(\frac{T}{V}\right)$

The level curves are of the form: $c = \left(\frac{520}{3}\right)\left(\frac{T}{V}\right)$

$$V = \frac{520}{3c}T$$

Thus, the level curves are lines through the origin with slope $\frac{520}{3c}$.

84. (a) $z = f(x, y) = 0.156x + 0.031y - 1.66$

Year	1998	1999	2000	2001	2002	2003
z	18.5	21.1	25.8	31.3	35.1	39.3
Model	18.16	21.36	26.26	30.60	34.91	39.42

(b) x has the greater influence because its coefficient (0.156) is larger than that of $y(0.031)$.

(c) $f(x, 55) = 0.156x + 0.031(55) - 1.66 = 0.156x + 0.045$

This function gives the shareholder's equity in terms of net sales x, assuming a constant total assets of $y = 55$ (billion).

85. (a) Highest pressure at C

(b) Lowest pressure at A

(c) Highest wind velocity at B

86. Southwest

87. (a) The boundaries between colors represent level curves.

(b) No, the colors represent intervals of different lengths, as indicated in the box

(c) You could use more colors, which means using smaller intervals.

88. (a) The different colors represent various amplitudes.

(b) No, the level curves are uneven and sporadically spaced.

89. False. Let

$f(x, y) = 2xy$

$f(1, 2) = f(2, 1)$, but $1 \neq 2$.

90. True

91. False. Let

$f(x, y) = 5$.

Then, $f(2x, 2y) = 5 \neq 2^2 f(x, y)$.

92. True

Section 13.2 Limits and Continuity

1. $\lim_{(x, y)\to(2, 3)} x = 2$. $f(x, y) = x, L = 2$

We need to show that for all $\varepsilon > 0$, there exists a δ-neighborhood about (2, 3) such that

$|f(x, y) - L| = |x - 2| < \varepsilon$

whenever $(x, y) \neq (2, 3)$ lies in the neighborhood.

From $0 < \sqrt{(x-2)^2 + (y-3)^2} < \delta$ it follows that $|x - 2| = \sqrt{(x-2)^2} \leq \sqrt{(x-2)^2 + (y-3)^2} < \delta$.

So, choose $\delta = \varepsilon$ and the limit is verified.

2. Let $\varepsilon > 0$ be given. We need to find $\delta > 0$ such that $|f(x, y) - L| = |x - 4| < \varepsilon$

whenever $0 < \sqrt{(x-a)^2 + (y-b)^2} = \sqrt{(x-4)^2 + (y+1)^2} < \delta$. Take $\delta = \varepsilon$.

Then if $0 < \sqrt{(x-4)^2 + (y+1)^2} < \delta = \varepsilon$, we have

$\sqrt{(x-4)^2} < \varepsilon$

$|x - 4| < \varepsilon$.

3. $\lim_{(x, y)\to(1, -3)} y = -3$. $f(x, y) = y, L = -3$

We need to show that for all $\varepsilon > 0$, there exists a δ-neighborhood about $(1, -3)$ such that

$$|f(x, y) - L| = |y + 3| < \varepsilon$$

whenever $(x, y) \neq (1, -3)$ lies in the neighborhood.

From $0 < \sqrt{(x-1)^2 + (y+3)^2} < \delta$ it follows that $|y + 3| = \sqrt{(y+3)^2} \leq \sqrt{(x-1)^2 + (y+3)^2} < \delta$.

So, choose $\delta = \varepsilon$ and the limit is verified.

4. Let $\varepsilon > 0$ be given. We need to find $\delta > 0$ such that $|f(x, y) - L| = |y - b| < \varepsilon$ whenever $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$. Take $\delta = \varepsilon$.

Then if $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta = \varepsilon$, we have

$$\sqrt{(y-b)^2} < \varepsilon$$

$$|y - b| < \varepsilon.$$

5. $\lim_{(x, y)\to(a, b)} [f(x, y) - g(x, y)] = \lim_{(x, y)\to(a, b)} f(x, y) - \lim_{(x, y)\to(a, b)} g(x, y) = 5 - 3 = 2$

6. $\lim_{(x, y)\to(a, b)} \left[\dfrac{4f(x, y)}{g(x, y)}\right] = \dfrac{4\left[\lim_{(x, y)\to(a, b)} f(x, y)\right]}{\lim_{(x, y)\to(a, b)} g(x, y)} = \dfrac{4(5)}{3} = \dfrac{20}{3}$

7. $\lim_{(x, y)\to(a, b)} [f(x, y)g(x, y)] = \left[\lim_{(x, y)\to(a, b)} f(x, y)\right]\left[\lim_{(x, y)\to(a, b)} g(x, y)\right] = 5(3) = 15$

8. $\lim_{(x, y)\to(a, b)} \left[\dfrac{f(x, y) - g(x, y)}{f(x, y)}\right] = \dfrac{\lim_{(x, y)\to(a, b)} f(x, y) - \lim_{(x, y)\to(a, b)} g(x, y)}{\lim_{(x, y)\to(a, b)} f(x, y)} = \dfrac{5-3}{5} = \dfrac{2}{5}$

9. $\lim_{(x, y)\to(2, 1)} (x + 3y^2) = 2 + 3(1)^2 = 5$

Continuous everywhere

10. $\lim_{(x, y)\to(0, 0)} (5x + y + 1) = 0 + 0 + 1 = 1$

Continuous everywhere

11. $\lim_{(x, y)\to(2, 4)} \dfrac{x + y}{x - y} = \dfrac{2 + 4}{2 - 4} = -3$

Continuous for $x \neq y$

12. $\lim_{(x, y)\to(1, 1)} \dfrac{x}{\sqrt{x + y}} = \dfrac{1}{\sqrt{1 + 1}} = \dfrac{\sqrt{2}}{2}$

Continuous for $x + y > 0$

13. $\lim_{(x, y)\to(0, 1)} \dfrac{\arcsin(x/y)}{1 + xy} = \arcsin 0 = 0$

Continuous for $xy \neq -1, y \neq 0, |x/y| \leq 1$

14. $\lim_{(x, y)\to(\pi/4, 2)} y \cos(xy) = 2 \cos \dfrac{\pi}{2} = 0$

Continuous everywhere

15. $\lim_{(x, y)\to(-1, 2)} e^{xy} = e^{-2} = \dfrac{1}{e^2}$

Continuous everywhere

16. $\lim_{(x, y)\to(1, 1)} \dfrac{xy}{x^2 + y^2} = \dfrac{1}{2}$

Continuous except at $(0, 0)$

17. $\lim_{(x, y, z)\to(1, 2, 5)} \sqrt{x + y + z} = \sqrt{8} = 2\sqrt{2}$

Continuous for $x + y + z \geq 0$

18. $\lim_{(x, y, z)\to(2, 0, 1)} xe^{yz} = 2e^0 = 2$

Continuous everywhere

19. The limit does not exist because along the line $y = 0$ you have

$$\lim_{(x,y)\to(0,0)} \frac{x+y}{x^2+y} = \lim_{(x,0)\to(0,0)} \frac{x}{x^2} = \lim_{(x,0)\to(0,0)} \frac{1}{x}$$

which does not exist.

20. The limit does not exist because along the line $x = y$ you have

$$\lim_{(x,y)\to(0,0)} \frac{x}{x^2-y^2} = \lim_{(x,x)\to(0,0)} \frac{x}{x^2-x^2} = \lim_{(x,x)\to(0,0)} \frac{x}{0}.$$

Since the denominator is 0, the limit does not exist.

21. $$\lim_{(x,y)\to(1,1)} \frac{xy-1}{1+xy} = \frac{1-1}{1+1} = 0$$

22. The limit does not exist because along the line $x = 0$ you have

$$\lim_{(x,y)\to(0,0)} \frac{x+y}{x+y^3} = \lim_{(0,y)\to(0,0)} \frac{y}{y^3} = \lim_{(0,y)\to(0,0)} \frac{1}{y^2}$$

which does not exist.

23. The limit does not exist because along the path $x = 0, y = 0$, you have

$$\lim_{(x,y,z)\to(0,0,0)} \frac{xy+yz+xz}{x^2+y^2+z^2} = \lim_{(0,0,z)\to(0,0,0)} \frac{0}{z^2} = 0$$

whereas along the path $x = y = z$, you have

$$\lim_{(x,y,z)\to(0,0,0)} \frac{xy+yz+xz}{x^2+y^2+z^2} = \lim_{(x,x,x)\to(0,0,0)} \frac{x^2+x^2+x^2}{x^2+x^2+x^2} = 1$$

24. The limit does not exist because along the path $y = z = 0$, you have

$$\lim_{(x,y,z)\to(0,0,0)} \frac{xy+yz^2+xz^2}{x^2+y^2+z^2} = \lim_{(x,0,0)\to(0,0,0)} \frac{0}{x^2} = 0$$

However, along the path $z = 0, x = y$, you have

$$\lim_{(x,y,z)\to(0,0,0)} \frac{xy+yz^2+xz^2}{x^2+y^2+z^2} = \lim_{(x,x,0)\to(0,0,0)} \frac{x^2}{x^2+x^2} = \frac{1}{2}$$

25. $$\lim_{(x,y)\to(0,0)} e^{xy} = 1$$

Continuous everywhere

26. $$f(x, y) = \frac{x^2}{(x^2+1)(y^2+1)}$$

$$\lim_{(x,y)\to(0,0)} \frac{x^2}{(x^2+1)(y^2+1)} = \frac{0}{(0+1)(0+1)} = 0$$

Continuous everywhere

27. $$\lim_{(x,y)\to(0,0)} \ln(x^2+y^2) = \ln(0) = -\infty$$

The limit does not exist.

Continuous except at $(0, 0)$

28. $$\lim_{(x,y)\to(0,0)} \left[1 - \frac{\cos(x^2+y^2)}{x^2+y^2}\right] = -\infty$$

The limit does not exist.

Continuous except at $(0, 0)$

29. $f(x, y) = \dfrac{xy}{x^2 + y^2}$

Continuous except at (0, 0)

Path: $y = 0$

(x, y)	(1, 0)	(0.5, 0)	(0.1, 0)	(0.01, 0)	(0.001, 0)
$f(x, y)$	0	0	0	0	0

Path: $y = x$

(x, y)	(1, 1)	(0.5, 0.5)	(0.1, 0.1)	(0.01, 0.01)	(0.001, 0.001)
$f(x, y)$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

The limit does not exist because along the path $y = 0$ the function equals 0, whereas along the path $y = x$ the function equals $\frac{1}{2}$.

30. $f(x, y) = \dfrac{y}{x^2 + y^2}$

Continuous except at (0, 0)

Path: $y = x$

(x, y)	(1, 1)	(0.5, 0.5)	(0.1, 0.1)	(0.01, 0.01)	(0.001, 0.001)
$f(x, y)$	$\frac{1}{2}$	1	5	50	500

Path: $y = 0$

(x, y)	(1, 0)	(0.5, 0)	(0.1, 0)	(0.01, 0)	(0.001, 0)
$f(x, y)$	0	0	0	0	0

The limit does not exist because along the path $y = 0$ the function equals 0, whereas along the path $y = x$ the function tends to infinity.

31. $f(x, y) = -\dfrac{xy^2}{x^2 + y^4}$

Continuous except at (0, 0)

Path: $x = y^2$

(x, y)	(1, 1)	(0.25, 0.5)	(0.01, 0.1)	(0.0001, 0.01)	(0.000001, 0.001)
$f(x, y)$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$

Path: $x = -y^2$

(x, y)	(−1, 1)	(−0.25, 0.5)	(−0.01, 0.1)	(−0.0001, 0.01)	(−0.000001, 0.001)
$f(x, y)$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

The limit does not exist because along the path $x = y^2$ the function equals $-\frac{1}{2}$, whereas along the path $x = -y^2$ the function equals $\frac{1}{2}$.

32. $f(x, y) = \dfrac{2x - y^2}{2x^2 + y}$

Continuous except at (0, 0)

Path: $y = 0$

(x, y)	(1, 0)	(0.25, 0)	(0.01, 0)	(0.001, 0)	(0.000001, 0)
$f(x, y)$	1	4	100	1000	1,000,000

Path: $y = x$

(x, y)	(1, 1)	(0.25, 0.25)	(0.01, 0.01)	(0.001, 0.001)	(0.0001, 0.0001)
$f(x, y)$	$\frac{1}{3}$	1.17	1.95	1.995	2.0

The limit does not exist because along the line $y = 0$ the function tends to infinity, whereas along the line $y = x$ the function tends to 2.

33. $\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \left(\dfrac{x^2 + 2xy^2 + y^2}{x^2 + y^2}\right)$

$= \lim_{(x,y)\to(0,0)} \left(1 + \dfrac{2xy^2}{x^2 + y^2}\right) = 1$

(same limit for g)

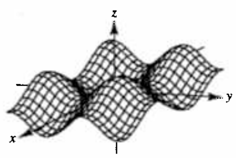
Thus, f is not continuous at (0, 0), whereas g is continuous at (0, 0).

34. $\lim_{(x,y)\to(0,0)} \dfrac{4x^2y^2}{x^2 + y^2} = 0$

Hence, $\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} g(x,y) = 0.$

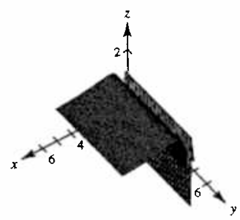
f is continuous at (0, 0), whereas g is not continuous at (0, 0).

35. $\lim_{(x,y)\to(0,0)} \sin x + \sin y = 0$



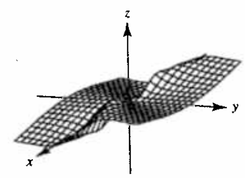
36. $\lim_{(x,y)\to(0,0)} \sin \dfrac{1}{x} + \cos \dfrac{1}{x}$

Does not exist



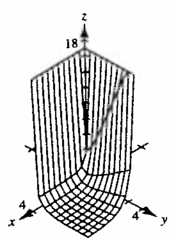
37. $\lim_{(x,y)\to(0,0)} \dfrac{x^2y}{x^4 + 4y^2}$

Does not exist



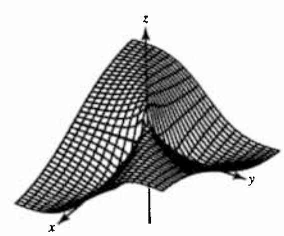
38. $\lim_{(x,y)\to(0,0)} \dfrac{x^2 + y^2}{x^2y}$

Does not exist



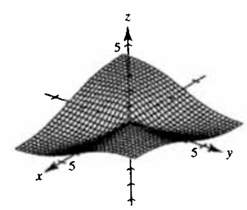
39. $f(x,y) = \dfrac{10xy}{2x^2 + 3y^2}$

The limit does not exist. Use the paths $x = 0$ and $x = y$.



40. $f(x,y) = \dfrac{2xy}{x^2 + y^2 + 1}$

The limit equals 0.



41. $\lim_{(x,y)\to(0,0)} \dfrac{\sin(x^2 + y^2)}{x^2 + y^2} = \lim_{r\to0} \dfrac{\sin r^2}{r^2} = \lim_{r\to0} \dfrac{2r\cos r^2}{2r} = \lim_{r\to0} \cos r^2 = 1$

42. $\lim_{(x,y)\to(0,0)} \dfrac{xy^2}{x^2 + y^2} = \lim_{r\to0} \dfrac{(r\cos\theta)(r^2\sin^2\theta)}{r^2} = \lim_{r\to0} (r\cos\theta\sin^2\theta) = 0$

43. $\lim_{(x,y)\to(0,0)} \dfrac{x^3 + y^3}{x^2 + y^2} = \lim_{r\to0} \dfrac{r^3(\cos^3\theta + \sin^3\theta)}{r^2} = \lim_{r\to0} r(\cos^3\theta + \sin^3\theta) = 0$

44. $\lim_{(x,y)\to(0,0)} \dfrac{x^2y^2}{x^2 + y^2} = \lim_{r\to0} \dfrac{r^4\cos^2\theta\sin^2\theta}{r^2} = \lim_{r\to0} r^2\cos^2\theta\sin^2\theta = 0$

45. $x = r\cos\theta,\ y = r\sin\theta,\ \sqrt{x^2 + y^2} = r,\ x^2 - y^2 = r^2(\cos^2\theta - \sin^2\theta)$

$\lim_{(x,y)\to(0,0)} \dfrac{x^2 - y^2}{\sqrt{x^2 + y^2}} = \lim_{r\to0} \dfrac{r^2(\cos^2\theta - \sin^2\theta)}{r} = \lim_{r\to0} r(\cos^2\theta - \sin^2\theta) = 0$

46. $\sqrt{x^2+y^2} = r$

$$\lim_{(x,y)\to(0,0)} \frac{\sin\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}} = \lim_{r\to 0^+} \frac{\sin(r)}{r} = 1$$

47. $x^2 + y^2 = r^2$

$$\lim_{(x,y)\to(0,0)} (x^2+y^2)\ln(x^2+y^2) = \lim_{r\to 0} r^2 \ln(r^2) = \lim_{r\to 0^+} 2r^2 \ln(r)$$

By L'Hôpital's Rule,

$$\lim_{r\to 0^+} 2r^2 \ln(r) = \lim_{r\to 0^+} \frac{2\ln(r)}{1/r^2} = \lim_{r\to 0^+} \frac{2/r}{-2/r^3} = \lim_{r\to 0^+} (-r^2) = 0$$

48. $x^2 + y^2 = r^2$

$$\lim_{(x,y)\to(0,0)} \frac{1-\cos(x^2+y^2)}{x^2+y^2} = \lim_{r\to 0} \frac{1-\cos(r^2)}{r^2} = 0$$

49. $f(x, y, z) = \dfrac{1}{\sqrt{x^2+y^2+z^2}}$

Continuous except at (0, 0, 0)

50. $f(x, y, z) = \dfrac{z}{x^2+y^2-9}$

Continuous for $x^2 + y^2 \neq 9$

51. $f(x, y, z) = \dfrac{\sin z}{e^x + e^y}$

Continuous everywhere

52. $f(x, y, z) = xy \sin z$

Continuous everywhere

53. For $xy \neq 0$, the function is clearly continuous.

For $xy \neq 0$, let $z = xy$. Then

$$\lim_{z\to 0} \frac{\sin z}{z} = 1$$

implies that f is continuous for all x, y.

54. For $x^2 \neq y^2$, the function is clearly continuous.

For $x^2 \neq y^2$, let $z = x^2 - y^2$. Then

$$\lim_{z\to 0} \frac{\sin(z)}{z} = 1$$

implies that f is continuous for all x, y.

55.

$$\begin{aligned} f(t) &= t^2 \\ g(x, y) &= 3x - 2y \\ f(g(x, y)) &= f(3x - 2y) \\ &= (3x - 2y)^2 \\ &= 9x^2 - 12xy + 4y^2 \end{aligned}$$

Continuous everywhere

56.

$$\begin{aligned} f(t) &= \frac{1}{t} \\ g(x, y) &= x^2 + y^2 \\ f(g(x, y)) &= f(x^2 + y^2) \\ &= \frac{1}{x^2 + y^2} \end{aligned}$$

Continuous except at (0, 0)

57.

$$f(t) = \frac{1}{t}$$

$$g(x, y) = 3x - 2y$$

$$f(g(x, y)) = f(3x - 2y) = \frac{1}{3x - 2y}$$

Continuous for $y \neq \dfrac{3x}{2}$

58.

$$f(t) = \frac{1}{4 - t}$$

$$g(x, y) = x^2 + y^2$$

$$f(g(x, y)) = f(x^2 + y^2) = \frac{1}{4 - x^2 - y^2}$$

Continuous for $x^2 + y^2 \neq 4$

59. $f(x, y) = x^2 - 4y$

(a) $$\lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} = \lim_{\Delta x \to 0} \frac{[(x + \Delta x)^2 - 4y] - (x^2 - 4y)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{2x\Delta x - (\Delta x)^2}{\Delta x} = \lim_{\Delta x \to 0} (2x - \Delta x) = 2x$$

(b) $$\lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} = \lim_{\Delta y \to 0} \frac{[x^2 - 4(y + \Delta y)] - (x^2 - 4y)}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{-4\Delta y}{\Delta y} = \lim_{\Delta y \to 0} (-4) = -4$$

60. $f(x, y) = x^2 + y^2$

(a) $$\lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} = \lim_{\Delta x \to 0} \frac{[(x + \Delta x)^2 + y^2] - (x^2 + y^2)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{2x\Delta x + (\Delta x)^2}{\Delta x} = \lim_{\Delta x \to 0} (2x + \Delta x) = 2x$$

(b) $$\lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} = \lim_{\Delta y \to 0} \frac{[x^2 + (y + \Delta y)^2] - (x^2 + y^2)}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{2y\Delta y + (\Delta y)^2}{\Delta y} = \lim_{\Delta y \to 0} (2y + \Delta y) = 2y$$

61. $f(x, y) = 2x + xy - 3y$

(a) $$\lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} = \lim_{\Delta x \to 0} \frac{[2(x + \Delta x) + (x + \Delta x)y - 3y] - (2x + xy - 3y)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{2\Delta x + \Delta xy}{\Delta x} = \lim_{\Delta x \to 0} (2 + y) = 2 + y$$

(b) $$\lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} = \lim_{\Delta y \to 0} \frac{[2x + x(y + \Delta y) - 3(y + \Delta y)] - (2x + xy - 3y)}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{x\Delta y - 3\Delta y}{\Delta y} = \lim_{\Delta y \to 0} (x - 3) = x - 3$$

62. $f(x, y) = \sqrt{y}(y + 1)$

(a) $$\lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\sqrt{y}(y + 1) - \sqrt{y}(y + 1)}{\Delta x} = 0$$

(b) $$\lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} = \lim_{\Delta y \to 0} \frac{(y + \Delta y)^{3/2} + (y + \Delta y)^{1/2} - (y^{3/2} + y^{1/2})}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{(y + \Delta y)^{3/2} - y^{3/2}}{\Delta y} + \lim_{\Delta y \to 0} \frac{(y + \Delta y)^{1/2} - y^{1/2}}{\Delta y}$$

$$= \frac{3}{2}y^{1/2} + \frac{1}{2}y^{-1/2} \qquad \text{(L'Hôpital's Rule)}$$

$$= \frac{3y + 1}{2\sqrt{y}}$$

63. True. Assuming $f(x, 0)$ exists for $x \neq 0$.

64. False. Let $f(x, y) = \dfrac{xy}{x^2 + y^2}$.

See Exercise 29.

65. False. Let

$$f(x, y) = \begin{cases} \ln(x^2 + y^2), & (x, y) \neq (0, 0) \\ 0, & x = 0, y = 0 \end{cases}.$$

See Exercise 27.

66. True

67. See the definition on page 897.

Show that the value of $\lim_{(x, y)\to(x_0, y_0)} f(x, y)$ is not the same for two different paths to (x_0, y_0).

68. See the definition on page 854.

69. No.

The existence of $f(2, 3)$ has no bearing on the existence of the limit as $(x, y)\to(2, 3)$.

70. No, $f(2, 3)$ can equal any number, or not even be defined.

71. $\lim_{(x, y)\to(0, 0)} \dfrac{x^2 + y^2}{xy}$

(a) Along $y = ax$:

$$\lim_{(x, ax)\to(0, 0)} \frac{x^2 + (ax)^2}{x(ax)} = \lim_{x\to 0} \frac{x^2(1 + a^2)}{ax^2} = \frac{1 + a^2}{a}, \quad a \neq 0$$

If $a = 0$, then $y = 0$ and the limit does not exist.

(b) Along $y = x^2$: $\lim_{(x, x^2)\to(0, 0)} \dfrac{x^2 + (x^2)^2}{x(x^2)} = \lim_{x\to 0} \dfrac{1 + x^2}{x}$

Limit does not exist.

(c) No, the limit does not exist. Different paths result in different limits.

72. $f(x, y) = \dfrac{x^2y}{x^4 + y^2}$

(a) $y = ax$: $f(x, ax) = \dfrac{x^2(ax)}{x^4 + (ax)^2} = \dfrac{ax}{x^2 + a^2}$

If $a \neq 0$, $\lim_{(x, ax)\to(0, 0)} \dfrac{ax}{x^2 + a^2} = 0.$

(b) $y = x^2$: $f(x, x^2) = \dfrac{x^2(x^2)}{x^4 + (x^2)^2} = \dfrac{x^4}{2x^4}$

$$\lim_{(x, x^2)} \frac{x^4}{2x^4} = \frac{1}{2}$$

(c) No, the limit does not exist. f approaches different numbers along different paths.

73. $\lim_{(x, y, z)\to(0, 0, 0)} \dfrac{xyz}{x^2 + y^2 + z^2}$

$$= \lim_{\rho\to 0^+} \frac{(\rho \sin\phi \cos\theta)(\rho \sin\phi \sin\theta)(\rho \cos\phi)}{\rho^2}$$

$$= \lim_{\rho\to 0^+} \rho[\sin^2\phi \cos\theta \sin\theta \cos\phi] = 0$$

74. $\lim_{(x, y, z)\to(0, 0, 0)} \tan^{-1}\left[\dfrac{1}{x^2 + y^2 + z^2}\right]$

$$= \lim_{\rho\to 0^+} \tan^{-1}\left[\frac{1}{\rho^2}\right] = \frac{\pi}{2}$$

75. As $(x, y)\to(0, 1)$, $x^2 + 1\to 1$ and $x^2 + (y - 1)^2\to 0$.

Thus,

$$\lim_{(x, y)\to(0, 1)} \tan^{-1}\left[\frac{x^2 + 1}{x^2 + (y - 1)^2}\right] = \frac{\pi}{2}.$$

76. $\lim_{(x, y)\to(0, 0)} f(x, y) = \lim_{r\to 0} (r\cos\theta)(r\sin\theta)\dfrac{r^2\cos^2\theta - r^2\sin^2\theta}{r^2}$

$$= \lim_{r\to 0} r^2[\cos\theta \sin\theta(\cos^2\theta - \sin^2\theta)] = 0$$

Hence, define $f(0, 0) = 0$.

77. Since $\lim_{(x, y)\to(a, b)} f(x, y) = L_1$, then for $\varepsilon/2 > 0$, there corresponds $\delta_1 > 0$ such that $|f(x, y) - L_1| < \varepsilon/2$ whenever

$$0 < \sqrt{(x - a)^2 + (y - b)^2} < \delta_1.$$

Since $\lim_{(x, y)\to(a, b)} g(x, y) = L_2$, then for $\varepsilon/2 > 0$, there corresponds $\delta_2 > 0$ such that $|g(x, y) - L_2| < \varepsilon/2$ whenever

$$0 < \sqrt{(x - a)^2 + (y - b)^2} < \delta_2.$$

Let δ be the smaller of δ_1 and δ_2. By the triangle inequality, whenever $\sqrt{(x - a)^2 + (y - b)^2} < \delta$, we have

$$|f(x, y) + g(x, y) - (L_1 + L_2)| = |(f(x, y) - L_1) + (g(x, y) - L_2)| \leq |f(x, y) - L_1| + |g(x, y) - L_2| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore, $\lim_{(x, y)\to(a, b)} [f(x, y) + g(x, y)] = L_1 + L_2$.

78. Given that $f(x, y)$ is continuous, then $\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b) < 0$, which means that for each $\varepsilon > 0$, there corresponds a $\delta > 0$ such that $|f(x, y) - f(a, b)| < \varepsilon$ whenever

$$0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta.$$

Let $\varepsilon = |f(a, b)|/2$, then $f(x, y) < 0$ for every point in the corresponding δ neighborhood since

$$|f(x, y) - f(a, b)| < \frac{|f(a, b)|}{2} \Rightarrow -\frac{|f(a, b)|}{2} < f(x, y) - f(a, b) < \frac{|f(a, b)|}{2}$$

$$\Rightarrow \frac{3}{2}f(a, b) < f(x, y) < \frac{1}{2}f(a, b) < 0.$$

Section 13.3 Partial Derivatives

1. $f_x(4, 1) < 0$

2. $f_y(-1, -2) < 0$

3. $f_y(4, 1) > 0$

4. $f_x(-1, -1) = 0$

5. $f(x, y) = 2x - 3y + 5$

$f_x(x, y) = 2$

$f_y(x, y) = -3$

6. $f(x, y) = x^2 - 3y^2 + 7$

$f_x(x, y) = 2x$

$f_y(x, y) = -6y$

7. $z = x\sqrt{y}$

$\frac{\partial z}{\partial x} = \sqrt{y}$

$\frac{\partial z}{\partial y} = \frac{x}{2\sqrt{y}}$

8. $z = 2y^2\sqrt{x}$

$\frac{\partial z}{\partial x} = \frac{y^2}{\sqrt{x}}$

$\frac{\partial z}{\partial y} = 4y\sqrt{x}$

9. $z = x^2 - 5xy + 3y^2$

$\frac{\partial z}{\partial x} = 2x - 5y$

$\frac{\partial z}{\partial y} = -5x + 6y$

10. $z = y^3 - 4xy^2 - 1$

$\frac{\partial z}{\partial x} = -4y^2$

$\frac{\partial z}{\partial y} = 3y^2 - 8xy$

11. $z = x^2e^{2y}$

$\frac{\partial z}{\partial x} = 2xe^{2y}$

$\frac{\partial z}{\partial y} = 2x^2e^{2y}$

12. $z = xe^{x/y}$

$\frac{\partial z}{\partial x} = \frac{x}{y}e^{x/y} + e^{x/y} = e^{x/y}\left(\frac{x}{y} + 1\right)$

$\frac{\partial z}{\partial y} = xe^{x/y}\left(-\frac{x}{y^2}\right) = -\frac{x^2}{y^2}e^{x/y}$

13. $z = \ln(x^2 + y^2)$

$\frac{\partial z}{\partial x} = \frac{2x}{x^2 + y^2}$

$\frac{\partial z}{\partial y} = \frac{2y}{x^2 + y^2}$

14. $z = \ln\sqrt{xy} = \frac{1}{2}\ln(xy)$

$\frac{\partial z}{\partial x} = \frac{1}{2}\frac{y}{xy} = \frac{1}{2x}$

$\frac{\partial z}{\partial y} = \frac{1}{2}\frac{x}{xy} = \frac{1}{2y}$

15. $z = \ln\frac{x+y}{x-y} = \ln(x + y) - \ln(x - y)$

$\frac{\partial z}{\partial x} = \frac{1}{x+y} - \frac{1}{x-y} = -\frac{2y}{x^2 - y^2}$

$\frac{\partial z}{\partial y} = \frac{1}{x+y} + \frac{1}{x-y} = \frac{2x}{x^2 - y^2}$

16. $z = \ln(x^2 - y^2)$

$\frac{\partial z}{\partial x} = \frac{1}{x^2 - y^2}(2x) = \frac{2x}{x^2 - y^2}$

$\frac{\partial z}{\partial y} = \frac{-2y}{x^2 - y^2}$

17. $z = \frac{x^2}{2y} + \frac{4y^2}{x}$

$\frac{\partial z}{\partial x} = \frac{2x}{2y} - \frac{4y^2}{x^2} = \frac{x^3 - 4y^3}{x^2y}$

$\frac{\partial z}{\partial y} = -\frac{x^2}{2y^2} + \frac{8y}{x} = \frac{-x^3 + 16y^3}{2xy^2}$

18. $f(x, y) = \frac{xy}{x^2 + y^2}$

$f_x(x, y) = \frac{(x^2 + y^2)(y) - (xy)(2x)}{(x^2 + y^2)^2} = \frac{y^3 - x^2y}{(x^2 + y^2)^2}$

$f_y(x, y) = \frac{(x^2 + y^2)(x) - (xy)(2y)}{(x^2 + y^2)^2} = \frac{x^3 - xy^2}{(x^2 + y^2)^2}$

19. $h(x, y) = e^{-(x^2+y^2)}$

$h_x(x, y) = -2xe^{-(x^2+y^2)}$

$h_y(x, y) = -2ye^{-(x^2+y^2)}$

20. $g(x, y) = \ln\sqrt{x^2 + y^2} = \frac{1}{2}\ln(x^2 + y^2)$

$g_x(x, y) = \frac{1}{2}\frac{2x}{x^2 + y^2} = \frac{x}{x^2 + y^2}$

$g_y(x, y) = \frac{1}{2}\frac{2y}{x^2 + y^2} = \frac{y}{x^2 + y^2}$

21. $f(x, y) = \sqrt{x^2 + y^2}$

$f_x(x, y) = \frac{1}{2}(x^2 + y^2)^{-1/2}(2x) = \frac{x}{\sqrt{x^2 + y^2}}$

$f_y(x, y) = \frac{1}{2}(x^2 + y^2)^{-1/2}(2y) = \frac{y}{\sqrt{x^2 + y^2}}$

22. $f(x, y) = \sqrt{2x + y^3}$

$\frac{\partial f}{\partial x} = \frac{1}{2}(2x + y^3)^{-1/2}(2) = \frac{1}{\sqrt{2x + y^3}}$

$\frac{\partial f}{\partial y} = \frac{1}{2}(2x + y^3)^{-1/2}(3y^2) = \frac{3y^2}{2\sqrt{2x + y^3}}$

23. $z = \tan(2x - y)$

$\frac{\partial z}{\partial x} = 2\sec^2(2x - y)$

$\frac{\partial z}{\partial y} = -\sec^2(2x - y)$

24. $z = \sin 3x \cos 3y$

$\frac{\partial z}{\partial x} = 3\cos 3x \cos 3y$

$\frac{\partial z}{\partial y} = -3\sin 3x \sin 3y$

25. $z = e^y \sin xy$

$\frac{\partial z}{\partial x} = ye^y \cos xy$

$\frac{\partial z}{\partial y} = e^y \sin xy + xe^y \cos xy$

$= e^y(x\cos xy + \sin xy)$

26. $z = \cos(x^2 + y^2)$

$\frac{\partial z}{\partial x} = -2x\sin(x^2 + y^2)$

$\frac{\partial z}{\partial y} = -2y\sin(x^2 + y^2)$

27. $f(x, y) = \int_x^y (t^2 - 1)\,dt$

$= \left[\frac{t^3}{3} - t\right]_x^y = \left(\frac{y^3}{3} - y\right) - \left(\frac{x^3}{3} - x\right)$

$f_x(x, y) = -x^2 + 1 = 1 - x^2$

$f_y(x, y) = y^2 - 1$

[You could also use the Second Fundamental Theorem of Calculus.]

28. $f(x, y) = \int_x^y (2t + 1)\,dt + \int_y^x (2t - 1)\,dt$

$= \int_x^y (2t + 1)\,dt - \int_x^y (2t - 1)\,dt$

$= \int_x^y 2\,dt = \left[2t\right]_x^y = 2y - 2x$

$f_x(x, y) = -2$

$f_y(x, y) = 2$

29. $f(x, y) = 2x + 3y$

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} = \lim_{\Delta x \to 0} \frac{2(x + \Delta x) + 3y - 2x - 3y}{\Delta x} = \lim_{\Delta x \to 0} \frac{2\Delta x}{\Delta x} = 2$$

$$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} = \lim_{\Delta y \to 0} \frac{2x + 3(y + \Delta y) - 2x - 3y}{\Delta y} = \lim_{\Delta y \to 0} \frac{3\Delta y}{\Delta y} = 3$$

30. $f(x, y) = x^2 - 2xy + y^2 = (x - y)^2$

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(x + \Delta x)^2 - 2(x + \Delta x)y + y^2 - x^2 + 2xy - y^2}{\Delta x} = \lim_{\Delta x \to 0} (2x + \Delta x - 2y) = 2(x - y)$$

$$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{x^2 - 2x(y + \Delta y) + (y + \Delta y)^2 - x^2 + 2xy - y^2}{\Delta y} = \lim_{\Delta y \to 0} (-2x + 2y + \Delta y) = 2(y - x)$$

31. $f(x, y) = \sqrt{x + y}$

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\sqrt{x + \Delta x + y} - \sqrt{x + y}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\left(\sqrt{x + \Delta x + y} - \sqrt{x + y}\right)\left(\sqrt{x + \Delta x + y} + \sqrt{x + y}\right)}{\Delta x\left(\sqrt{x + \Delta x + y} + \sqrt{x + y}\right)}$$

$$= \lim_{\Delta x \to 0} \frac{1}{\sqrt{x + \Delta x + y} + \sqrt{x + y}} = \frac{1}{2\sqrt{x + y}}$$

$$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} = \lim_{\Delta y \to 0} \frac{\sqrt{x + y + \Delta y} - \sqrt{x + y}}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{\left(\sqrt{x + y + \Delta y} - \sqrt{x + y}\right)\left(\sqrt{x + y + \Delta y} + \sqrt{x + y}\right)}{\Delta y\left(\sqrt{x + y + \Delta y} + \sqrt{x + y}\right)}$$

$$= \lim_{\Delta y \to 0} \frac{1}{\sqrt{x + y + \Delta y} + \sqrt{x + y}} = \frac{1}{2\sqrt{x + y}}$$

32. $f(x, y) = \dfrac{1}{x + y}$

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\dfrac{1}{x + \Delta x + y} - \dfrac{1}{x + y}}{\Delta x} = \lim_{\Delta x \to 0} \frac{-1}{(x + \Delta x + y)(x + y)} = \frac{-1}{(x + y)^2}$$

$$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} = \lim_{\Delta y \to 0} \frac{\dfrac{1}{x + y + \Delta y} - \dfrac{1}{x + y}}{\Delta y} = \lim_{\Delta y \to 0} \frac{-1}{(x + y + \Delta y)(x + y)} = \frac{-1}{(x + y)^2}$$

33. $f(x, y) = \arctan \dfrac{y}{x}$

$$f_x(x, y) = \frac{1}{1 + (y^2/x^2)}\left(-\frac{y}{x^2}\right) = \frac{-y}{x^2 + y^2}$$

At $(2, -2)$: $f_x(2, -2) = \dfrac{1}{4}$

$$f_y(x, y) = \frac{1}{1 + (y^2/x^2)}\left(\frac{1}{x}\right) = \frac{x}{x^2 + y^2}$$

At $(2, -2)$: $f_y(2, -2) = \dfrac{1}{4}$

34. $f(x, y) = \arccos(xy)$

$$f_x(x, y) = \frac{-y}{\sqrt{1 - x^2y^2}}$$

At $(1, 1)$, f_x is undefined.

$$f_y(x, y) = \frac{-x}{\sqrt{1 - x^2y^2}}$$

At $(1, 1)$, f_y is undefined.

35. $f(x, y) = \dfrac{xy}{x - y}$

$f_x(x, y) = \dfrac{y(x - y) - xy}{(x - y)^2} = \dfrac{-y^2}{(x - y)^2}$

At $(2, -2)$: $f_x(2, -2) = -\dfrac{1}{4}$

$f_y(x, y) = \dfrac{x(x - y) + xy}{(x - y)^2} = \dfrac{x^2}{(x - y)^2}$

At $(2, -2)$: $f_y(2, -2) = \dfrac{1}{4}$

36. $f(x, y) = \dfrac{6xy}{\sqrt{4x^2 + 5y^2}}$

$f_x(x, y) = \dfrac{30y^3}{(4x^2 + 5y^2)^{3/2}}$

At $(1, 1)$, $f_x(1, 1) = \dfrac{30}{27} = \dfrac{10}{9}$

$f_y(x, y) = \dfrac{24x^3}{(4x^2 + 5y^2)^{3/2}}$

At $(1, 1)$, $f_y(1, 1) = \dfrac{8}{9}$

37. $g(x, y) = 4 - x^2 - y^2$

$g_x(x, y) = -2x$

At $(1, 1)$: $g_x(1, 1) = -2$

$g_y(x, y) = -2y$

At $(1, 1)$: $g_y(1, 1) = -2$

38. $h(x, y) = x^2 - y^2$

$h_x(x, y) = 2x$

At $(-2, 1)$: $h_x(-2, 1) = -4$

$h_y(x, y) = -2y$

At $(-2, 1)$: $h_y(-2, 1) = -2$

39. $z = e^{-x} \cos y$

$\dfrac{\partial z}{\partial x} = -e^{-x} \cos y$

At $(0, 0)$: $\dfrac{\partial z}{\partial x} = -1$

$\dfrac{\partial z}{\partial y} = -e^{-x} \sin y$

At $(0, 0)$: $\dfrac{\partial z}{\partial y} = 0$

40. $z = \cos(2x - y)$

$\dfrac{\partial z}{\partial x} = -2 \sin(2x - y)$

At $\left(\dfrac{\pi}{4}, \dfrac{\pi}{3}\right)$, $\dfrac{\partial z}{\partial x} = -2 \sin\left(\dfrac{\pi}{6}\right) = -1$

$\dfrac{\partial z}{\partial y} = -\sin(2x - y)(-1) = \sin(2x - y)$

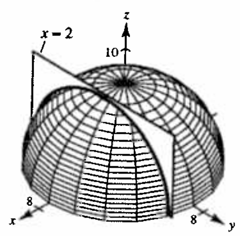
At $\left(\dfrac{\pi}{4}, \dfrac{\pi}{3}\right)$, $\dfrac{\partial z}{\partial y} = \sin\left(\dfrac{\pi}{6}\right) = \dfrac{1}{2}$

41. $z = \sqrt{49 - x^2 - y^2}$, $x = 2$, $(2, 3, 6)$

Intersecting curve: $z = \sqrt{45 - y^2}$

$\dfrac{\partial z}{\partial y} = \dfrac{-y}{\sqrt{45 - y^2}}$

At $(2, 3, 6)$: $\dfrac{\partial z}{\partial y} = \dfrac{-3}{\sqrt{45 - 9}} = -\dfrac{1}{2}$

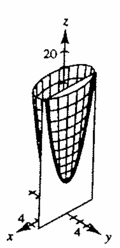


42. $z = x^2 + 4y^2$, $y = 1$, $(2, 1, 8)$

Intersecting curve: $z = x^2 + 4$

$\dfrac{\partial z}{\partial x} = 2x$

At $(2, 1, 8)$: $\dfrac{\partial z}{\partial x} = 2(2) = 4$

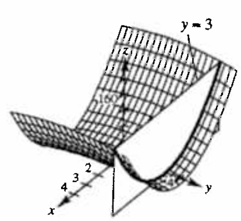


43. $z = 9x^2 - y^2$, $y = 3$, $(1, 3, 0)$

Intersecting curve: $z = 9x^2 - 9$

$\dfrac{\partial z}{\partial x} = 18x$

At $(1, 3, 0)$: $\dfrac{\partial z}{\partial x} = 18(1) = 18$

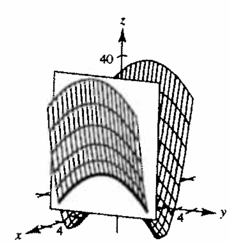


44. $z = 9x^2 - y^2$, $x = 1$, $(1, 3, 0)$

Intersecting curve: $z = 9 - y^2$

$\dfrac{\partial z}{\partial y} = -2y$

At $(1, 3, 0)$: $\dfrac{\partial z}{\partial y} = -2(3) = -6$



45. $f_x(x, y) = 2x + 4y - 4$, $f_y(x, y) = 4x + 2y + 16$

$f_x = f_y = 0$: $2x + 4y = 4$

$4x + 2y = -16$

Solving for x and y,

$x = -6$ and $y = 4$.

46. $f_x(x, y) = 9x^2 - 12y$, $f_y(x, y) = -12x + 3y^2$

$f_x = f_y = 0$: $9x^2 - 12y = 0 \Rightarrow 3x^2 = 4y$

$3y^2 - 12x = 0 \Rightarrow y^2 = 4x$

Solving for x in the second equation, $x = y^2/4$, you obtain $3(y^2/4)^2 = 4y$.

$$3y^4 = 64y \Rightarrow y = 0 \text{ or } y = \frac{4}{3^{1/3}}$$

$$\Rightarrow x = 0 \text{ or } x = \frac{1}{4}\left(\frac{16}{3^{2/3}}\right)$$

Points: $(0, 0)$, $\left(\frac{4}{3^{2/3}}, \frac{4}{3^{1/3}}\right)$

47. $f_x(x, y) = -\frac{1}{x^2} + y$, $f_y(x, y) = -\frac{1}{y^2} + x$

$f_x = f_y = 0$: $-\frac{1}{x^2} + y = 0$ and $-\frac{1}{y^2} + x = 0$

$y = \frac{1}{x^2}$ and $x = \frac{1}{y^2}$

$y = y^4 \Rightarrow y = 1 = x$

Points: $(1, 1)$

48. $f_x(x, y) = \frac{2x}{x^2 + y^2 + 1} = 0 \Rightarrow x = 0$

$f_y(x, y) = \frac{2y}{x^2 + y^2 + 1} = 0 \Rightarrow y = 0$

Points: $(0, 0)$

49. (a) The graph is that of f_y.

(b) The graph is that of f_x.

50. (a) The graph is that of f_x.

(b) The graph is that of f_y.

51. $w = \sqrt{x^2 + y^2 + z^2}$

$$\frac{\partial w}{\partial x} = \frac{x}{\sqrt{x^2 + y^2 + z^2}}$$

$$\frac{\partial w}{\partial y} = \frac{y}{\sqrt{x^2 + y^2 + z^2}}$$

$$\frac{\partial w}{\partial z} = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

52. $w = \frac{3xz}{x + y}$

$$\frac{\partial w}{\partial x} = \frac{(x + y)(3z) - 3xz}{(x + y)^2} = \frac{3yz}{(x + y)^2}$$

$$\frac{\partial w}{\partial y} = \frac{-3xz}{(x + y)^2}$$

$$\frac{\partial w}{\partial z} = \frac{3x}{x + y}$$

53. $F(x, y, z) = \ln \sqrt{x^2 + y^2 + z^2}$

$$= \frac{1}{2}\ln(x^2 + y^2 + z^2)$$

$$F_x(x, y, z) = \frac{x}{x^2 + y^2 + z^2}$$

$$F_y(x, y, z) = \frac{y}{x^2 + y^2 + z^2}$$

$$F_z(x, y, z) = \frac{z}{x^2 + y^2 + z^2}$$

54. $G(x, y, z) = \frac{1}{\sqrt{1 - x^2 - y^2 - z^2}}$

$$G_x(x, y, z) = \frac{x}{(1 - x^2 - y^2 - z^2)^{3/2}}$$

$$G_y(x, y, z) = \frac{y}{(1 - x^2 - y^2 - z^2)^{3/2}}$$

$$G_z(x, y, z) = \frac{z}{(1 - x^2 - y^2 - z^2)^{3/2}}$$

55. $H(x, y, z) = \sin(x + 2y + 3z)$

$H_x(x, y, z) = \cos(x + 2y + 3z)$

$H_y(x, y, z) = 2\cos(x + 2y + 3z)$

$H_z(x, y, z) = 3\cos(x + 2y + 3z)$

56. $f(x, y, z) = 3x^2y - 5xyz + 10yz^2$

$f_x(x, y, z) = 6xy - 5yz$

$f_y(x, y, z) = 3x^2 - 5xz + 10z^2$

$f_z(x, y, z) = -5xy + 20yz$

57. $\sqrt{3x^2 + y^2 - 2z^2}$

$$f_x(x, y, z) = \frac{6x}{2\sqrt{3x^2 + y^2 - 2z^2}} = \frac{3x}{\sqrt{3x^2 + y^2 - 2z^2}}$$

$$f_x(1, -2, 1) = \frac{6}{2\sqrt{3 + 4 - 2}} = \frac{3}{\sqrt{5}} = \frac{3\sqrt{5}}{5}$$

$$f_y(x, y, z) = \frac{2y}{2\sqrt{3x^2 + y^2 - 2z^2}} = \frac{y}{\sqrt{3x^2 + y^2 - 2z^2}}$$

$$f_y(1, -2, 1) = \frac{-2}{\sqrt{5}} = \frac{-2\sqrt{5}}{5}$$

$$f_z(x, y, z) = \frac{-4z}{2\sqrt{3x^2 + y^2 - 2z^2}} = \frac{-2z}{\sqrt{3x^2 + y^2 - 2z^2}}$$

$$f_z(1, -2, 1) = \frac{-2}{\sqrt{5}} = \frac{-2\sqrt{5}}{5}$$

58. $f(x, y, z) = \dfrac{xy}{x + y + z}$

$$f_x(x, y, z) = \frac{(x + y + z)y - xy}{(x + y + z)^2} = \frac{y^2 + yz}{(x + y + z)^2}$$

$$f_x(3, 1, -1) = \frac{1 - 1}{3^2} = 0$$

$$f_y(x, y, z) = \frac{(x + y + z)x - xy}{(x + y + z)^2} = \frac{x^2 + xz}{(x + y + z)^2}$$

$$f_y(3, 1, -1) = \frac{9 - 3}{3^2} = \frac{2}{3}$$

$$f_z(x, y, z) = \frac{(x + y + z)(0) - xy}{(x + y + z)^2} = \frac{-xy}{(x + y + z)^2}$$

$$f_z(3, 1, -1) = \frac{-3}{9} = \frac{-1}{3}$$

59. $f(x, y, z) = z\sin(x + y)$

$f_x(x, y, z) = z\cos(x + y)$

$$f_x\left(0, \frac{\pi}{2}, -4\right) = -4\cos\frac{\pi}{2} = 0$$

$f_y(x, y, z) = z\cos(x + y)$

$$f_y\left(0, \frac{\pi}{2}, -4\right) = -4\cos\frac{\pi}{2} = 0$$

$f_z(x, y, z) = \sin(x + y)$

$$f_z\left(0, \frac{\pi}{2}, -4\right) = \sin\frac{\pi}{2} = 1$$

60. $f(x, y, z) = x^2y^3 + 2xyz - 3yz$

$f_x(x, y, z) = 2xy^3 + 2yz$

$f_x(-2, 1, 2) = -4 + 4 = 0$

$f_y(x, y, z) = 3x^2y^2 + 2xz - 3z$

$f_y(-2, 1, 2) = 12 - 8 - 6 = -2$

$f_z(x, y, z) = 2xy - 3y$

$f_z(-2, 1, 2) = -4 - 3 = -7$

61. $z = x^2 - 2xy + 3y^2$

$$\frac{\partial z}{\partial x} = 2x - 2y$$

$$\frac{\partial^2 z}{\partial x^2} = 2$$

$$\frac{\partial^2 z}{\partial y \partial x} = -2$$

$$\frac{\partial z}{\partial y} = -2x + 6y$$

$$\frac{\partial^2 z}{\partial y^2} = 6$$

$$\frac{\partial^2 z}{\partial x \partial y} = -2$$

62. $z = x^4 - 3x^2y^2 + y^4$

$$\frac{\partial z}{\partial x} = 4x^3 - 6xy^2$$

$$\frac{\partial^2 z}{\partial x^2} = 12x^2 - 6y^2$$

$$\frac{\partial^2 z}{\partial y \partial x} = -12xy$$

$$\frac{\partial z}{\partial y} = -6x^2y + 4y^3$$

$$\frac{\partial^2 z}{\partial y^2} = -6x^2 + 12y^2$$

$$\frac{\partial^2 z}{\partial x \partial y} = -12xy$$

63. $z = \sqrt{x^2 + y^2}$

$$\frac{\partial z}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{y^2}{(x^2 + y^2)^{3/2}}$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{-xy}{(x^2 + y^2)^{3/2}}$$

$$\frac{\partial z}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{x^2}{(x^2 + y^2)^{3/2}}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{-xy}{(x^2 + y^2)^{3/2}}$$

64. $z = \ln(x - y)$

$$\frac{\partial z}{\partial x} = \frac{1}{x - y}$$

$$\frac{\partial^2 z}{\partial x^2} = -\frac{1}{(x - y)^2}$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{1}{(x - y)^2}$$

$$\frac{\partial z}{\partial y} = \frac{-1}{x - y} = \frac{1}{y - x}$$

$$\frac{\partial^2 z}{\partial y^2} = -\frac{1}{(x - y)^2}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{1}{(x - y)^2}$$

Therefore, $\dfrac{\partial^2 z}{\partial y \partial x} = \dfrac{\partial^2 z}{\partial x \partial y}$.

65. $z = e^x \tan y$

$$\frac{\partial z}{\partial x} = e^x \tan y$$

$$\frac{\partial^2 z}{\partial x^2} = e^x \tan y$$

$$\frac{\partial^2 z}{\partial y \partial x} = e^x \sec^2 y$$

$$\frac{\partial z}{\partial y} = e^x \sec^2 y$$

$$\frac{\partial^2 z}{\partial y^2} = 2e^x \sec^2 y \tan y$$

$$\frac{\partial^2 z}{\partial x \partial y} = e^x \sec^2 y$$

66. $z = 2xe^y - 3ye^{-x}$

$$\frac{\partial z}{\partial x} = 2e^y + 3ye^{-x}$$

$$\frac{\partial^2 z}{\partial x^2} = -3ye^{-x}$$

$$\frac{\partial^2 z}{\partial y \partial x} = 2e^y + 3ye^{-x}$$

$$\frac{\partial z}{\partial y} = 2xe^y - 3e^{-x}$$

$$\frac{\partial^2 z}{\partial y^2} = 2xe^y$$

$$\frac{\partial^2 z}{\partial x \partial y} = 2e^y + 3e^{-x}$$

67. $z = \arctan \dfrac{y}{x}$

$$\frac{\partial z}{\partial x} = \frac{1}{1 + (y^2/x^2)}\left(-\frac{y}{x^2}\right) = \frac{-y}{x^2 + y^2}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{2xy}{(x^2 + y^2)^2}$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{-(x^2 + y^2) + y(2y)}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

$$\frac{\partial z}{\partial y} = \frac{1}{1 + (y^2/x^2)}\left(\frac{1}{x}\right) = \frac{x}{x^2 + y^2}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{-2xy}{(x^2 + y^2)^2}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{(x^2 + y^2) - x(2x)}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

68. $z = \sin(x - 2y)$

$$\frac{\partial z}{\partial x} = \cos(x - 2y)$$

$$\frac{\partial^2 z}{\partial x^2} = -\sin(x - 2y)$$

$$\frac{\partial^2 z}{\partial y \partial x} = 2\sin(x - 2y)$$

$$\frac{\partial z}{\partial y} = -2\cos(x - 2y)$$

$$\frac{\partial^2 z}{\partial y^2} = -4\sin(x - 2y)$$

$$\frac{\partial^2 z}{\partial x \partial y} = 2\sin(x - 2y)$$

69. $z = x \sec y$

$$\frac{\partial z}{\partial x} = \sec y$$

$$\frac{\partial^2 z}{\partial x^2} = 0$$

$$\frac{\partial^2 z}{\partial y \partial x} = \sec y \tan y$$

$$\frac{\partial z}{\partial y} = x \sec y \tan y$$

$$\frac{\partial^2 z}{\partial y^2} = x \sec y(\sec^2 y + \tan^2 y)$$

$$\frac{\partial^2 z}{\partial x \partial y} = \sec y \tan y$$

Therefore, $\dfrac{\partial^2 z}{\partial y \partial x} = \dfrac{\partial^2 z}{\partial x \partial y}$.

There are no points for which $z_x = 0 = z_y$, because

$$\frac{\partial z}{\partial x} = \sec y \neq 0.$$

70. $z = \sqrt{9 - x^2 - y^2}$

$$\frac{\partial z}{\partial x} = \frac{-x}{\sqrt{9 - x^2 - y^2}}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{y^2 - 9}{(9 - x^2 - y^2)^{3/2}}$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{-xy}{(9 - x^2 - y^2)^{3/2}}$$

$$\frac{\partial z}{\partial y} = \frac{-y}{\sqrt{9 - x^2 - y^2}}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{x^2 - 9}{(9 - x^2 - y^2)^{3/2}}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{-xy}{(9 - x^2 - y^2)^{3/2}}$$

Therefore, $\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y}$.

$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} = 0$ if $x = y = 0$

71. $z = \ln\left(\frac{x}{x^2 + y^2}\right) = \ln x - \ln(x^2 + y^2)$

$$\frac{\partial z}{\partial x} = \frac{1}{x} - \frac{2x}{x^2 + y^2} = \frac{y^2 - x^2}{x(x^2 + y^2)}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{x^4 - 4x^2y^2 - y^4}{x^2(x^2 + y^2)^2}$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{4xy}{(x^2 + y^2)^2}$$

$$\frac{\partial z}{\partial y} = -\frac{2y}{x^2 + y^2}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{2(y^2 - x^2)}{(x^2 + y^2)^2}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{4xy}{(x^2 + y^2)^2}$$

There are no points for which $z_x = z_y = 0$.

72. $z = \frac{xy}{x - y}$

$$\frac{\partial z}{\partial x} = \frac{y(x - y) - xy}{(x - y)^2} = \frac{-y^2}{(x - y)^2}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{2y^2}{(x - y)^3}$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{(x - y)^2(-2y) + y^2(2)(x - y)(-1)}{(x - y)^4} = \frac{-2xy}{(x - y)^3}$$

$$\frac{\partial z}{\partial y} = -\frac{x(x - y) + xy}{(x - y)^2} = \frac{x^2}{(x - y)^2}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{2x^2}{(x - y)^3}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{(x - y)^2(2x) - x^2(2)(x - y)}{(x - y)^4} = \frac{-2xy}{(x - y)^3}$$

There are no points for which $z_x = z_y = 0$.

73. $f(x, y, z) = xyz$

$f_x(x, y, z) = yz$

$f_y(x, y, z) = xz$

$f_{yy}(x, y, z) = 0$

$f_{xy}(x, y, z) = z$

$f_{yx}(x, y, z) = z$

$f_{yyx}(x, y, z) = 0$

$f_{xyy}(x, y, z) = 0$

$f_{yxy}(x, y, z) = 0$

Therefore, $f_{xyy} = f_{yxy} = f_{yyx} = 0$.

74. $f(x, y, z) = x^2 - 3xy + 4yz + z^3$

$f_x(x, y, z) = 2x - 3y$

$f_y(x, y, z) = -3x + 4z$

$f_{yy}(x, y, z) = 0$

$f_{xy}(x, y, z) = -3$

$f_{yx}(x, y, z) = -3$

$f_{yyx}(x, y, z) = 0$

$f_{xyy}(x, y, z) = 0$

$f_{yxy}(x, y, z) = 0$

Therefore, $f_{xyy} = f_{yxy} = f_{yyx} = 0$.

75. $f(x, y, z) = e^{-x} \sin yz$

$f_x(x, y, z) = -e^{-x} \sin yz$

$f_y(x, y, z) = ze^{-x} \cos yz$

$f_{yy}(x, y, z) = -z^2e^{-x} \sin yz$

$f_{xy}(x, y, z) = -ze^{-x} \cos yz$

$f_{yx}(x, y, z) = -ze^{-x} \cos yz$

$f_{yyx}(x, y, z) = z^2e^{-x} \sin yz$

$f_{xyy}(x, y, z) = z^2e^{-x} \sin yz$

$f_{yxy}(x, y, z) = z^2e^{-x} \sin yz$

Therefore, $f_{xyy} = f_{yxy} = f_{yyx}$.

76. $f(x, y, z) = \dfrac{2z}{x + y}$

$f_x(x, y, z) = \dfrac{-2z}{(x + y)^2}$

$f_y(x, y, z) = \dfrac{-2z}{(x + y)^2}$

$f_{yy}(x, y, z) = \dfrac{4z}{(x + y)^3}$

$f_{xy}(x, y, z) = \dfrac{4z}{(x + y)^3}$

$f_{yx}(x, y, z) = \dfrac{4z}{(x + y)^3}$

$f_{yyx}(x, y, z) = \dfrac{-12z}{(x + y)^4}$

$f_{xyy}(x, y, z) = \dfrac{-12z}{(x + y)^4}$

$f_{yxy}(x, y, z) = \dfrac{-12z}{(x + y)^4}$

77. $z = 5xy$

$\dfrac{\partial z}{\partial x} = 5y$

$\dfrac{\partial^2 z}{\partial x^2} = 0$

$\dfrac{\partial z}{\partial y} = 5x$

$\dfrac{\partial^2 z}{\partial y^2} = 0$

Therefore, $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = 0 + 0 = 0.$

78. $z = \sin x\left(\dfrac{e^y - e^{-y}}{2}\right)$

$\dfrac{\partial z}{\partial x} = \cos x\left(\dfrac{e^y - e^{-y}}{2}\right)$

$\dfrac{\partial^2 z}{\partial x^2} = -\sin x\left(\dfrac{e^y - e^{-y}}{2}\right)$

$\dfrac{\partial z}{\partial y} = \sin x\left(\dfrac{e^y + e^{-y}}{2}\right)$

$\dfrac{\partial^2 z}{\partial y^2} = \sin x\left(\dfrac{e^y - e^{-y}}{2}\right)$

Therefore,

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = -\sin x\left(\frac{e^y - e^{-y}}{2}\right) + \sin x\left(\frac{e^y - e^{-y}}{2}\right) = 0.$$

79. $z = e^x \sin y$

$\dfrac{\partial z}{\partial x} = e^x \sin y$

$\dfrac{\partial^2 z}{\partial x^2} = e^x \sin y$

$\dfrac{\partial z}{\partial y} = e^x \cos y$

$\dfrac{\partial^2 z}{\partial y^2} = -e^x \sin y$

Therefore, $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = e^x \sin y - e^x \sin y = 0.$

80. $z = \arctan \dfrac{y}{x}$

From Exercise 67, we have

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{2xy}{(x^2 + y^2)^2} + \frac{-2xy}{(x^2 + y^2)^2} = 0.$$

81. $z = \sin(x - ct)$

$\dfrac{\partial z}{\partial t} = -c\cos(x - ct)$

$\dfrac{\partial^2 z}{\partial t^2} = -c^2 \sin(x - ct)$

$\dfrac{\partial z}{\partial x} = \cos(x - ct)$

$\dfrac{\partial^2 z}{\partial x^2} = -\sin(x - ct)$

Therefore, $\dfrac{\partial^2 z}{\partial t^2} = c^2\left(\dfrac{\partial^2 z}{\partial x^2}\right).$

82. $z = \cos(4x + 4ct)$

$\dfrac{\partial z}{\partial t} = -4c\sin(4x + 4ct)$

$\dfrac{\partial^2 z}{\partial t^2} = -16c^2 \cos(4x + 4ct)$

$\dfrac{\partial z}{\partial x} = -4\sin(4x + 4ct)$

$\dfrac{\partial^2 z}{\partial x^2} = -16\cos(4x + 4ct)$

$$\frac{\partial^2 z}{\partial t^2} = c^2(-16\cos(4x + 4ct)) = c^2\left(\frac{\partial^2 z}{2x^2}\right)$$

83. $z = \ln(x + ct)$

$\dfrac{\partial z}{\partial t} = \dfrac{c}{x + ct}$

$\dfrac{\partial^2 z}{\partial t^2} = \dfrac{-c^2}{(x + ct)^2}$

$\dfrac{\partial z}{\partial x} = \dfrac{1}{x + ct}$

$\dfrac{\partial^2 z}{\partial x^2} = \dfrac{-1}{(x + ct)^2}$

$$\frac{\partial^2 z}{\partial t^2} = \frac{-c^2}{(x + ct)^2} = c^2\left(\frac{\partial^2 z}{\partial x^2}\right)$$

84. $z = \sin(wct)\sin(wx)$

$$\frac{\partial z}{\partial t} = wc\cos(wct)\sin(wx)$$

$$\frac{\partial^2 z}{\partial t^2} = -w^2c^2\sin(wct)\sin(wx)$$

$$\frac{\partial z}{\partial x} = w\sin(wct)\cos(wx)$$

$$\frac{\partial^2 z}{\partial x^2} = -w^2\sin(wct)\sin(wx)$$

Therefore, $\dfrac{\partial^2 z}{\partial t^2} = c^2\left(\dfrac{\partial^2 z}{\partial x^2}\right)$.

85. $z = e^{-t}\cos\dfrac{x}{c}$

$$\frac{\partial z}{\partial t} = -e^{-t}\cos\frac{x}{c}$$

$$\frac{\partial z}{\partial x} = -\frac{1}{c}e^{-t}\sin\frac{x}{c}$$

$$\frac{\partial^2 z}{\partial x^2} = -\frac{1}{c^2}e^{-t}\cos\frac{x}{c}$$

Therefore, $\dfrac{\partial z}{\partial t} = c^2\left(\dfrac{\partial^2 z}{\partial x^2}\right)$.

86. $z = e^{-t}\sin\dfrac{x}{c}$

$$\frac{\partial z}{\partial t} = -e^{-t}\sin\frac{x}{c}$$

$$\frac{\partial z}{\partial x} = \frac{1}{c}e^{-t}\cos\frac{x}{c}$$

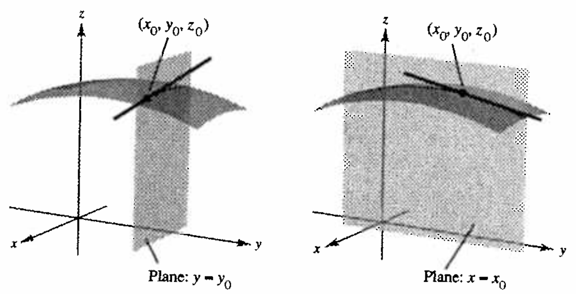
$$\frac{\partial^2 z}{\partial x^2} = -\frac{1}{c^2}e^{-t}\sin\frac{x}{c}$$

Therefore, $\dfrac{\partial z}{\partial t} = c^2\left(\dfrac{\partial^2 z}{\partial x^2}\right)$.

87. See the definition on page 906.

88. If $z = f(x, y)$, then to find f_x you consider y constant and differentiate with respect to x. Similarly, to find f_y, you consider x constant and differentiate with respect to y.

89.

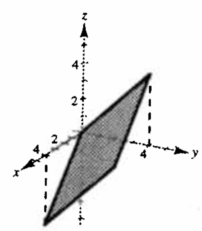


$\dfrac{\partial f}{\partial x}$ denotes the slope of the surface in the x-direction.

$\dfrac{\partial f}{\partial y}$ denotes the slope of the surface in the y-direction.

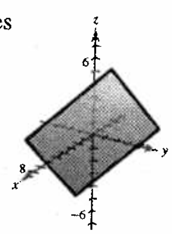
90. The plane $z = -x + y = f(x, y)$ satisfies

$$\frac{\partial f}{\partial x} < 0 \text{ and } \frac{\partial f}{\partial y} > 0.$$



91. The plane $z = x + y = f(x, y)$ satisfies

$$\frac{\partial f}{\partial x} > 0 \text{ and } \frac{\partial f}{\partial y} > 0.$$



92. In this case, the mixed partials are equal, $f_{xy} = f_{yx}$. See Theorem 13.3.

93. (a) $C = 32\sqrt{xy} + 175x + 205y + 1050$

$$\frac{\partial C}{\partial x} = 16\sqrt{\frac{y}{x}} + 175$$

$$\left.\frac{\partial C}{\partial x}\right]_{(80,\,20)} = 16\sqrt{\frac{1}{4}} + 175 = 183$$

$$\frac{\partial C}{\partial y} = 16\sqrt{\frac{x}{y}} + 205$$

$$\left.\frac{\partial C}{\partial y}\right]_{(80,\,20)} = 16\sqrt{4} + 205 = 237$$

(b) The fireplace-insert stove results in the cost increasing at a faster rate because

$$\frac{\partial C}{\partial y} > \frac{\partial C}{\partial x}.$$

94. $f(x, y) = 200x^{0.7}y^{0.3}$

(a) $\dfrac{\partial f}{\partial x} = 140x^{-0.3}y^{0.3} = 140\left(\dfrac{y}{x}\right)^{0.3}$

At $(x, y) = (1000, 500)$, $\dfrac{\partial f}{\partial x} = 140\left(\dfrac{500}{1000}\right)^{0.3} = 140\left(\dfrac{1}{2}\right)^{0.3} \approx 113.72$

(b) $\dfrac{\partial f}{\partial x} = 60x^{0.7}y^{-0.7} = 60\left(\dfrac{x}{y}\right)^{0.7}$

At $(x, y) = (1000, 500)$, $\dfrac{\partial f}{\partial x} = 60\left(\dfrac{1000}{500}\right)^{0.7} = 60(2)^{0.7} \approx 97.47$

95. An increase in either price will cause a decrease in demand.

96. $$V(I, R) = 1000\left[\frac{1 + 0.10(1 - R)}{1 + I}\right]^{10}$$

$$V_I(I, R) = 10{,}000\left[\frac{1 + 0.10(1 - R)}{1 + I}\right]^9\left[-\frac{1 + 0.10(1 - R)}{(1 + I)^2}\right] = -10{,}000\frac{[1 + 0.10(1 - R)]^{10}}{(1 + I)^{11}}$$

$V_I(0.03, 0.28) \approx -14{,}478.99$

$$V_R(I, R) = 10{,}000\left[\frac{1 + 0.10(1 - R)}{1 + I}\right]^9\left[\frac{-0.10}{1 + I}\right] = -1000\frac{[1 + 0.10(1 - R)]^9}{(1 + I)^{10}}$$

$V_R(0.03, 0.28) \approx -1391.17$

The rate of inflation has the greater negative influence on the growth of the investment. (See Exercise 67 in Section 12.1.)

97. $T = 500 - 0.6x^2 - 1.5y^2$

$\dfrac{\partial T}{\partial x} = -1.2x, \dfrac{\partial T}{\partial x}(2, 3) = -2.4°/\text{m}$

$\dfrac{\partial T}{\partial y} = -3y = \dfrac{\partial T}{\partial y}(2, 3) = -9°/\text{m}$

98. $A = 0.885t - 22.4h + 1.20th - 0.544$

(a) $\dfrac{\partial A}{\partial t} = 0.885 + 1.20h$

$\dfrac{\partial A}{\partial t}(30°, 0.80) = 0.885 + 1.20(0.80) = 1.845$

$\dfrac{\partial A}{\partial h} = -22.4 + 1.20t$

$\dfrac{\partial A}{\partial h}(30°, 0.80) = -22.4 + 1.20(30°) = 13.6$

(b) The humidity has a greater effect on A since its coefficient -22.4 is larger than that of t.

99. $$PV = mRT$$

$$T = \frac{PV}{mR} \Rightarrow \frac{\partial T}{\partial P} = \frac{V}{mR}$$

$$P = \frac{mRT}{V} \Rightarrow \frac{\partial P}{\partial V} = -\frac{mRT}{V^2}$$

$$V = \frac{mRT}{P} \Rightarrow \frac{\partial V}{\partial T} = \frac{mR}{P}$$

$$\frac{\partial T}{\partial P} \cdot \frac{\partial P}{\partial V} \cdot \frac{\partial V}{\partial T} = \left(\frac{V}{mR}\right)\left(-\frac{mRT}{V^2}\right)\left(\frac{mR}{P}\right)$$

$$= -\frac{mRT}{VP} = -\frac{mRT}{mRT} = -1$$

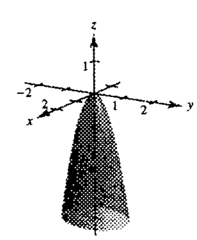
100. $U = -5x^2 + xy - 3y^2$

(a) $U_x = -10x + y$

(b) $U_y = x - 6y$

(c) $U_x(2, 3) = -17$ and $U_y(2, 3) = -16$. The person should consume one more unit of y because the rate of decrease of satisfaction is less for y.

(d)



101. $z = -0.04x + 0.64y + 3.4$

(a) $\dfrac{\partial z}{\partial x} = -0.04$

$\dfrac{\partial z}{\partial y} = 0.64$

(b) As the consumption of skim milk (x) increases, the consumption of whole milk (z) decreases. As the consumption of reduced-fat milk increases, the consumption of whole milk increases.

102. $z = -1.3520x^2 - 0.0025y^2 + 56.080x + 1.537y - 562.23$

(a) $\dfrac{\partial z}{\partial x} = -2.704x + 56.08$

$\dfrac{\partial^2 z}{\partial x^2} = -2.704$

$\dfrac{\partial z}{\partial y} = -0.005y + 1.537$

$\dfrac{\partial^2 z}{\partial y^2} = -0.005$

(b) Concave downward $\left(\dfrac{\partial^2 z}{\partial x^2} < 0\right)$

The rate of increase of Medicare expenses (z) is declining with respect to worker's compensation expenses (x).

(c) Concave downward $\left(\dfrac{\partial^2 z}{\partial y^2} < 0\right)$

The rate of increase of Medicare expenses (z) is slightly declining with respect to public assistance expenses.

103. False

Let $z = x + y + 1$.

104. True

105. True

106. True

107. $f(x, y) = \begin{cases} \dfrac{xy(x^2 - y^2)}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$

(a) $f_x(x, y) = \dfrac{(x^2 + y^2)(3x^2y - y^3) - (x^3y - xy^3)(2x)}{(x^2 + y^2)^2} = \dfrac{y(x^4 + 4x^2y^2 - y^4)}{(x^2 + y^2)^2}$

$f_y(x, y) = \dfrac{(x^2 + y^2)(x^3 - 3xy^2) - (x^3y - xy^3)(2y)}{(x^2 + y^2)^2} = \dfrac{x(x^4 - 4x^2y^2 - y^4)}{(x^2 + y^2)^2}$

(b) $f_x(0, 0) = \lim\limits_{\Delta x \to 0} \dfrac{f(\Delta x, 0) - f(0, 0)}{\Delta x} = \lim\limits_{\Delta x \to 0} \dfrac{0/[(\Delta x)^2] - 0}{\Delta x} = 0$

$f_y(0, 0) = \lim\limits_{\Delta y \to 0} \dfrac{f(0, \Delta y) - f(0, 0)}{\Delta y} = \lim\limits_{\Delta y \to 0} \dfrac{0/[(\Delta y)^2] - 0}{\Delta y} = 0$

(c) $f_{xy}(0, 0) = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right)\Bigg|_{(0,0)} = \lim\limits_{\Delta y \to 0} \dfrac{f_x(0, \Delta y) - f_x(0, 0)}{\Delta y} = \lim\limits_{\Delta y \to 0} \dfrac{\Delta y(-(\Delta y)^4)}{((\Delta y)^2)^2(\Delta y)} = \lim\limits_{\Delta y \to 0} (-1) = -1$

$f_{yx}(0, 0) = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial y}\right)\Bigg|_{(0,0)} = \lim\limits_{\Delta x \to 0} \dfrac{f_y(\Delta x, 0) - f_y(0, 0)}{\Delta x} = \lim\limits_{\Delta x \to 0} \dfrac{\Delta x((\Delta x)^4)}{((\Delta x)^2)^2(\Delta x)} = \lim\limits_{\Delta x \to 0} 1 = 1$

(d) f_{yx} or f_{xy} or both are not continuous at $(0, 0)$.

108. $f(x, y) = \displaystyle\int_x^y \sqrt{1 + t^3}\, dt$

By the Second Fundamental Theorem of Calculus,

$$\frac{\partial f}{\partial x} = \frac{d}{dx}\int_x^y \sqrt{1 + t^3}\, dt = -\frac{d}{dx}\int_y^x \sqrt{1 + t^3}\, dt = -\sqrt{1 + x^3}$$

$$\frac{\partial f}{\partial y} = \frac{d}{dy}\int_x^y \sqrt{1 + t^3}\, dt = \sqrt{1 + y^3}.$$

109. $f(x, y) = (x^3 + y^3)^{1/3}$

(a) $f_y(0, 0) = \lim_{\Delta y \to 0} \dfrac{f(0, 0 + \Delta y) - f(0, 0)}{\Delta y}$

$= \lim_{\Delta y \to 0} \dfrac{\Delta y}{\Delta y} = 1$

(b) $f_y(x, y) = \dfrac{y^2}{(x^3 + y^3)^{2/3}}$ fails to exist for $y = -x, x \neq 0$.

110. $f(x, y) = (x^2 + y^2)^{2/3}$

For $(x, y) \neq (0, 0)$,

$$f_x(x, y) = \frac{2}{3}(x^2 + y^2)^{-1/3}(2x) = \frac{4x}{3(x^2 + y^2)^{1/3}}.$$

For $(x, y) = (0, 0)$, use the definition of partial derivative.

$$f_x(0, 0) = \lim_{\Delta x \to 0} \frac{f(0 + \Delta x) - f(0, 0)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(\Delta x)^{4/3}}{\Delta x} = \lim_{\Delta x \to 0} (\Delta x)^{1/3} = 0$$

Section 13.4 Differentials

1. $z = 3x^2y^3$

$dz = 6xy^3\,dx + 9x^2y^2\,dy$

2. $z = \dfrac{x^2}{y}$

$dz = \dfrac{2x}{y}\,dx - \dfrac{x^2}{y^2}\,dy$

3. $z = \dfrac{-1}{x^2 + y^2}$

$$dz = \frac{2x}{(x^2 + y^2)^2}\,dx + \frac{2y}{(x^2 + y^2)^2}\,dy$$

$$= \frac{2}{(x^2 + y^2)^2}(x\,dx + y\,dy)$$

4. $w = \dfrac{x + y}{z - 2y}$

$$dw = \frac{1}{z - 2y}\,dx + \frac{z + 2x}{(z - 2y)^2}\,dy - \frac{x + y}{(z - 2y)^2}\,dz$$

5. $z = x\cos y - y\cos x$

$dz = (\cos y + y\sin x)\,dx + (-x\sin y - \cos x)\,dy = (\cos y + y\sin x)\,dx - (x\sin y + \cos x)\,dy$

6. $z = \left(\dfrac{1}{2}\right)(e^{x^2+y^2} - e^{-x^2-y^2})$

$$dz = 2x\left(\frac{e^{x^2+y^2} + e^{-x^2-y^2}}{2}\right)dx + 2y\left(\frac{e^{x^2+y^2} + e^{-x^2-y^2}}{2}\right)dy = (e^{x^2+y^2} + e^{-x^2-y^2})(x\,dx + y\,dy)$$

7. $z = e^x \sin y$

$dz = (e^x \sin y)\,dx + (e^x \cos y)\,dy$

8. $w = e^y \cos x + z^2$

$dw = -e^y \sin x\,dx + e^y \cos x\,dy + 2z\,dz$

9. $w = 2z^3y\sin x$

$dw = 2z^3y\cos x\,dx + 2z^3\sin x\,dy + 6z^2y\sin x\,dz$

10. $w = x^2yz^2 + \sin yz$

$dw = 2xyz^2\,dx + (x^2z^2 + z\cos yz)\,dy +$
$(2x^2yz + y\cos yz)dz$

11. (a) $f(1, 2) = 4$

$f(1.05, 2.1) = 3.4875$

$\Delta z = f(1.05, 2.1) - f(1, 2) = -0.5125$

(b) $dz = -2x\,dx - 2y\,dy$

$= -2(0.05) - 4(0.1) = -0.5$

12. (a) $f(1, 2) = \sqrt{5} \approx 2.2361$

$f(1.05, 2.1) = \sqrt{5.5125} \approx 2.3479$

$\Delta z = 0.11180$

(b) $dz = \dfrac{x}{\sqrt{x^2 + y^2}}\,dx + \dfrac{y}{\sqrt{x^2 + y^2}}\,dy$

$$= \frac{x\,dx + y\,dy}{\sqrt{x^2 + y^2}} = \frac{0.05 + 2(0.1)}{\sqrt{5}} \approx 0.11180$$

13. (a) $f(1, 2) = \sin 2$

$f(1.05, 2.1) = 1.05 \sin 2.1$

$\Delta z = f(1.05, 2.1) - f(1, 2) \approx -0.00293$

(b) $dz = \sin y\, dx + x \cos y\, dy$

$= (\sin 2)(0.05) + (\cos 2)(0.1) \approx 0.00385$

14. (a) $f(1, 2) = e^2 \approx 7.3891$

$f(1.05, 2.1) = 1.05e^{2.1} \approx 8.5745$

$\Delta z = 1.1854$

(b) $dz = e^y\, dx + xe^y\, dy$

$= e^2(0.05) + e^2(0.1) \approx 1.1084$

15. (a) $f(1, 2) = -5$

$f(1.05, 2.1) = -5.25$

$\Delta z = -0.25$

(b) $dz = 3\, dx - 4\, dy$

$= 3(0.05) - 4(0.1) = -0.25$

16. (a) $f(1, 2) = \dfrac{1}{2} = 0.5$

$f(1.05, 2.1) = \dfrac{1.05}{2.1} = 0.5$

$\Delta z = 0$

(b) $dz = \dfrac{1}{y}\, dx - \dfrac{x}{y^2}\, dy$

$= \dfrac{1}{2}(0.05) - \dfrac{1}{4}(0.1) = 0$

17. Let $z = \sqrt{x^2 + y^2}$, $x = 5$, $y = 3$, $dx = 0.05$, $dy = 0.1$. Then: $dz = \dfrac{x}{\sqrt{x^2 + y^2}}\, dx + \dfrac{y}{\sqrt{x^2 + y^2}}\, dy$

$$\sqrt{(5.05)^2 + (3.1)^2} - \sqrt{5^2 + 3^2} \approx \frac{5}{\sqrt{5^2 + 3^2}}(0.05) + \frac{3}{\sqrt{5^2 + 3^2}}(0.1) = \frac{0.55}{\sqrt{34}} \approx 0.094$$

18. Let $z = x^2(1 + y)^3$, $x = 2$, $y = 9$, $dx = 0.03$, $dy = -0.1$. Then: $dz = 2x(1 + y)^3\, dx + 3x^2(1 + y)^2\, dy$

$$(2.03)^2(1 + 8.9)^3 - 2^2(1 + 9)^3 \approx 2(2)(1 + 9)^3(0.03) + 3(2)^2(1 + 9)^2(-0.1) = 0$$

19. Let $z = (1 - x^2)/y^2$, $x = 3$, $y = 6$, $dx = 0.05$, $dy = -0.05$. Then: $dz = -\dfrac{2x}{y^2}\, dx + \dfrac{-2(1 - x^2)}{y^3}\, dy$

$$\frac{1 - (3.05)^2}{(5.95)^2} - \frac{1 - 3^2}{6^2} \approx -\frac{2(3)}{6^2}(0.05) - \frac{2(1 - 3^2)}{6^3}(-0.05) \approx -0.012$$

20. Let $z = \sin(x^2 + y^2)$, $x = y = 1$, $dx = 0.05$, $dy = -0.05$. Then: $dz = 2x \cos(x^2 + y^2)\, dx + 2y \cos(x^2 + y^2)\, dy$

$$\sin[(1.05)^2 + (0.95)^2] - \sin 2 \approx 2(1) \cos(1^2 + 1^2)(0.05) + 2(1) \cos(1^2 + 1^2)(-0.05) = 0$$

21. See the definition on page 916.

22. In general, the accuracy worsens as Δx and Δy increase.

23. The tangent plane to the surface $z = f(x, y)$ at the point P is a linear approximation of z.

24. If $z = f(x, y)$, then $\Delta z \approx dz$ is the propagated error, and $\dfrac{\Delta z}{z} \approx \dfrac{dz}{z}$ is the relative error.

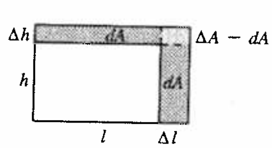
25. $A = lh$

$dA = l\, dh + h\, dl$

$\Delta A = (l + dl)(h + dh) - lh$

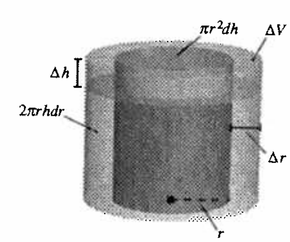
$= h\, dl + l\, dh + dl\, dh$

$\Delta A - dA = dl\, dh$



26. $V = \pi r^2 h$

$dV = 2\pi rh\, dr + \pi r^2\, dh$



27. $V = \dfrac{\pi r^2 h}{3}$

$r = 3$

$h = 6$

$$dV = \frac{2\pi rh}{3}\,dr + \frac{\pi r^2}{3}\,dh = \frac{\pi r}{3}(2h\,dr + r\,dh)$$

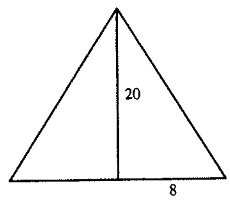
$$\Delta V = \frac{\pi}{3}[(r + \Delta r)^2(h + \Delta h) - r^2 h]$$

$$= \frac{\pi}{3}[(3 + \Delta r)^2(6 + \Delta h) - 54]$$

Δr	Δh	dV	ΔV	$\Delta V - dV$
0.1	0.1	4.7124	4.8391	0.1267
0.1	−0.1	2.8274	2.8264	−0.0010
0.001	0.002	0.0565	0.0566	0.0001
−0.0001	0.0002	−0.0019	−0.0019	0.0000

28. $S = \pi r\sqrt{r^2 + h^2}$

$r = 8, h = 20$



$$\frac{dS}{dr} = \pi(r^2 + h^2)^{1/2} + \pi r^2(r^2 + h^2)^{-1/2}$$

$$= \frac{\pi(r^2 + h^2) + \pi r^2}{(r^2 + h^2)^{1/2}} = \pi\frac{2r^2 + h^2}{\sqrt{r^2 + h^2}}$$

$$\frac{dS}{dh} = \pi r(r^2 + h^2)^{-1/2}h = \pi\frac{rh}{\sqrt{r^2 + h^2}}$$

$$dS = \pi\frac{2r^2 + h^2}{\sqrt{r^2 + h^2}}\,dr + \pi\frac{rh}{\sqrt{r^2 + h^2}}\,dh$$

$$= \frac{\pi}{\sqrt{r^2 + h^2}}[(2r^2 + h^2)\,dr + (rh)\,dh]$$

$S(8, 20) = 541.3758$

Δr	Δh	dS	ΔS	$\Delta S - dS$
0.1	0.1	10.0341	10.0768	0.0427
0.1	−0.1	5.3671	5.3596	−0.0075
0.001	0.002	0.12368	0.12368	0.683×10^{-5}
−0.0001	0.0002	−0.00303	−0.00303	-0.286×10^{-7}

29. $z = -0.04x + 0.64y + 3.4$

(a) $dz = -0.04\,dx + 0.64\,dy$

(b) $dz = \dfrac{\partial z}{\partial x}dx + \dfrac{\partial z}{\partial y}dy = -0.04(\pm 0.25) + 0.64(\pm 0.25) = \mp 0.01 \pm 0.16$

Maximum error: ± 0.17

Relative error: $\dfrac{dz}{z} = \dfrac{\pm 0.17}{-0.04(6.2) + 0.64(7.5) + 3.4} = \dfrac{\pm 0.17}{7.952} \approx 0.21$ or 2.1%

30. $(x, y) = (8.5, 3.2)$, $|dx| \le 0.05$, $|dy| \le 0.05$

$$r = \sqrt{x^2 + y^2} \Longrightarrow dr = \frac{x}{\sqrt{x^2 + y^2}}\,dx + \frac{y}{\sqrt{x^2 + y^2}}\,dy$$

$$= \frac{8.5}{\sqrt{8.5^2 + 3.2^2}}\,dx + \frac{3.2}{\sqrt{8.5^2 + 3.2^2}}\,dy \approx 0.9359\,dx + 0.3523\,dy$$

$|dr| \le (1.288)(0.05) \approx 0.064$

$$\theta = \arctan\left(\frac{y}{x}\right) \Longrightarrow d\theta = \frac{-\dfrac{y}{x^2}}{1 + \left(\dfrac{y}{x}\right)^2}\,dx + \frac{\dfrac{1}{x}}{1 + \left(\dfrac{y}{x}\right)^2}\,dy$$

$$= \frac{-y}{x^2 + y^2}\,dx + \frac{x}{x^2 + y^2}\,dy = \frac{-3.2}{8.5^2 + 3.2^2}\,dx + \frac{8.5}{8.5^2 + 3.2^2}\,dy$$

Using the worst case scenario, $dx = -0.05$ and $dy = 0.05$, you see that

$|d\theta| \le 0.00194 + 0.00515 = 0.0071.$

31. $V = \pi r^2 h \implies dV = (2\pi rh)\,dr + (\pi r^2)\,dh$

$$\frac{dV}{V} = 2\frac{dr}{r} + \frac{dh}{h}$$

$$= 2(0.04) + (0.02) = 0.10 = 10\%$$

32. $A = \frac{1}{2}ab \sin C$

$$dA = \tfrac{1}{2}[(b \sin C)\,da + (a \sin C)\,db + (ab \cos C)\,dC]$$

$$= \tfrac{1}{2}[4(\sin 45°)(\pm\tfrac{1}{16}) + 3(\sin 45°)(\pm\tfrac{1}{16}) + 12(\cos 45°)(\pm 0.02)] \approx \pm 0.24 \text{ in.}^2$$

33. $C = 35.74 + 0.6215T - 35.75v^{0.16} + 0.4275Tv^{0.16}$

$$\frac{\partial C}{\partial T} = 0.6215 + 0.4275v^{0.16}$$

$$\frac{\partial C}{\partial v} = -5.72v^{-0.84} + 0.0684Tv^{-0.84}$$

$$dC = \frac{\partial C}{\partial T}dT + \frac{\partial C}{\partial v}dv$$

$$= (0.6215 + 0.4275(23)^{0.16})(\pm 1) + (-5.72(23)^{-0.84} + 0.0684(8)(23)^{-0.84})(\pm 3)$$

$$= \pm 1.3275 \mp 1.1143$$

$= \pm 2.4418$ Maximum propagated error

$$\frac{dC}{C} = \frac{\pm 2.4418}{-12.6807} \approx \pm 0.19 \text{ or } 19\%$$

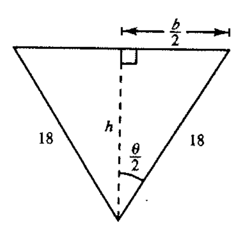
34. $a = \dfrac{v^2}{r}$

$$da = \frac{2v}{r}dv - \frac{v^2}{r^2}dr$$

$$\frac{da}{a} = 2\frac{dv}{v} - \frac{dr}{r} = 2(0.03) - (-0.02) = 0.08 = 8\%$$

Note: The maximum error will occur when dv and dr differ in signs.

35. (a) $V = \dfrac{1}{2}bhl$



$$= \left(18 \sin\frac{\theta}{2}\right)\left(18 \cos\frac{\theta}{2}\right)(16)(12)$$

$$= 31{,}104 \sin\theta \text{ in.}^3$$

$$= 18 \sin\theta \text{ ft}^3$$

V is maximum when $\sin\theta = 1$ or $\theta = \pi/2$.

(b) $V = \dfrac{s^2}{2}(\sin\theta)l$

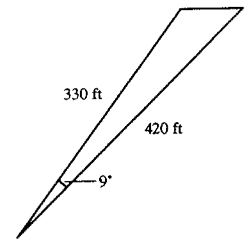
$$dV = s(\sin\theta)l\,ds + \frac{s^2}{2}l(\cos\theta)\,d\theta + \frac{s^2}{2}(\sin\theta)\,dl$$

$$= 18\left(\sin\frac{\pi}{2}\right)(16)(12)\left(\frac{1}{2}\right) + \frac{18^2}{2}(16)(12)\left(\cos\frac{\pi}{2}\right)\left(\frac{\pi}{90}\right) + \frac{18^2}{2}\left(\sin\frac{\pi}{2}\right)\left(\frac{1}{2}\right)$$

$$= 1809 \text{ in}^3 \approx 1.047 \text{ ft}^3$$

36. (a) Using the Law of Cosines:

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$= 330^2 + 420^2 - 2(330)(420)\cos 9°$$
$$a \approx 107.3 \text{ ft.}$$



(b) $a = \sqrt{b^2 + 420^2 - 2b(420)\cos\theta}$

$$da = \frac{1}{2}\left[b^2 + 420^2 - 840b\cos\theta\right]^{-1/2}[(2b - 840\cos\theta)\,db + 840b\sin\theta\,d\theta]$$

$$= \frac{1}{2}\left[330^2 + 420^2 - 840(330)\left(\cos\frac{\pi}{20}\right)\right]^{-1/2}\left[\left(2(330) - 840\cos\frac{\pi}{20}\right)(6) + 840(330)\left(\sin\frac{\pi}{20}\right)\left(\frac{\pi}{180}\right)\right]$$

$$\approx \frac{1}{2}[11512.79]^{-1/2}[\pm 1774.79] \approx \pm 8.27 \text{ ft}$$

37. $P = \dfrac{E^2}{R}$

$$dP = \frac{2E}{R}\,dE - \frac{E^2}{R^2}\,dR$$

$$\frac{dP}{P} = \frac{\frac{2E}{R}\,dE - \frac{E^2}{R^2}\,dR}{P} = 2\frac{dE}{E} - \frac{dR}{R} = 2(0.02) - (-0.03) = 0.07 = 7\%$$

38. $\dfrac{1}{R} = \dfrac{1}{R_1} + \dfrac{1}{R_2}$

$$R = \frac{R_1R_2}{R_1 + R_2}$$

$$dR_1 = \Delta R_1 = 0.5$$

$$dR_2 = \Delta R_2 = -2$$

$$\Delta R \approx dR = \frac{\partial R}{\partial R_1}\,dR + \frac{\partial R}{\partial R_2}\,dR_2 = \frac{R_2{}^2}{(R_1 + R_2)^2}\,\Delta R_1 + \frac{R_1{}^2}{(R_1 + R_2)^2}\,\Delta R_2$$

When $R_1 = 10$ and $R_2 = 15$, we have $\Delta R \approx \dfrac{15^2}{(10 + 15)^2}(0.5) + \dfrac{10^2}{(10 + 15)^2}(-2) = -0.14$ ohm.

39. $L = 0.00021\left(\ln\dfrac{2h}{r} - 0.75\right)$

$$dL = 0.00021\left[\frac{dh}{h} - \frac{dr}{r}\right] = 0.00021\left[\frac{(\pm 1/100)}{100} - \frac{(\pm 1/16)}{2}\right] \approx (\pm 6.6) \times 10^{-6}$$

$$L = 0.00021(\ln 100 - 0.75) \pm dL \approx 8.096 \times 10^{-4} \pm dL = 8.096 \times 10^{-4} \pm 6.6 \times 10^{-6} \text{ micro–henrys}$$

40. $T = 2\pi\sqrt{\dfrac{L}{g}}$

$$dg = 32.23 - 32.09 = 0.14$$

$$dL = 2.48 - 2.50 = -0.02$$

$$\Delta T \approx dT = \frac{\partial T}{\partial g}\,dg + \frac{\partial T}{\partial L}\,dL = \frac{-\pi}{g}\sqrt{\frac{L}{g}}\,dg + \frac{\pi}{\sqrt{Lg}}\,dL$$

When $g = 32.09$ and $L = 2.50$, $\Delta T \approx \dfrac{-\pi}{32.09}\sqrt{\dfrac{2.5}{32.09}}(0.14) + \dfrac{\pi}{\sqrt{(2.5)(32.09)}}(-0.02) \approx -0.0108$ seconds.

41. $z = f(x, y) = x^2 - 2x + y$

$$\begin{aligned}\Delta z &= f(x + \Delta x, y + \Delta y) - f(x, y)\\ &= (x^2 + 2x(\Delta x) + (\Delta x)^2 - 2x - 2(\Delta x) + y + (\Delta y)) - (x^2 - 2x + y)\\ &= 2x(\Delta x) + (\Delta x)^2 - 2(\Delta x) + (\Delta y)\\ &= (2x - 2)\,\Delta x + \Delta y + \Delta x(\Delta x) + 0(\Delta y)\\ &= f_x(x, y)\,\Delta x + f_y(x, y)\,\Delta y + \varepsilon_1\Delta x + \varepsilon_2\Delta y \text{ where } \varepsilon_1 = \Delta x \text{ and } \varepsilon_2 = 0.\end{aligned}$$

As $(\Delta x, \Delta y) \to (0, 0)$, $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$.

42. $z = f(x, y) = x^2 + y^2$

$$\begin{aligned}\Delta z &= f(x + \Delta x,\ y + \Delta y) - f(x, y)\\ &= x^2 + 2x(\Delta x) + (\Delta x)^2 + y^2 + 2y(\Delta y) + (\Delta y)^2 - (x^2 + y^2)\\ &= 2x(\Delta x) + 2y(\Delta y) + \Delta x(\Delta x) + \Delta y(\Delta y)\\ &= f_x(x, y)\,\Delta x + f_y(x, y)\,\Delta y + \varepsilon_1\Delta x + \varepsilon_2\Delta y \text{ where } \varepsilon_1 = \Delta x \text{ and } \varepsilon_2 = \Delta y.\end{aligned}$$

As $(\Delta x, \Delta y) \to (0, 0)$, $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$.

43. $z = f(x, y) = x^2y$

$$\begin{aligned}\Delta z &= f(x + \Delta x,\ y + \Delta y) - f(x, y)\\ &= (x^2 + 2x(\Delta x) + (\Delta x)^2)(y + \Delta y) - x^2y\\ &= 2xy(\Delta x) + y(\Delta x)^2 + x^2\Delta y + 2x(\Delta x)(\Delta y) + (\Delta x)^2\,\Delta y\\ &= 2xy(\Delta x) + x^2\Delta y + (y\Delta x)\,\Delta x + [2x\Delta x + (\Delta x)^2]\,\Delta y\\ &= f_x(x, y)\,\Delta x + f_y(x, y)\,\Delta y + \varepsilon_1\Delta x + \varepsilon_2\Delta y \text{ where } \varepsilon_1 = y(\Delta x) \text{ and } \varepsilon_2 = 2x\Delta x + (\Delta x)^2.\end{aligned}$$

As $(\Delta x, \Delta y) \to (0, 0)$, $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$.

44. $z = f(x, y) = 5x - 10y + y^3$

$$\begin{aligned}\Delta z &= f(x + \Delta x,\ y + \Delta y) - f(x, y)\\ &= 5x + 5\Delta x - 10y - 10\Delta y + y^3 + 3y^2(\Delta y) + 3y(\Delta y)^2 + (\Delta y)^3 - (5x - 10y + y^3)\\ &= 5(\Delta x) + (3y^2 - 10)(\Delta y) + 0(\Delta x) + (3y(\Delta y) + (\Delta y)^2)\,\Delta y\\ &= f_x(x, y)\,\Delta x + f_y(x, y)\,\Delta y + \varepsilon_1\Delta x + \varepsilon_2\Delta y \text{ where } \varepsilon_1 = 0 \text{ and } \varepsilon_2 = 3y(\Delta y) + (\Delta y)^2.\end{aligned}$$

As $(\Delta x, \Delta y) \to (0, 0)$, $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$.

45. $f(x, y) = \begin{cases} \dfrac{3x^2y}{x^4 + y^2}, & (x, y) \neq (0, 0)\\ 0, & (x, y) = (0, 0)\end{cases}$

(a) $$f_x(0, 0) = \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0, 0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\dfrac{0}{(\Delta x)^4} - 0}{\Delta x} = 0$$

$$f_y(0, 0) = \lim_{\Delta y \to 0} \frac{f(0, \Delta y) - f(0, 0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{\dfrac{0}{(\Delta y)^2} - 0}{\Delta y} = 0$$

Thus, the partial derivatives exist at (0, 0).

(b) Along the line $y = x$: $\displaystyle\lim_{(x, y) \to (0, 0)} f(x, y) = \lim_{x \to 0} \frac{3x^3}{x^4 + x^2} = \lim_{x \to 0} \frac{3x}{x^2 + 1} = 0$

Along the curve $y = x^2$: $\displaystyle\lim_{(x, y) \to (0, 0)} f(x, y) = \frac{3x^4}{2x^4} = \frac{3}{2}$

f is not continuous at (0, 0). Therefore, f is not differentiable at (0, 0). (See Theorem 12.5)

46. $f(x, y) = \begin{cases} \dfrac{5x^2y}{x^3 + y^3}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$

(a) $f_x(0, 0) = \lim_{\Delta x \to 0} \dfrac{f(\Delta x, 0) - f(0, 0)}{\Delta x} = \lim_{\Delta x \to 0} \dfrac{0 - 0}{\Delta x} = 0$

$f_y(0, 0) = \lim_{\Delta y \to 0} \dfrac{f(0, \Delta y) - f(0, 0)}{\Delta y} = \lim_{\Delta y \to 0} \dfrac{0 - 0}{\Delta y} = 0$

Thus, the partial derivatives exist at (0, 0).

(b) Along the line $y = x$: $\lim_{(x, y) \to (0, 0)} f(x, y) = \lim_{x \to 0} \dfrac{5x^3}{2x^3} = \dfrac{5}{2}$.

Along the line $x = 0$, $\lim_{(x, y) \to (0, 0)} f(x, y) = 0$.

Thus, f is not continuous at (0, 0). Therefore f is not differentiable at (0, 0).

47. Essay. For example, we can use the equation $F = ma$:

$$dF = \frac{\partial F}{\partial m} dm + \frac{\partial F}{\partial a} da = a\, dm + m\, da.$$

Section 13.5 Chain Rules for Functions of Several Variables

1. $w = x^2 + y^2$

$x = e^t$

$y = e^{-t}$

$\dfrac{dw}{dt} = \dfrac{\partial w}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial w}{\partial y}\dfrac{dy}{dt} = 2xe^t + 2y(-e^{-t}) = 2(e^{2t} - e^{-2t})$

2. $w = \sqrt{x^2 + y^2}$

$x = \cos t,\ y = e^t$

$\dfrac{dw}{dt} = \dfrac{\partial w}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial w}{\partial y}\dfrac{dy}{dt}$

$= \dfrac{x}{\sqrt{x^2 + y^2}}(-\sin t) + \dfrac{y}{\sqrt{x^2 + y^2}} e^t$

$= \dfrac{-x \sin t + ye^t}{\sqrt{x^2 + y^2}} = \dfrac{-\cos t \sin t + e^{2t}}{\sqrt{\cos^2 t + e^{2t}}}$

3. $w = x \sec y$

$x = e^t$

$y = \pi - t$

$\dfrac{dw}{dt} = \dfrac{\partial w}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial w}{\partial y}\dfrac{dy}{dt} = (\sec y)(e^t) + (x \sec y \tan y)(-1)$

$= e^t \sec(\pi - t)[1 - \tan(\pi - t)]$

$= -e^t (\sec t + \sec t \tan t)$

4. $w = \ln \dfrac{y}{x}$

$x = \cos t$

$y = \sin t$

$\dfrac{dw}{dt} = \left(\dfrac{-1}{x}\right)(-\sin t) + \left(\dfrac{1}{y}\right)(\cos t)$

$= \tan t + \cot t = \dfrac{1}{\sin t \cos t}$

5. $w = xy,\ x = 2 \sin t,\ y = \cos t$

(a) $\dfrac{dw}{dt} = \dfrac{\partial w}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial w}{\partial y}\dfrac{dy}{dt} = 2y \cos t + x(-\sin t) = 2y \cos t - x \sin t$

$= 2(\cos^2 t - \sin^2 t) = 2 \cos 2t$

(b) $w = 2 \sin t \cos t = \sin 2t,\ \dfrac{dw}{dt} = 2 \cos 2t$

6. $w = \cos(x - y),\ x = t^2,\ y = 1$

(a) $\dfrac{dw}{dt} = -\sin(x - y)(2t) + \sin(x - y)(0)$

$= -2t \sin(x - y) = -2t \sin(t^2 - 1)$

(b) $w = \cos(t^2 - 1),\ \dfrac{dw}{dt} = -2t \sin(t^2 - 1)$

7. $w = x^2 + y^2 + z^2$

$x = e^t \cos t$

$y = e^t \sin t$

$z = e^t$

(a) $\dfrac{dw}{dt} = \dfrac{\partial w}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial w}{\partial y}\dfrac{dy}{dt} + \dfrac{\partial w}{\partial z}\dfrac{dz}{dt} = 2x(-e^t \sin t + e^t \cos t) + 2y(e^t \cos t + e^t \sin t) + 2ze^t = 4e^{2t}$

(b) $w = (e^t \cos t)^2 + (e^t \sin t)^2 + (e^t)^2 = 2e^{2t}, \ \dfrac{dw}{dt} = 4e^{2t}$

8. $w = xy \cos z$

$x = t$

$y = t^2$

$z = \arccos t$

(a) $\dfrac{dw}{dt} = (y \cos z)(1) + (x \cos z)(2t) + (-xy \sin z)\left(-\dfrac{1}{\sqrt{1-t^2}}\right)$

$= t^2(t) + t(t)(2t) - t(t^2)\sqrt{1-t^2}\left(\dfrac{-1}{\sqrt{1-t^2}}\right) = t^3 + 2t^3 + t^3 = 4t^3$

(b) $w = t^4, \ \dfrac{dw}{dt} = 4t^3$

9. $w = xy + xz + yz, \ x = t - 1, \ y = t^2 - 1, \ z = t$

(a) $\dfrac{dw}{dt} = \dfrac{\partial w}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial w}{\partial y}\dfrac{dy}{dt} + \dfrac{\partial w}{\partial z}\dfrac{dz}{dt} = (y + z) + (x + z)(2t) + (x + y)$

$= (t^2 - 1 + t) + (t - 1 + t)(2t) + (t - 1 + t^2 - 1) = 3(2t^2 - 1)$

(b) $w = (t - 1)(t^2 - 1) + (t - 1)t + (t^2 - 1)t$

$\dfrac{dw}{dt} = 2t(t - 1) + (t^2 - 1) + 2t - 1 + 3t^2 - 1 = 3(2t^2 - 1)$

10. $w = xyz, \ x = t^2, \ y = 2t, \ z = e^{-t}$

(a) $\dfrac{dw}{dt} = yz(2t) + xz(2) + (xy)(-e^{-t}) = (2t)(e^{-t})(2t) + (t^2)(e^{-t})(2) + (t^2)(2t)(-e^{-t})$

$= 2t^2e^{-t}(2 + 1 - t) = 2t^2e^{-t}(3 - t)$

(b) $w = (t^2)(2t)(e^{-t}) = 2t^3e^{-t}$

$\dfrac{dw}{dt} = (2t^3)(-e^{-t}) + (e^{-t})(6t^2) = 2t^2e^{-t}(-t + 3)$

11. Distance $= f(t) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{(10 \cos 2t - 7 \cos t)^2 + (6 \sin 2t - 4 \sin t)^2}$

$f'(t) = \dfrac{1}{2}[(10 \cos 2t - 7 \cos t)^2 + (6 \sin 2t - 4 \sin t)^2]^{-1/2}$

$[[2(10 \cos 2t - 7 \cos t)(-20 \sin 2t + 7 \sin t)] + [2(6 \sin 2t - 4 \sin t)(12 \cos 2t - 4 \cos t)]]$

$f'\left(\dfrac{\pi}{2}\right) = \dfrac{1}{2}[(-10)^2 + 4^2]^{-1/2}[[2(-10)(7)] + (2(-4)(-12)]$

$= \dfrac{1}{2}(116)^{-1/2}(-44) = \dfrac{-22}{2\sqrt{29}} = \dfrac{-11\sqrt{29}}{29} \approx -2.04$

12. Distance $= f(t) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{\left[48 + \left(\sqrt{3} - \sqrt{2}\right)\right]^2 + \left[48t\left(1 - \sqrt{2}\right)\right]^2}$

$$= 48t\sqrt{8 - 2\sqrt{2} - 2\sqrt{6}}$$

$$f'(t) = 48\sqrt{8 - 2\sqrt{2} - 2\sqrt{6}} = f'(1)$$

13. $w = \arctan(2xy),\ x = \cos t,\ y = \sin t,\ t = 0$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$

$$= \frac{2y}{1 + (4x^2y^2)}(-\sin t) + \frac{2x}{1 + (4x^2y^2)}(\cos t)$$

$$= \frac{2\sin t}{1 + 4\cos^2 t \sin^2 t}(-\sin t) + \frac{2\cos t}{1 + 4\cos^2 t \sin^2 t}(\cos t)$$

$$= \frac{2\cos^2 t - 2\sin^2 t}{1 + 4\cos^2 t \sin^2 t}$$

$$\frac{d^2w}{dt^2} = \frac{(1 + 4\cos^2 t \sin^2 t)(-8\cos t \sin t) - (2\cos^2 t - 2\sin^2 t)(8\cos^3 t \sin t - 8\sin^3 t \cos t)}{(1 + 4\cos^2 t \sin^2 t)^2}$$

$$= \frac{-8\cos t \sin t(1 + 2\sin^4 t + 2\cos^4 t)}{(1 + 4\cos^2 t \sin^2 t)^2}$$

At $t = 0, \dfrac{d^2w}{dt^2} = 0.$

14. $w = \dfrac{x^2}{y}$

$x = t^2$

$y = t + 1$

$t = 1$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$

$$= \frac{2x}{y}(2t) + \frac{-x^2}{y^2}(1)$$

$$= \frac{2t^2(2t)}{t + 1} - \frac{t^4}{(t + 1)^2}$$

$$= \frac{(t + 1)(4t^3) - t^4}{(t + 1)^2}$$

$$= \frac{3t^4 + 4t^3}{(t + 1)^2}$$

$$\frac{d^2w}{dt^2} = \frac{(t + 1)^2(12t^3 + 12t^2) - (3t^4 + 4t^3)2(t + 1)}{(t + 1)^4}$$

At $t = 1$: $\dfrac{d^2w}{dt^2} = \dfrac{4(24) - (7)(4)}{16} = \dfrac{68}{16} = 4.25$

15. $w = x^2 + y^2$

$x = s + t$

$y = s - t$

$$\frac{\partial w}{\partial s} = 2x + 2y = 2(x + y) = 4s$$

$$\frac{\partial w}{\partial t} = 2x + 2y(-1) = 2(x - y) = 4t$$

When $s = 2$ and $t = -1$,

$$\frac{\partial w}{\partial s} = 8 \text{ and } \frac{\partial w}{\partial t} = -4.$$

16. $w = y^3 - 3x^2y$

$x = e^s$

$y = e^t$

$$\frac{\partial w}{\partial s} = -6xy(e^s) + (3y^2 - 3x^2)(0) = -6e^{2s+t}$$

$$\frac{\partial w}{\partial t} = -6xy(0) + (3y^2 - 3x^2)(e^t)$$

$$= 3e^t(e^{2t} - e^{2s})$$

When $s = 0$ and $t = 1$, $\frac{\partial w}{\partial s} = -6e$ and $\frac{\partial w}{\partial t} = 3e(e^2 - 1)$.

17. $w = x^2 - y^2$

$x = s\cos t$

$y = s\sin t$

$$\frac{\partial w}{\partial s} = 2x\cos t - 2y\sin t$$

$$= 2s\cos^2 t - 2s\sin^2 t = 2s\cos 2t$$

$$\frac{\partial w}{\partial t} = 2x(-s\sin t) - 2y(s\cos t) = -2s^2\sin 2t$$

When $s = 3$ and $t = \frac{\pi}{4}$, $\frac{\partial w}{\partial s} = 0$ and $\frac{\partial w}{\partial t} = -18$.

18. $w = \sin(2x + 3y)$

$x = s + t$

$y = s - t$

$$\frac{\partial w}{\partial s} = 2\cos(2x + 3y) + 3\cos(2x + 3y)$$

$$= 5\cos(2x + 3y) = 5\cos(5s - t)$$

$$\frac{\partial w}{\partial t} = 2\cos(2x + 3y) - 3\cos(2x + 3y)$$

$$= -\cos(2x + 3y) = -\cos(5s - t)$$

When $s = 0$ and $t = \frac{\pi}{2}$, $\frac{\partial w}{\partial s} = 0$ and $\frac{\partial w}{\partial t} = 0$.

19. $w = x^2 - 2xy + y^2$, $x = r + \theta$, $y = r - \theta$

(a) $$\frac{\partial w}{\partial r} = (2x - 2y)(1) + (-2x + 2y)(1) = 0$$

$$\frac{\partial w}{\partial \theta} = (2x - 2y)(1) + (-2x + 2y)(-1)$$

$$= 4x - 4y = 4(x - y)$$

$$= 4[(r + \theta) - (r - \theta)] = 8\theta$$

(b) $$w = (r + \theta)^2 - 2(r + \theta)(r - \theta) + (r - \theta)^2$$

$$= (r^2 + 2r\theta + \theta^2) - 2(r^2 - \theta^2) + (r^2 - 2r\theta + \theta^2)$$

$$= 4\theta^2$$

$$\frac{\partial w}{\partial r} = 0$$

$$\frac{\partial w}{\partial \theta} = 8\theta$$

20. $w = \sqrt{25 - 5x^2 - 5y^2}$, $x = r\cos\theta$, $y = r\sin\theta$

(a) $$\frac{\partial w}{\partial r} = \frac{-5x}{\sqrt{25 - 5x^2 - 5y^2}}\cos\theta + \frac{-5y}{\sqrt{25 - 5x^2 - 5y^2}}\sin\theta$$

$$= \frac{-5r\cos^2\theta - 5r\sin^2\theta}{\sqrt{25 - 5x^2 - 5y^2}} = \frac{-5r}{\sqrt{25 - 5r^2}}$$

$$\frac{\partial w}{\partial \theta} = \frac{-5x}{\sqrt{25 - 5x^2 - 5y^2}}(-r\sin\theta) + \frac{-5y}{\sqrt{25 - 5x^2 - 5y^2}}(r\cos\theta)$$

$$= \frac{-5r^2\sin^2\theta\cos\theta - 5r^2\sin\theta\cos\theta}{\sqrt{25 - 5x^2 - 5y^2}} = 0$$

(b) $w = \sqrt{25 - 5r^2}$

$$\frac{\partial w}{\partial r} = \frac{-5r}{\sqrt{25 - 5r^2}};\ \frac{\partial w}{\partial \theta} = 0$$

21. $w = \arctan \frac{y}{x},\ x = r\cos\theta,\ y = r\sin\theta$

(a) $\frac{\partial w}{\partial r} = \frac{-y}{x^2+y^2}\cos\theta + \frac{x}{x^2+y^2}\sin\theta = \frac{-r\sin\theta\cos\theta}{r^2} + \frac{r\cos\theta\sin\theta}{r^2} = 0$

$\frac{\partial w}{\partial \theta} = \frac{-y}{x^2+y^2}(-r\sin\theta) + \frac{x}{x^2+y^2}(r\cos\theta) = \frac{-(r\sin\theta)(-r\sin\theta)}{r^2} + \frac{(r\cos\theta)(r\cos\theta)}{r^2} = 1$

(b) $w = \arctan\frac{r\sin\theta}{r\cos\theta} = \arctan(\tan\theta) = \theta$

$\frac{\partial w}{\partial r} = 0$

$\frac{\partial w}{\partial \theta} = 1$

22. $w = \frac{yz}{x},\ x = \theta^2,\ y = r+\theta,\ z = r-\theta$

(a) $\frac{\partial w}{\partial r} = \frac{-yz}{x^2}(0) + \frac{z}{x}(1) + \frac{y}{x}(1) = \frac{z+y}{x} = \frac{2r}{\theta^2}$

$\frac{\partial w}{\partial \theta} = \frac{-yz}{x^2}(2\theta) + \frac{z}{x}(1) + \frac{y}{x}(-1)$

$= \frac{-(r+\theta)(r-\theta)}{\theta^4}(2\theta) + \frac{(r-\theta)-(r+\theta)}{\theta^2}$

$= \frac{2(\theta^2-r^2)}{\theta^3} - \frac{2}{\theta} = \frac{-2r^2}{\theta^3}$

(b) $w = \frac{yz}{x} = \frac{(r+\theta)(r-\theta)}{\theta^2} = \frac{r^2}{\theta^2} - 1$

$\frac{\partial w}{\partial r} = \frac{2r}{\theta^2}$

$\frac{\partial w}{\partial \theta} = \frac{-2r^2}{\theta^3}$

23. $w = xyz,\ x = s+t,\ y = s-t,\ z = st^2$

$\frac{\partial w}{\partial s} = yz(1) + xz(1) + xy(t^2)$

$= (s-t)st^2 + (s+t)st^2 + (s+t)(s-t)t^2$

$= 2s^2t^2 + s^2t^2 - t^4 = 3s^2t^2 - t^4 = t^2(3s^2 - t^2)$

$\frac{\partial w}{\partial t} = yz(1) + xz(-1) + xy(2st)$

$= (s-t)st^2 - (s+t)st^2 + (s+t)(s-t)(2st)$

$= -2st^3 + 2s^3t - 2st^3 = 2s^3t - 4st^3 = 2st(s^2 - 2t^2)$

24. $w = x\cos yz,\ x = s^2,\ y = t^2,\ z = s - 2t$

$\frac{\partial w}{\partial s} = \cos(yz)(2s) - xz\sin(yz)(0) - xy\sin(yz)(1)$

$= \cos(st^2 - 2t^3)2s - s^2t^2\sin(st^2 - 2t^3)$

$\frac{\partial w}{\partial t} = \cos(yz)(0) - xz\sin(yz)(2t) - xy\sin(yz)(-2)$

$= -2s^2t(s-2t)\sin(st^2 - 2t^3) + 2s^2t^2\sin(st^2 - 2t^3)$

$= (6s^2t^2 - 2s^3t)\sin(st^2 - 2t^3)$

25. $w = ze^{x/y},\ x = s-t,\ y = s+t,\ z = st$

$\frac{\partial w}{\partial s} = \frac{z}{y}e^{x/y}(1) + -\frac{zx}{y^2}e^{x/y}(1) + e^{x/y}(t)$

$= e^{(s-t)/(s+t)}\left[\frac{st}{s+t} - \frac{(s-t)st}{(s+t)^2} + t\right]$

$= e^{(s-t)/(s+t)}\left[\frac{st(s+t) - s^2t + st^2 + t(s+t)^2}{(s+t)^2}\right]$

$= e^{(s-t)/(s+t)}\frac{t(s^2 + 4st + t^2)}{(s+t)^2}$

$\frac{\partial w}{\partial t} = \frac{z}{y}e^{x/y}(-1) + -\frac{zx}{y^2}e^{x/y}(1) + e^{x/y}(s)$

$= e^{(s-t)/(s+t)}\left[-\frac{st}{s+t} - \frac{st(s-t)}{(s+t)^2} + s\right]$

$= e^{(s-t)/(s+t)}\left[\frac{-st(s+t) - st(s-t) + s(s+t)^2}{(s+t)^2}\right]$

$= e^{(s-t)/(s+t)}\frac{s(s^2 + t^2)}{(s+t)^2}$

26. $w = x^2 + y^2 + z^2, x = t\sin s, y = t\cos s, z = st^2$

$$\frac{\partial w}{\partial s} = 2x + \cos s + 2y(-t\sin s) + 2z(t^2)$$

$$= 2t^2 \sin s \cos s - 2t^2 \sin s \cos s + 2st^4 = 2st^4$$

$$\frac{\partial w}{\partial t} = 2x\sin s + 2y\cos s + 2z(2st)$$

$$= 2t\sin^2 s + 2t\cos^2 s + 4s^2t^3 = 2t + 4s^2t^3$$

27. $x^2 - 3xy + y^2 - 2x + y - 5 = 0$

$$\frac{dy}{dx} = -\frac{F_x(x, y)}{F_y(x, y)} = -\frac{2x - 3y - 2}{-3x + 2y + 1}$$

$$= \frac{3y - 2x + 2}{2y - 3x + 1}$$

28. $\cos x + \tan xy + 5 = 0$

$$\frac{dy}{dx} = -\frac{F_x(x, y)}{F_y(x, y)} = -\frac{-\sin x + y\sec^2 xy}{x\sec^2 xy}$$

29. $\ln\sqrt{x^2 + y^2} + xy = 4$

$$\frac{1}{2}\ln(x^2 + y^2) + xy - 4 = 0$$

$$\frac{dy}{dx} = -\frac{F_x(x, y)}{F_y(x, y)} = -\frac{\dfrac{x}{x^2 + y^2} + y}{\dfrac{y}{x^2 + y^2} + x} = -\frac{x + x^2y + y^3}{y + xy^2 + x^3}$$

30. $\dfrac{x}{x^2 + y^2} - y^2 - 6 = 0$

$$\frac{dy}{dx} = -\frac{F_x(x, y)}{F_y(x, y)}$$

$$= -\frac{(y^2 - x^2)/(x^2 + y^2)^2}{(-2xy)/(x^2 + y^2)^2 - 2y}$$

$$= \frac{y^2 - x^2}{2xy + 2y(x^2 + y^2)^2}$$

$$= \frac{y^2 - x^2}{2xy + 2yx^4 + 4x^2y^3 + 2y^5}$$

31. $F(x, y, z) = x^2 + y^2 + z^2 - 25$

$F_x = 2x$

$F_y = 2y$

$F_z = 2z$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{x}{z}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{y}{z}$$

32. $F(x, y, z) = xz + yz + xy$

$F_x = z + y$

$F_y = z + x$

$F_z = x + y$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{y + z}{x + y}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{x + z}{x + y}$$

33. $F(x, y, z) = \tan(x + y) + \tan(y + z) - 1$

$F_x = \sec^2(x + y)$

$F_y = \sec^2(x + y) + \sec^2(y + z)$

$F_z = \sec^2(y + z)$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{\sec^2(x + y)}{\sec^2(y + z)}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{\sec^2(x + y) + \sec^2(y + z)}{\sec^2(y + z)}$$

$$= -\left(\frac{\sec^2(x + y)}{\sec^2(y + z)} + 1\right)$$

34. $F(x, y, z) = e^x \sin(y + z) - z$

$F_x = e^x \sin(y + z)$

$F_y = e^x \cos(y + z)$

$F_z = e^x \cos(y + z) - 1$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = \frac{e^x \sin(y + z)}{1 - e^x \cos(y + x)}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = \frac{e^x \cos(y + z)}{1 - e^x \cos(y + z)}$$

35. $F(x, y, z) = x^2 + 2yz + z^2 - 1 = 0$

$$\frac{\partial z}{\partial x} = -\frac{F_x(x, y, z)}{F_z(x, y, z)} = \frac{-2x}{2y + 2z} = \frac{-x}{y + z}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y(x, y, z)}{F_z(x, y, z)} = \frac{-2z}{2y + 2z} = \frac{-z}{y + z}$$

36. $x + \sin(y + z) = 0$

(i) $1 + \frac{\partial z}{\partial x}\cos(y + z) = 0$ implies

$$\frac{\partial z}{\partial x} = -\frac{1}{\cos(y + z)} = -\sec(y + z).$$

(ii) $\left(1 + \frac{\partial z}{\partial y}\right)\cos(y + z) = 0$ implies $\frac{\partial z}{\partial y} = -1.$

37. $F(x, y, z) = e^{xz} + xy = 0$

$$\frac{\partial z}{\partial x} = -\frac{F_x(x, y, z)}{F_z(x, y, z)} = -\frac{ze^{xz} + y}{xe^{xz}}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y(x, y, z)}{F_z(x, y, z)} = \frac{-x}{xe^{xz}} = \frac{-1}{e^{xz}} = -e^{-xz}$$

38. $x \ln y + y^2 z + z^2 - 8 = 0$

(i) $\frac{\partial z}{\partial x} = \frac{-F_x(x, y, z)}{F_z(x, y, z)} = \frac{-\ln y}{y^2 + 2z}$

(ii) $\frac{\partial z}{\partial y} = \frac{-F_y(x, y, z)}{F_z(x, y, z)} = -\frac{\frac{x}{y} + 2yz}{y^2 + 2z} = -\frac{x + 2y^2 z}{y^3 + 2yz}$

39. $F(x, y, z, w) = xyz + xzw - yzw + w^2 - 5$

$F_x = yz + zw$

$F_y = xz - zw$

$F_z = xy + xw - yw$

$F_w = xz - yz + 2w$

$$\frac{\partial w}{\partial x} = -\frac{F_x}{F_w} = -\frac{z(y + w)}{xz - yz + 2w}$$

$$\frac{\partial w}{\partial y} = -\frac{F_y}{F_w} = -\frac{z(x - w)}{xz - yz + 2w}$$

$$\frac{\partial w}{\partial z} = -\frac{F_z}{F_w} = -\frac{xy + xw - yw}{xz - yz + 2w}$$

40. $x^2 + y^2 - z^2 - 5yw + 10w^2 - 2 = F(x, y, z, w)$

$F_x = 2x, F_y = 2y - 5w, F_z = 2z, F_w = -5y + 20w$

$$\frac{\partial w}{\partial x} = -\frac{F_x}{F_w} = \frac{-2x}{-5y + 20w} = \frac{2x}{5y - 20w}$$

$$\frac{\partial w}{\partial y} = -\frac{F_y}{F_w} = \frac{5w - 2y}{20w - 5y}$$

$$\frac{\partial w}{\partial z} = -\frac{F_z}{F_w} = \frac{2z}{5y - 20w}$$

41. $F(x, y, z, w) = \cos xy + \sin yz + wz - 20$

$$\frac{\partial w}{\partial x} = \frac{-F_x}{F_w} = \frac{y \sin xy}{z}$$

$$\frac{\partial w}{\partial y} = \frac{-F_y}{F_w} = \frac{x \sin xy - z \cos yz}{z}$$

$$\frac{\partial w}{\partial z} = \frac{-F_z}{F_w} = -\frac{y \cos zy + w}{z}$$

42. $F(x, y, z, w) = w - \sqrt{x - y} - \sqrt{y - z} = 0$

$$\frac{\partial w}{\partial x} = \frac{-F_x}{F_w} = \frac{\frac{1}{2}(x - y)^{-1/2}}{1} = \frac{1}{2\sqrt{x - y}}$$

$$\frac{\partial w}{\partial y} = \frac{-F_y}{F_w} = \frac{-1}{2}(x - y)^{-1/2} + \frac{1}{2}(y - z)^{-1/2}$$

$$= \frac{-1}{2\sqrt{x - y}} + \frac{1}{2\sqrt{y - z}}$$

$$\frac{\partial w}{\partial z} = \frac{-F_z}{F_w} = \frac{-1}{2\sqrt{y - z}}$$

43. $f(x, y) = \frac{xy}{\sqrt{x^2 + y^2}}$

$$f(tx, ty) = \frac{(tx)(ty)}{\sqrt{(tx)^2 + (ty)^2}} = t\left(\frac{xy}{\sqrt{x^2 + y^2}}\right) = tf(x, y)$$

Degree: 1

$$xf_x(x, y) + yf_y(x, y) = x\left(\frac{y^3}{(x^2 + y^2)^{3/2}}\right) + y\left(\frac{x^3}{(x^2 + y^2)^{3/2}}\right)$$

$$= \frac{xy}{\sqrt{x^2 + y^2}} = 1f(x, y)$$

44. $f(x, y) = x^3 - 3xy^2 + y^3$

$$f(tx, ty) = (tx)^3 - 3(tx)(ty)^2 + (ty)^3$$

$$= t^3(x^3 - 3xy^2 + y^3) = t^3 f(x, y)$$

Degree: 3

$$xf_x(x, y) + yf_y(x, y) = x(3x^2 - 3y^2) + y(-6xy + 3y^2)$$

$$= 3x^3 - 9xy^2 + 3y^3 = 3f(x, y)$$

45. $f(x, y) = e^{x/y}$

$$f(tx, ty) = e^{tx/ty} = e^{x/y} = f(x, y)$$

Degree: 0

$$xf_x(x, y) + yf_y(x, y) = x\left(\frac{1}{y}e^{x/y}\right) + y\left(-\frac{x}{y^2}e^{x/y}\right) = 0$$

46. $f(x, y) = \dfrac{x^2}{\sqrt{x^2 + y^2}}$

$$f(tx, ty) = \frac{(tx)^2}{\sqrt{(tx)^2 + (ty)^2}} = t\left(\frac{x^2}{\sqrt{x^2 + y^2}}\right) = tf(x, y)$$

Degree: 1

$$xf_x(x, y) + yf_y(x, y) = x\left[\frac{x^3 + 2xy^2}{(x^2 + y^2)^{3/2}}\right] + y\left[\frac{-x^2y}{(x^2 + y^2)^{3/2}}\right]$$

$$= \frac{x^4 + x^2y^2}{(x^2 + y^2)^{3/2}} = \frac{x^2(x^2 + y^2)}{(x^2 + y^2)^{3/2}}$$

$$= \frac{x^2}{\sqrt{x^2 + y^2}} = f(x, y)$$

47. $\dfrac{dw}{dt} = \dfrac{\partial w}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial w}{\partial y}\dfrac{dy}{dt}$ (Page 923)

48. $\dfrac{\partial w}{\partial s} = \dfrac{\partial w}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial w}{\partial y}\dfrac{\partial y}{\partial s}$

$\dfrac{\partial w}{\partial t} = \dfrac{\partial w}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial w}{\partial y}\dfrac{\partial y}{\partial t}$ (Page 925)

49. $w = f(x, y)$ is the explicit form of a function of two variables, as in $z = x^2 + y^2$. The implicit form is of the form $F(x, y, z) = 0$, as in $z - x^2 - y^2 = 0$.

50. $\dfrac{dy}{dx} = -\dfrac{f_x(x, y)}{f_y(x, y)}$

$\dfrac{\partial z}{\partial x} = -\dfrac{f_x(x, y, z)}{f_z(x, y, z)}$

$\dfrac{\partial z}{\partial y} = -\dfrac{f_y(x, y, z)}{f_z(x, y, z)}$

51. $V = \pi r^2 h$

$$\frac{dV}{dt} = \pi\left(2rh\frac{dr}{dt} + r^2\frac{dh}{dt}\right) = \pi r\left(2h\frac{dr}{dt} + r\frac{dh}{dt}\right) = \pi(12)[2(36)(6) + 12(-4)] = 4608\pi \text{ in.}^3/\text{min}$$

$S = 2\pi r(r + h)$

$$\frac{dS}{dt} = 2\pi\left[(2r + h)\frac{dr}{dt} + r\frac{dh}{dt}\right] = 2\pi[(24 + 36)(6) + 12(-4)] = 624\pi \text{ in.}^2/\text{min}$$

52. $V = \dfrac{1}{3}\pi r^2 h$

$$\frac{dV}{dt} = \frac{1}{3}\pi\left(2rh\frac{dr}{dt} + r^2\frac{dh}{dt}\right) = \frac{1}{3}\pi[2(12)(36)(6) + (12)^2(-4)] = 1536\pi \text{ in.}^3/\text{min}$$

$S = \pi r\sqrt{r^2 + h^2} + \pi r^2$ (Surface area includes base.)

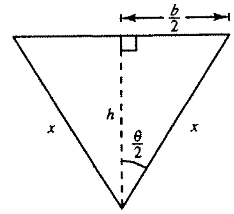
$$\frac{dS}{dt} = \pi\left[\left(\sqrt{r^2 + h^2} + \frac{r^2}{\sqrt{r^2 + h^2}} + 2r\right)\frac{dr}{dt} + \frac{rh}{\sqrt{r^2 + h^2}}\frac{dh}{dt}\right]$$

$$= \pi\left[\left(\sqrt{12^2 + 36^2} + \frac{144}{\sqrt{12^2 + 36^2}} + 2(12)\right)(6) + \frac{36(12)}{\sqrt{12^2 + 36^2}}(-4)\right]$$

$$= \pi\left[\left(12\sqrt{10} + \frac{12}{\sqrt{10}}\right)(6) + 144 + \frac{36}{\sqrt{10}}(-4)\right]$$

$$= \frac{648\pi}{\sqrt{10}} + 144\pi \text{ in.}^2/\text{min} = \frac{36\pi}{5}\left(20 + 9\sqrt{10}\right) \text{ in.}^2/\text{min}$$

53. $A = \frac{1}{2}bh = \left(x \sin \frac{\theta}{2}\right)\left(x \cos \frac{\theta}{2}\right) = \frac{x^2}{2} \sin \theta$



$$\frac{dA}{dt} = \frac{\partial A}{\partial x}\frac{dx}{dt} + \frac{\partial A}{\partial \theta}\frac{d\theta}{dt} = x \sin \theta \frac{dx}{dt} + \frac{x^2}{2} \cos \theta \frac{d\theta}{dt}$$

$$= 6\left(\sin \frac{\pi}{4}\right)\left(\frac{1}{2}\right) + \frac{6^2}{2}\left(\cos \frac{\pi}{4}\right)\left(\frac{\pi}{90}\right) = \frac{3\sqrt{2}}{2} + \frac{\pi\sqrt{2}}{10} \text{ m}^2/\text{hr}$$

$$\approx 2.566 \text{ m}^2/\text{hr}$$

54. (a) $V = \frac{\pi}{3}(r^2 + rR + R^2)h$

$$\frac{dV}{dt} = \frac{\pi}{3}\left[(2r + R)h\frac{dr}{dt} + (r + 2R)h\frac{dR}{dt} + (r^2 + rR + R^2)\frac{dh}{dt}\right]$$

$$= \frac{\pi}{3}\left[[2(15) + 25](10)(4) + [15 + 2(25)](10)(4) + [(15)^2 + (15)(25) + (25)^2](12)\right]$$

$$= \frac{\pi}{3}(19{,}500) = 6{,}500\pi \text{ cm}^3/\text{min}$$

(b) $S = \pi(R + r)\sqrt{(R - r)^2 + h^2}$

$$\frac{dS}{dt} = \pi\left\{\left[\sqrt{(R - r)^2 + h^2} - (R + r)\frac{(R - r)}{\sqrt{(R - r)^2 + h^2}}\right]\frac{dr}{dt} + \left[\sqrt{(R - r)^2 + h^2} + (R + r)\frac{(R - r)}{\sqrt{(R - r)^2 + h^2}}\right]\frac{dR}{dt} + (R + r)\frac{h}{\sqrt{(R - r)^2 + h^2}}\frac{dh}{dt}\right\}$$

$$= \pi\left\{\left[\sqrt{(25 - 15)^2 + 10^2} - (25 + 15)\frac{25 - 15}{\sqrt{(25 - 15)^2 + 10^2}}\right](4) + \left[\sqrt{(25 - 15)^2 + 10^2} + (25 + 15)\frac{25 - 15}{\sqrt{(25 - 15)^2 + 10^2}}\right](4) + (25 + 15)\left[\frac{10}{\sqrt{(25 - 15)^2 + 10^2}}(12)\right]\right\}$$

$$= 320\sqrt{2}\pi \text{ cm}^2/\text{min}$$

55. $I = \frac{1}{2}m(r_1^2 + r_2^2)$

$$\frac{dI}{dt} = \frac{1}{2}m\left[2r_1\frac{dr_1}{dt} + 2r_2\frac{dr_2}{dt}\right]$$

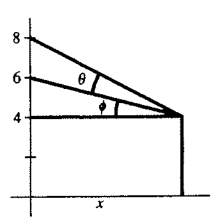
$$= m[(6)(2) + (8)(2)] = 28m \text{ cm}^2/\text{sec}$$

56. $pV = mRT$

$$T = \frac{1}{mR}(pV)$$

$$\frac{dT}{dt} = \frac{1}{mR}\left[V\frac{dp}{dt} + p\frac{dV}{dt}\right]$$

57. (a) $\tan \phi = \frac{2}{x}$



$$\tan(\theta + \phi) = \frac{4}{x}$$

$$\frac{\tan \theta + \tan \phi}{1 - \tan \theta \tan \phi} = \frac{4}{x}$$

$$\frac{\tan \theta + (2/x)}{1 - (2/x)\tan \theta} = \frac{4}{x}$$

$$x \tan \theta + 2 = 4 - \frac{8}{x}\tan \theta$$

$$x^2 \tan \theta - 2x + 8 \tan \theta = 0$$

—CONTINUED—

57. —CONTINUED—

(b) $F(x, \theta) = (x^2 + 8)\tan\theta - 2x = 0$

$$\frac{d\theta}{dx} = -\frac{F_x}{F_\theta} = -\frac{2x\tan\theta - 2}{\sec^2\theta(x^2 + 8)} = \frac{2\cos^2\theta - 2x\sin\theta\cos\theta}{x^2 + 8}$$

(c) $\dfrac{d\theta}{dx} = 0 \Rightarrow 2\cos^2\theta = 2x\sin\theta\cos\theta \Rightarrow \cos\theta = x\sin\theta \Rightarrow \tan\theta = \dfrac{1}{x}$

Thus, $x^2\left(\dfrac{1}{x}\right) - 2x + 8\left(\dfrac{1}{x}\right) = 0 \Rightarrow \dfrac{8}{x} = x \Rightarrow x = 2\sqrt{2}$ ft.

58. $g(t) = f(xt, yt) = t^n f(x, y)$

Let $u = xt,\ v = yt,$ then

$$g'(t) = \frac{\partial f}{\partial u}\cdot\frac{du}{dt} + \frac{\partial f}{\partial v}\cdot\frac{dv}{dt} = \frac{\partial f}{\partial u}x + \frac{\partial f}{\partial v}y$$

and $g'(t) = nt^{n-1}f(x, y)$.

Now, let $t = 1$ and we have $u = x,\ v = y$. Thus,

$$\frac{\partial f}{\partial x}x + \frac{\partial f}{\partial y}y = nf(x, y).$$

59. $w = f(x, y)$

$x = u - v$

$y = v - u$

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{dx}{du} + \frac{\partial w}{\partial y}\frac{dy}{du} = \frac{\partial w}{\partial x} - \frac{\partial w}{\partial y}$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{dx}{dv} + \frac{\partial w}{\partial y}\frac{dy}{dv} = -\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y}$$

$$\frac{\partial w}{\partial u} + \frac{\partial w}{\partial v} = 0$$

60. $w = (x - y)\sin(y - x)$

$$\frac{\partial w}{\partial x} = -(x - y)\cos(y - x) + \sin(y - x)$$

$$\frac{\partial w}{\partial y} = (x - y)\cos(y - x) - \sin(y - x)$$

$$\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} = 0$$

61. $w = f(x, y),\ x = r\cos\theta,\ y = r\sin\theta$

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\cos\theta + \frac{\partial w}{\partial y}\sin\theta$$

$$\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x}(-r\sin\theta) + \frac{\partial w}{\partial y}(r\cos\theta)$$

(a)
$$r\cos\theta\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}r\cos^2\theta + \frac{\partial w}{\partial y}r\sin\theta\cos\theta$$

$$-\sin\theta\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x}(r\sin^2\theta) - \frac{\partial w}{\partial y}r\sin\theta\cos\theta$$

$$r\cos\theta\frac{\partial w}{\partial r} - \sin\theta\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x}(r\cos^2\theta + r\sin^2\theta)$$

$$r\frac{\partial w}{\partial x} = \frac{\partial w}{\partial r}(r\cos\theta) - \frac{\partial w}{\partial \theta}\sin\theta$$

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial r}\cos\theta - \frac{\partial w}{\partial \theta}\frac{\sin\theta}{r} \qquad \text{(First Formula)}$$

$$r\sin\theta\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}r\sin\theta\cos\theta + \frac{\partial w}{\partial y}r\sin^2\theta$$

$$\cos\theta\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x}(-r\sin\theta\cos\theta) + \frac{\partial w}{\partial y}(r\cos^2\theta)$$

—CONTINUED—

61. —CONTINUED—

$$r \sin\theta \frac{\partial w}{\partial r} + \cos\theta \frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial y}(r\sin^2\theta + r\cos^2\theta)$$

$$r\frac{\partial w}{\partial y} = \frac{\partial w}{\partial r} r\sin\theta + \frac{\partial w}{\partial \theta}\cos\theta$$

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial r}\sin\theta + \frac{\partial w}{\partial \theta}\frac{\cos\theta}{r} \qquad \text{(Second Formula)}$$

(b) $$\left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 = \left(\frac{\partial w}{\partial x}\right)^2\cos^2\theta + 2\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\sin\theta\cos\theta + \left(\frac{\partial w}{\partial y}\right)^2\sin^2\theta + \left(\frac{\partial w}{\partial x}\right)^2\sin^2\theta -$$
$$2\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\sin\theta\cos\theta + \left(\frac{\partial w}{\partial y}\right)^2\cos^2\theta = \left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2$$

62. $w = \arctan\frac{y}{x},\ x = r\cos\theta,\ y = r\sin\theta$

$$= \arctan\left(\frac{r\sin\theta}{r\cos\theta}\right) = \arctan(\tan\theta) = \theta \text{ for } -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$\frac{\partial w}{\partial x} = \frac{-y}{x^2+y^2},\ \frac{\partial w}{\partial y} = \frac{x}{x^2+y^2},\ \frac{\partial w}{\partial r} = 0,\ \frac{\partial w}{\partial \theta} = 1$$

$$\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 = \frac{y^2}{(x^2+y^2)^2} + \frac{x^2}{(x^2+y^2)^2} = \frac{1}{x^2+y^2} = \frac{1}{r^2}$$

$$\left(\frac{\partial w}{\partial r}\right)^2 + \left(\frac{1}{r^2}\right)\left(\frac{\partial w}{\partial \theta}\right)^2 = 0 + \frac{1}{r^2}(1) = \frac{1}{r^2}$$

Therefore, $\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 = \left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2.$

63. Given $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$, $x = r\cos\theta$ and $y = r\sin\theta$.

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta = \frac{\partial v}{\partial y}\cos\theta - \frac{\partial v}{\partial x}\sin\theta$$

$$\frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial x}(-r\sin\theta) + \frac{\partial v}{\partial y}(r\cos\theta) = r\left[\frac{\partial v}{\partial y}\cos\theta - \frac{\partial v}{\partial x}\sin\theta\right]$$

Therefore, $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$.

$$\frac{\partial v}{\partial r} = \frac{\partial v}{\partial x}\cos\theta + \frac{\partial v}{\partial y}\sin\theta = -\frac{\partial u}{\partial y}\cos\theta + \frac{\partial u}{\partial x}\sin\theta$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}(-r\sin\theta) + \frac{\partial u}{\partial y}(r\cos\theta) = -r\left[-\frac{\partial u}{\partial y}\cos\theta + \frac{\partial u}{\partial x}\sin\theta\right]$$

Therefore, $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$.

64. Note first that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{x}{x^2 + y^2}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = \frac{y}{x^2 + y^2}.$$

$$\frac{\partial u}{\partial r} = \frac{x}{x^2 + y^2}\cos\theta + \frac{y}{x^2 + y^2}\sin\theta = \frac{r\cos^2\theta + r\sin^2\theta}{r^2} = \frac{1}{r}$$

$$\frac{\partial v}{\partial \theta} = \frac{-y}{x^2 + y^2}(-r\sin\theta) + \frac{x}{x^2 + y^2}(r\cos\theta) = \frac{r^2\sin^2\theta + r^2\cos^2\theta}{r^2} = 1$$

Thus, $\dfrac{\partial u}{\partial r} = \dfrac{1}{r}\dfrac{\partial v}{\partial \theta}$.

$$\frac{\partial v}{\partial r} = \frac{-y}{x^2 + y^2}\cos\theta + \frac{x}{x^2 + y^2}\sin\theta = \frac{-r\sin\theta\cos\theta + r\sin\theta\cos\theta}{r^2} = 0$$

$$\frac{\partial u}{\partial \theta} = \frac{x}{x^2 + y^2}(-r\sin\theta) + \frac{y}{x^2 + y^2}(r\cos\theta) = \frac{-r^2\sin\theta\cos\theta + r^2\sin\theta\cos\theta}{r^2} = 0$$

Thus, $\dfrac{\partial v}{\partial r} = -\dfrac{1}{r}\dfrac{\partial u}{\partial \theta}$.

Section 13.6 Directional Derivatives and Gradients

1. $f(x, y) = 3x - 4xy + 5y$

$$\mathbf{v} = \frac{1}{2}\left(\mathbf{i} + \sqrt{3}\,\mathbf{j}\right)$$

$$\nabla f(x, y) = (3 - 4y)\mathbf{i} + (-4x + 5)\mathbf{j}$$

$$\nabla f(1, 2) = -5\mathbf{i} + \mathbf{j}$$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}$$

$$D_{\mathbf{u}}f(1, 2) = \nabla f(1, 2)\cdot\mathbf{u} = \frac{1}{2}\left(-5 + \sqrt{3}\right)$$

2. $f(x, y) = x^3 - y^3,\ \mathbf{v} = \dfrac{\sqrt{2}}{2}(\mathbf{i} + \mathbf{j})$

$$\nabla f(x, y) = 3x^2\mathbf{i} - 3y^2\mathbf{j}$$

$$\nabla f(4, 3) = 48\mathbf{i} - 27\mathbf{j}$$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}$$

$$D_{\mathbf{u}}f(4, 3) = \nabla f(4, 3)\cdot\mathbf{u} = 24\sqrt{2} - \frac{27}{2}\sqrt{2} = \frac{21}{2}\sqrt{2}$$

3. $f(x, y) = xy$

$$\mathbf{v} = \mathbf{i} + \mathbf{j}$$

$$\nabla f(x, y) = y\mathbf{i} + x\mathbf{j}$$

$$\nabla f(2, 3) = 3\mathbf{i} + 2\mathbf{j}$$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}$$

$$D_{\mathbf{u}}f(2, 3) = \nabla f(2, 3)\cdot\mathbf{u} = \frac{5\sqrt{2}}{2}$$

4. $f(x, y) = \dfrac{x}{y}$

$$\mathbf{v} = -\mathbf{j}$$

$$\nabla f(x, y) = \frac{1}{y}\mathbf{i} - \frac{x}{y^2}\mathbf{j}$$

$$\nabla f(1, 1) = \mathbf{i} - \mathbf{j}$$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = -\mathbf{j}$$

$$D_{\mathbf{u}}f(1, 1) = \nabla f(1, 1)\cdot\mathbf{u} = 1$$

5. $g(x, y) = \sqrt{x^2 + y^2}$

$\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$

$$\nabla g = \frac{x}{\sqrt{x^2 + y^2}}\mathbf{i} + \frac{y}{\sqrt{x^2 + y^2}}\mathbf{j}$$

$$\nabla g(3, 4) = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}$$

$$D_{\mathbf{u}}\, g(3, 4) = \nabla g(3, 4) \cdot \mathbf{u} = -\frac{7}{25}$$

6. $g(x, y) = \arccos xy, \mathbf{v} = \mathbf{i} + 5\mathbf{j}$

$$\nabla g(x, y) = \frac{-y}{\sqrt{1 - (xy)^2}}\mathbf{i} + \frac{-x}{\sqrt{1 - (xy)^2}}\mathbf{j}$$

$$\nabla g(1, 0) = -\mathbf{j}$$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\sqrt{26}}\mathbf{i} + \frac{5}{\sqrt{26}}\mathbf{j}$$

$$D_{\mathbf{u}}g(1, 0) = \nabla g(1, 0) \cdot \mathbf{u} = \frac{-5}{\sqrt{26}} = \frac{-5\sqrt{26}}{26}$$

7. $h(x, y) = e^x \sin y$

$\mathbf{v} = -\mathbf{i}$

$$\nabla h = e^x \sin y\mathbf{i} + e^x \cos y\mathbf{j}$$

$$\nabla h\left(1, \frac{\pi}{2}\right) = e\mathbf{i}$$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = -\mathbf{i}$$

$$D_{\mathbf{u}}h\left(1, \frac{\pi}{2}\right) = \nabla h\left(1, \frac{\pi}{2}\right) \cdot \mathbf{u} = -e$$

8. $h(x, y) = e^{-(x^2+y^2)}$

$\mathbf{v} = \mathbf{i} + \mathbf{j}$

$$\nabla h = -2xe^{-(x^2+y^2)}\mathbf{i} - 2ye^{-(x^2+y^2)}\mathbf{j}$$

$$\nabla h(0, 0) = \mathbf{0}$$

$$D_{\mathbf{u}}h(0, 0) = \nabla h(0, 0) \cdot \mathbf{u} = 0$$

9. $f(x, y, z) = xy + yz + xz$

$\mathbf{v} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$

$$\nabla f(x, y, z) = (y + z)\mathbf{i} + (x + z)\mathbf{j} + (x + y)\mathbf{k}$$

$$\nabla f(1, 1, 1) = 2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\sqrt{6}}{3}\mathbf{i} + \frac{\sqrt{6}}{6}\mathbf{j} - \frac{\sqrt{6}}{6}\mathbf{k}$$

$$D_{\mathbf{u}}\, f(1, 1, 1) = \nabla f(1, 1, 1) \cdot \mathbf{u} = \frac{2\sqrt{6}}{3}$$

10. $f(x, y, z) = x^2 + y^2 + z^2$

$\mathbf{v} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$

$$\nabla f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$$

$$\nabla f(1, 2, -1) = 2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\sqrt{14}}\mathbf{i} - \frac{2}{\sqrt{14}}\mathbf{j} + \frac{3}{\sqrt{14}}\mathbf{k}$$

$$D_{\mathbf{u}}\, f(1, 2, -1) = \nabla f(1, 2, -1) \cdot \mathbf{u} = -\frac{6}{7}\sqrt{14}$$

11. $h(x, y, z) = x \arctan yz$

$\mathbf{v} = \langle 1, 2, -1 \rangle$

$$\nabla h(x, y, z) = \arctan yz\mathbf{i} + \frac{xz}{1 + (yz)^2}\mathbf{j} + \frac{xy}{1 + (yz)^2}\mathbf{k}$$

$$\nabla h(4, 1, 1) = \frac{\pi}{4}\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \left\langle \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}} \right\rangle$$

$$D_{\mathbf{u}}h(4, 1, 1) = \nabla h(4, 1, 1) \cdot \mathbf{u} = \frac{\pi + 8}{4\sqrt{6}} = \frac{(\pi + 8)\sqrt{6}}{24}$$

12. $h(x, y, z) = xyz$

$\mathbf{v} = \langle 2, 1, 2 \rangle$

$$\nabla h = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$$

$$\nabla h(2, 1, 1) = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$$

$$D_{\mathbf{u}}\, h(2, 1, 1) = \nabla h(2, 1, 1) \cdot \mathbf{u} = \frac{8}{3}$$

13. $f(x, y) = x^2 + y^2$

$$\mathbf{u} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$$

$$\nabla f = 2x\mathbf{i} + 2y\mathbf{j}$$

$$D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} = \frac{2}{\sqrt{2}}x + \frac{2}{\sqrt{2}}y = \sqrt{2}(x + y)$$

14. $f(x, y) = \dfrac{y}{x + y}$

$$\mathbf{u} = \frac{\sqrt{3}}{2}\mathbf{i} - \frac{1}{2}\mathbf{j}$$

$$\nabla f = -\frac{y}{(x + y)^2}\mathbf{i} + \frac{x}{(x + y)^2}\mathbf{j}$$

$$D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} = -\frac{\sqrt{3}y}{2(x + y)^2} - \frac{x}{2(x + y)^2}$$

$$= -\frac{1}{2(x + y)^2}\left(\sqrt{3}y + x\right)$$

15. $f(x, y) = \sin(2x - y)$

$$\mathbf{u} = \frac{1}{2}\mathbf{i} - \frac{\sqrt{3}}{2}\mathbf{j}$$

$$\nabla f = 2\cos(2x - y)\mathbf{i} - \cos(2x - y)\mathbf{j}$$

$$D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} = \cos(2x - y) + \frac{\sqrt{3}}{2}\cos(2x - y)$$

$$= \left(\frac{2 + \sqrt{3}}{2}\right)\cos(2x - y)$$

16. $g(x, y) = xe^y$

$$\mathbf{u} = -\frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}$$

$$\nabla g = e^y\mathbf{i} + xe^y\mathbf{j}$$

$$D_{\mathbf{u}} g = -\frac{1}{2}e^y + \frac{\sqrt{3}}{2}xe^y = \frac{e^y}{2}\left(\sqrt{3}x - 1\right)$$

17. $f(x, y) = x^2 + 4y^2$

$$\mathbf{v} = -2\mathbf{i} - 2\mathbf{j}$$

$$\nabla f = 2x\mathbf{i} + 8y\mathbf{j}$$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = -\frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}$$

$$D_{\mathbf{u}} f = -\frac{2}{\sqrt{2}}x - \frac{8}{\sqrt{2}}y = -\sqrt{2}(x + 4y)$$

At $P = (3, 1)$, $D_{\mathbf{u}} f = -7\sqrt{2}$.

18. $f(x, y) = \cos(x + y)$

$$\mathbf{v} = \frac{\pi}{2}\mathbf{i} - \pi\mathbf{j}$$

$$\nabla f = -\sin(x + y)\mathbf{i} - \sin(x + y)\mathbf{j}$$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\sqrt{5}}\mathbf{i} - \frac{2}{\sqrt{5}}\mathbf{j}$$

$$D_{\mathbf{u}} f = -\frac{1}{\sqrt{5}}\sin(x + y) + \frac{2}{\sqrt{5}}\sin(x + y)$$

$$= \frac{1}{\sqrt{5}}\sin(x + y) = \frac{\sqrt{5}}{5}\sin(x + y)$$

At $(0, \pi)$, $D_{\mathbf{u}} f = 0$.

19. $h(x, y, z) = \ln(x + y + z)$

$$\mathbf{v} = 3\mathbf{i} + 3\mathbf{j} + \mathbf{k}$$

$$\nabla h = \frac{1}{x + y + z}(\mathbf{i} + \mathbf{j} + \mathbf{k})$$

At $(1, 0, 0)$, $\nabla h = \mathbf{i} + \mathbf{j} + \mathbf{k}$.

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\sqrt{19}}(3\mathbf{i} + 3\mathbf{j} + \mathbf{k})$$

$$D_{\mathbf{u}} h = \nabla h \cdot \mathbf{u} = \frac{7}{\sqrt{19}} = \frac{7\sqrt{19}}{19}$$

20. $g(x, y, z) = xye^z$

$$\mathbf{v} = -2\mathbf{i} - 4\mathbf{j}$$

$$\nabla g = ye^z\mathbf{i} + xe^z\mathbf{j} + xye^z\mathbf{k}$$

At $(2, 4, 0)$, $\nabla g = 4\mathbf{i} + 2\mathbf{j} + 8\mathbf{k}$.

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = -\frac{1}{\sqrt{5}}\mathbf{i} - \frac{2}{\sqrt{5}}\mathbf{j}$$

$$D_{\mathbf{u}} g = \nabla g \cdot \mathbf{u} = -\frac{4}{\sqrt{5}} - \frac{4}{\sqrt{5}} = -\frac{8}{\sqrt{5}}$$

21. $f(x, y) = 3x - 5y^2 + 10$

$$\nabla f(x, y) = 3\mathbf{i} - 10y\mathbf{j}$$

$$\nabla f(2, 1) = 3\mathbf{i} - 10\mathbf{j}$$

22. $g(x, y) = 2xe^{y/x}$

$$\nabla g(x, y) = \left(-\frac{2y}{x}e^{y/x} + 2e^{y/x}\right)\mathbf{i} + 2e^{y/x}\mathbf{j}$$

$$\nabla g(2, 0) = 2\mathbf{i} + 2\mathbf{j}$$

23. $z = \cos(x^2 + y^2)$

$\nabla z(x, y) = -2x\sin(x^2 + y^2)\mathbf{i} - 2y\sin(x^2 + y^2)\mathbf{j}$

$\nabla z(3, -4) = -6\sin 25\mathbf{i} + 8\sin 25\mathbf{j} \approx 0.7941\mathbf{i} - 1.0588\mathbf{j}$

24. $z = \ln(x^2 - y)$

$\nabla z(x, y) = \frac{2x}{x^2 - y}\mathbf{i} - \frac{1}{x^2 - y}\mathbf{j}$

$\nabla z(2, 3) = 4\mathbf{i} - \mathbf{j}$

25. $w = 3x^2y - 5yz + z^2$

$\nabla w(x, y, z) = 6xy\mathbf{i} + (3x^2 - 5z)\mathbf{j} + (2z - 5y)\mathbf{k}$

$\nabla w(1, 1, -2) = 6\mathbf{i} + 13\mathbf{j} - 9\mathbf{k}$

26. $w = x\tan(y + z)$

$\nabla w(x, y, z) = \tan(y + z)\mathbf{i} + x\sec^2(y + z)\mathbf{j} + x\sec^2(y + z)\mathbf{k}$

$\nabla w(4, 3, -1) = \tan 2\mathbf{i} + 4\sec^2 2\mathbf{j} + 4\sec^2 2\mathbf{k}$

27. $\overrightarrow{PQ} = 2\mathbf{i} + 4\mathbf{j}, \mathbf{u} = \frac{1}{\sqrt{5}}\mathbf{i} + \frac{2}{\sqrt{5}}\mathbf{j}$

$\nabla g(x, y) = 2x\mathbf{i} + 2y\mathbf{j}, \nabla g(1, 2) = 2\mathbf{i} + 4\mathbf{j}$

$D_{\mathbf{u}}g = \nabla g \cdot \mathbf{u} = \frac{2}{\sqrt{5}} + \frac{8}{\sqrt{5}} = \frac{10}{\sqrt{5}} = 2\sqrt{5}$

28. $\overrightarrow{PQ} = -2\mathbf{i} + 7\mathbf{j}, \mathbf{u} = -\frac{2}{\sqrt{53}}\mathbf{i} + \frac{7}{\sqrt{53}}\mathbf{j}$

$\nabla f(x, y) = 6x\mathbf{i} - 2y\mathbf{j}, \nabla f(3, 1) = 18\mathbf{i} - 2\mathbf{j}$

$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = -\frac{36}{\sqrt{53}} - \frac{14}{\sqrt{53}} = -\frac{50}{\sqrt{53}} = -\frac{50\sqrt{53}}{53}$

29. $\overrightarrow{PQ} = 2\mathbf{i} + \mathbf{j}, \mathbf{u} = \frac{2}{\sqrt{5}}\mathbf{i} + \frac{1}{\sqrt{5}}\mathbf{j}$

$\nabla f(x, y) = -e^{-x}\cos y\mathbf{i} - e^{-x}\sin y\mathbf{j}$

$\nabla f(0, 0) = -\mathbf{i}$

$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = -\frac{2}{\sqrt{5}} = -\frac{2\sqrt{5}}{5}$

30. $\overrightarrow{PQ} = \frac{\pi}{2}\mathbf{i} + \pi\mathbf{j}, \mathbf{u} = \frac{1}{\sqrt{5}}\mathbf{i} + \frac{2}{\sqrt{5}}\mathbf{j}$

$\nabla f(x, y) = 2\cos 2x\cos y\mathbf{i} - \sin 2x\sin y\mathbf{j}$

$\nabla f(0, 0) = 2\mathbf{i}$

$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$

31. $h(x, y) = x\tan y$

$\nabla h(x, y) = \tan y\mathbf{i} + x\sec^2 y\mathbf{j}$

$\nabla h\left(2, \frac{\pi}{4}\right) = \mathbf{i} + 4\mathbf{j}$

$\left\|\nabla h\left(2, \frac{\pi}{4}\right)\right\| = \sqrt{17}$

32. $h(x, y) = y\cos(x - y)$

$\nabla h(x, y) = -y\sin(x - y)\mathbf{i} + [\cos(x - y) + y\sin(x - y)]\mathbf{j}$

$\nabla h\left(0, \frac{\pi}{3}\right) = \frac{\sqrt{3}\pi}{6}\mathbf{i} + \left(\frac{3 - \sqrt{3}\pi}{6}\right)\mathbf{j}$

$\left\|\nabla h\left(0, \frac{\pi}{3}\right)\right\| = \sqrt{\frac{3\pi^2}{36} + \frac{9 - 6\sqrt{3}\pi + 3\pi^2}{36}} = \frac{\sqrt{3\left(2\pi^2 - 2\sqrt{3}\pi + 3\right)}}{6}$

33. $g(x, y) = \ln \sqrt[3]{x^2 + y^2} = \frac{1}{3}\ln(x^2 + y^2)$

$$\nabla g(x, y) = \frac{1}{3}\left[\frac{2x}{x^2 + y^2}\mathbf{i} + \frac{2y}{x^2 + y^2}\mathbf{j}\right]$$

$$\nabla g(1, 2) = \frac{1}{3}\left(\frac{2}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}\right) = \frac{2}{15}(\mathbf{i} + 2\mathbf{j})$$

$$\|\nabla g(1, 2)\| = \frac{2\sqrt{5}}{15}$$

34. $g(x, y) = ye^{-x^2}$

$$\nabla g(x, y) = -2xye^{-x^2}\mathbf{i} + e^{-x^2}\mathbf{j}$$

$$\nabla g(0, 5) = \mathbf{j}$$

$$\|\nabla g(0, 5)\| = 1$$

35. $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$

$$\nabla f(x, y, z) = \frac{1}{\sqrt{x^2 + y^2 + z^2}}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$$

$$\nabla f(1, 4, 2) = \frac{1}{\sqrt{21}}(\mathbf{i} + 4\mathbf{j} + 2\mathbf{k})$$

$$\|\nabla f(1, 4, 2)\| = 1$$

36. $w = \dfrac{1}{\sqrt{1 - x^2 - y^2 - z^2}}$

$$\nabla w = \frac{1}{\left(\sqrt{1 - x^2 - y^2 - z^2}\right)^3}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$$

$$\nabla w(0, 0, 0) = \mathbf{0}$$

$$\|\nabla w(0, 0, 0)\| = 0$$

37. $f(x, y, z) = xe^{yz}$

$$\nabla f(x, y, z) = e^{yz}\mathbf{i} + xze^{yz}\mathbf{j} + xye^{yz}\mathbf{k}$$

$$\nabla f(2, 0, -4) = \mathbf{i} - 8\mathbf{j}$$

$$\|\nabla f(2, 0, -4)\| = \sqrt{65}$$

38. $w = xy^2z^2$

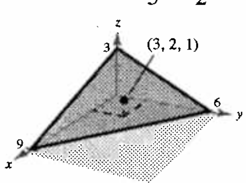
$$\nabla w = y^2z^2\mathbf{i} + 2xyz^2\mathbf{j} + 2xy^2z\mathbf{k}$$

$$\nabla w(2, 1, 1) = \mathbf{i} + 4\mathbf{j} + 4\mathbf{k}$$

$$\|\nabla w(2, 1, 1)\| = \sqrt{33}$$

For Exercises 39–46, $f(x, y) = 3 - \frac{x}{3} - \frac{y}{2}$ and $D_\theta f(x, y) = -\left(\frac{1}{3}\right)\cos\theta - \left(\frac{1}{2}\right)\sin\theta$.

39. $f(x, y) = 3 - \frac{x}{3} - \frac{y}{2}$



40. (a) $D_{\pi/4}f(3, 2) = -\left(\frac{1}{3}\right)\frac{\sqrt{2}}{2} - \left(\frac{1}{2}\right)\frac{\sqrt{2}}{2} = -\frac{5\sqrt{2}}{12}$

(b) $D_{2\pi/3}f(3, 2) = -\left(\frac{1}{3}\right)\left(-\frac{1}{2}\right) - \left(\frac{1}{2}\right)\frac{\sqrt{3}}{2} = \frac{2 - 3\sqrt{3}}{12}$

41. (a) $D_{4\pi/3}f(3, 2) = -\left(\frac{1}{3}\right)\left(-\frac{1}{2}\right) - \left(\frac{1}{2}\right)\left(-\frac{\sqrt{3}}{2}\right)$

$$= \frac{2 + 3\sqrt{3}}{12}$$

(b) $D_{-\pi/6}f(3, 2) = -\left(\frac{1}{3}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)$

$$= \frac{3 - 2\sqrt{3}}{12}$$

42. (a) $\mathbf{u} = \left(\frac{1}{\sqrt{2}}\right)(\mathbf{i} + \mathbf{j})$

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u}$$

$$= -\left(\frac{1}{3}\right)\frac{1}{\sqrt{2}} - \left(\frac{1}{2}\right)\frac{1}{\sqrt{2}} = -\frac{5\sqrt{2}}{12}$$

(b) $\mathbf{v} = -3\mathbf{i} - 4\mathbf{j}$

$$\|\mathbf{v}\| = \sqrt{9 + 16} = 5$$

$$\mathbf{u} = -\frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}$$

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = \frac{1}{5} + \frac{2}{5} = \frac{3}{5}$$

43. (a) $\mathbf{v} = -3\mathbf{i} + 4\mathbf{j}$

$\|\mathbf{v}\| = \sqrt{9 + 16} = 5$

$\mathbf{u} = -\frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$

$D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} = \frac{1}{5} - \frac{2}{5} = -\frac{1}{5}$

(b) $\mathbf{v} = \mathbf{i} + 3\mathbf{j}$

$\|\mathbf{v}\| = \sqrt{10}$

$\mathbf{u} = \frac{1}{\sqrt{10}}\mathbf{i} + \frac{3}{\sqrt{10}}\mathbf{j}$

$D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} = \frac{-11}{6\sqrt{10}} = -\frac{11\sqrt{10}}{60}$

44. $\nabla f = -\left(\frac{1}{3}\right)\mathbf{i} - \left(\frac{1}{2}\right)\mathbf{j}$

45. $\|\nabla f\| = \sqrt{\frac{1}{9} + \frac{1}{4}} = \frac{1}{6}\sqrt{13}$

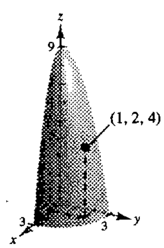
46. $\nabla f = -\frac{1}{3}\mathbf{i} - \frac{1}{2}\mathbf{j}$

$\frac{\nabla f}{\|\nabla f\|} = \frac{1}{\sqrt{13}}(-2\mathbf{i} - 3\mathbf{j})$

Therefore, $\mathbf{u} = \left(1/\sqrt{13}\right)(3\mathbf{i} - 2\mathbf{j})$ and $D_{\mathbf{u}} f(3, 2) = \nabla f \cdot \mathbf{u} = 0$. ∇f is the direction of greatest rate of change of f. Hence, in a direction orthogonal to ∇f, the rate of change of f is 0.

For Exercises 47–50, $f(x, y) = 9 - x^2 - y^2$ and $D_\theta f(x, y) = -2x\cos\theta - 2y\sin\theta = -2(x\cos\theta + y\sin\theta)$.

47. $f(x, y) = 9 - x^2 - y^2$



48. (a) $D_{-\pi/4} f(1, 2) = -2\left(\frac{\sqrt{2}}{2} - \sqrt{2}\right) = \sqrt{2}$

(b) $D_{\pi/3} f(1, 2) = -2\left(\frac{1}{2} + \sqrt{3}\right) = -\left(1 + 2\sqrt{3}\right)$

49. $\nabla f(1, 2) = -2\mathbf{i} - 4\mathbf{j}$

$\|\nabla f(1, 2)\| = \sqrt{4 + 16} = \sqrt{20} = 2\sqrt{5}$

50. $\nabla f(1, 2) = -2\mathbf{i} - 4\mathbf{j}$

$\frac{\nabla f(1, 2)}{\|\nabla f(1, 2)\|} = \frac{1}{\sqrt{5}}(-\mathbf{i} - 2\mathbf{j})$

Therefore,

$\mathbf{u} = \left(1/\sqrt{5}\right)(-2\mathbf{i} + \mathbf{j})$ and

$D_{\mathbf{u}} f(1, 2) = \nabla f(1, 2) \cdot \mathbf{u} = 0.$

51. (a) In the direction of the vector $-4\mathbf{i} + \mathbf{j}$

(b) $\nabla f = \frac{1}{10}(2x - 3y)\mathbf{i} + \frac{1}{10}(-3x + 2y)\mathbf{j}$

$\nabla f(1, 2) = \frac{1}{10}(-4)\mathbf{i} + \frac{1}{10}(1)\mathbf{j} = -\frac{2}{5}\mathbf{i} + \frac{1}{10}\mathbf{j}$

(Same direction as in part (a))

(c) $-\nabla f = \frac{2}{5}\mathbf{i} - \frac{1}{10}\mathbf{j}$, the direction opposite that of the gradient

52. (a) In the direction of the vector $\mathbf{i} + \mathbf{j}$

(b) $\nabla f = \frac{1}{2}y\frac{1}{2\sqrt{x}}\mathbf{i} + \frac{1}{2}\sqrt{x}\mathbf{j} = \frac{y}{4\sqrt{x}}\mathbf{i} + \frac{1}{2}\sqrt{x}\mathbf{j}$

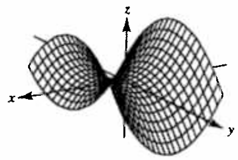
$\nabla f(1, 2) = \frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}$

(Same direction as in part (a))

(c) $-\nabla f = -\frac{1}{2}\mathbf{i} - \frac{1}{2}\mathbf{j}$, the direction opposite that of the gradient

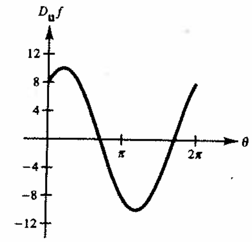
53. $f(x, y) = x^2 - y^2,\ (4, -3, 7)$

(a)



(b) $D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u} = 2x \cos \theta - 2y \sin \theta$

$D_{\mathbf{u}} f(4, -3) = 8 \cos \theta + 6 \sin \theta$



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(c) Zeros: $\theta \approx 2.21, 5.36$

These are the angles θ for which $D_{\mathbf{u}} f(4, 3)$ equals zero.

(d) $g(\theta) = D_{\mathbf{u}} f(4, -3) = 8 \cos \theta + 6 \sin \theta$

$g'(\theta) = -8 \sin \theta + 6 \cos \theta$

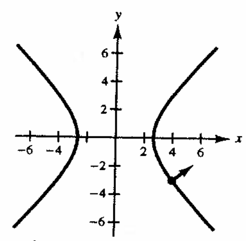
Critical numbers: $\theta \approx 0.64, 3.79$

These are the angles for which $D_{\mathbf{u}} f(4, -3)$ is a maximum (0.64) and minimum (3.79).

(e) $\|\nabla f(4, -3)\| = \|2(4)\mathbf{i} - 2(-3)\mathbf{j}\| = \sqrt{64 + 36} = 10$, the maximum value of $D_{\mathbf{u}} f(4, -3)$, at $\theta \approx 0.64$.

(f) $f(x, y) = x^2 - y^2 = 7$

$\nabla f(4, -3) = 8\mathbf{i} + 6\mathbf{j}$ is perpendicular to the level curve at $(4, -3)$.



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54. (a) $f(x, y) = \dfrac{8y}{1 + x^2 + y^2} = 2$

$\Rightarrow 4y = 1 + x^2 + y^2$

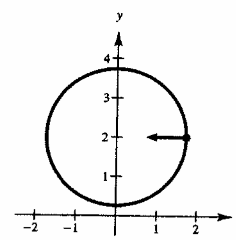
$4 = y^2 - 4y + 4 + x^2 + 1$

$(y - 2)^2 + x^2 = 3$

Circle: center: $(0, 2)$, radius: $\sqrt{3}$

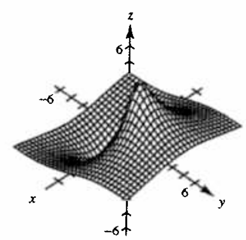
(b) $\nabla f = \dfrac{-16xy}{(1 + x^2 + y^2)^2}\mathbf{i} + \dfrac{8 + 8x^2 - 8y^2}{(1 + x^2 + y^2)^2}\mathbf{j}$

$\nabla f(\sqrt{3}, 2) = \dfrac{-\sqrt{3}}{2}\mathbf{i}$



(c) The directional derivative of f is 0 in the directions $\pm\mathbf{j}$.

(d)



55. $f(x, y) = x^2 + y^2$

$c = 25,\ P = (3, 4)$

$\nabla f(x, y) = 2x\mathbf{i} + 2y\mathbf{j}$

$x^2 + y^2 = 25$

$\nabla f(3, 4) = 6\mathbf{i} + 8\mathbf{j}$

56. $f(x, y) = 6 - 2x - 3y$

$c = 6,\ P = (0, 0)$

$\nabla f(x, y) = -2\mathbf{i} - 3\mathbf{j}$

$6 - 2x - 3y = 6$

$0 = 2x + 3y$

$\nabla f(0, 0) = -2\mathbf{i} - 3\mathbf{j}$

57. $f(x, y) = \dfrac{x}{x^2 + y^2}$

$c = \dfrac{1}{2}, \; P = (1, 1)$

$\nabla f(x, y) = \dfrac{y^2 - x^2}{(x^2 + y^2)^2}\mathbf{i} - \dfrac{2xy}{(x^2 + y^2)^2}\mathbf{j}$

$\dfrac{x}{x^2 + y^2} = \dfrac{1}{2}$

$x^2 + y^2 - 2x = 0$

$\nabla f(1, 1) = -\dfrac{1}{2}\mathbf{j}$

58. $f(x, y) = xy$

$c = -3, \; P = (-1, 3)$

$\nabla f(x, y) = y\mathbf{i} + x\mathbf{j}$

$xy = -3$

$\nabla f(-1, 3) = 3\mathbf{i} - \mathbf{j}$

59. $4x^2 - y = 6$

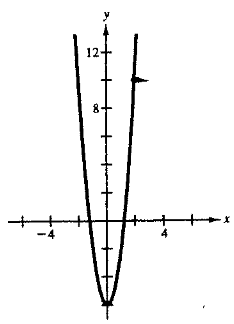
$f(x, y) = 4x^2 - y$

$\nabla f(x, y) = 8x\mathbf{i} - \mathbf{j}$

$\nabla f(2, 10) = 16\mathbf{i} - \mathbf{j}$

$\dfrac{\nabla f(2, 10)}{\|\nabla f(2, 10)\|} = \dfrac{1}{\sqrt{257}}(16\mathbf{i} - \mathbf{j})$

$= \dfrac{\sqrt{257}}{257}(16\mathbf{i} - \mathbf{j})$



60. $3x^2 - 2y^2 = 1$

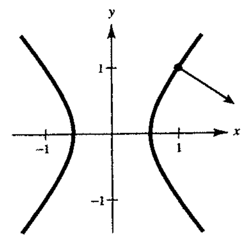
$f(x, y) = 3x^2 - 2y^2$

$\nabla f(x, y) = 6x\mathbf{i} - 4y\mathbf{j}$

$\nabla f(1, 1) = 6\mathbf{i} - 4\mathbf{j}$

$\dfrac{\nabla f(1, 1)}{\|\nabla f(1, 1)\|} = \dfrac{1}{\sqrt{13}}(3\mathbf{i} - 2\mathbf{j})$

$= \dfrac{\sqrt{13}}{13}(3\mathbf{i} - 2\mathbf{j})$



61. $9x^2 + 4y^2 = 40$

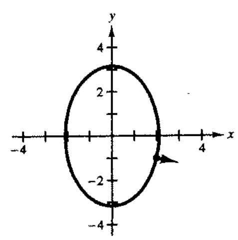
$f(x, y) = 9x^2 + 4y^2$

$\nabla f(x, y) = 18x\mathbf{i} + 8y\mathbf{j}$

$\nabla f(2, -1) = 36\mathbf{i} - 8\mathbf{j}$

$\dfrac{\nabla f(2, -1)}{\|\nabla f(2, -1)\|} = \dfrac{1}{\sqrt{85}}(9\mathbf{i} - 2\mathbf{j})$

$= \dfrac{\sqrt{85}}{85}(9\mathbf{i} - 2\mathbf{j})$



62. $xe^y - y = 5$

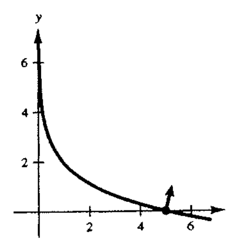
$f(x, y) = xe^y - y$

$\nabla f(x, y) = e^y\mathbf{i} + (xe^y - 1)\mathbf{j}$

$\nabla f(5, 0) = \mathbf{i} + 4\mathbf{j}$

$\dfrac{\nabla f(5, 0)}{\|\nabla f(5, 0)\|} = \dfrac{1}{\sqrt{17}}(\mathbf{i} + 4\mathbf{j})$

$= \dfrac{\sqrt{17}}{17}(\mathbf{i} + 4\mathbf{j})$



63. $T = \dfrac{x}{x^2 + y^2}$

$\nabla T = \dfrac{y^2 - x^2}{(x^2 + y^2)^2}\mathbf{i} - \dfrac{2xy}{(x^2 + y^2)^2}\mathbf{j}$

$\nabla T(3, 4) = \dfrac{7}{625}\mathbf{i} - \dfrac{24}{625}\mathbf{j} = \dfrac{1}{625}(7\mathbf{i} - 24\mathbf{j})$

64. $h(x, y) = 5000 - 0.001x^2 - 0.004y^2$

$\nabla h = -0.002x\mathbf{i} - 0.008y\mathbf{j}$

$\nabla h(500, 300) = -\mathbf{i} - 2.4\mathbf{j}$ or

$5\nabla h = -(5\mathbf{i} + 12\mathbf{j})$

65. See the definition, page 932.

66. The directional derivative gives the slope of a surface at a point in an arbitrary direction $\mathbf{u} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$.

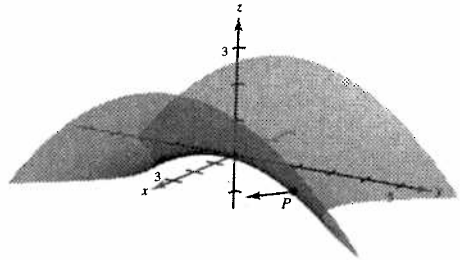
67. Let $f(x, y)$ be a function of two variables and $\mathbf{u} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$ a unit vector.

(a) If $\theta = 0°$, then $D_{\mathbf{u}}f = \dfrac{\partial f}{\partial x}$.

(b) If $\theta = 90°$, then $D_{\mathbf{u}}f = \dfrac{\partial f}{\partial y}$.

68. See the definition, pages 934 and 935.

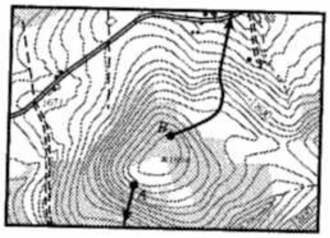
69.



70. The gradient vector is normal to the level curves.

See Theorem 13.12.

71.



72. The wind speed is greatest at B.

73. $T(x, y) = 400 - 2x^2 - y^2, \quad P = (10, 10)$

$\dfrac{dx}{dt} = -4x \qquad \dfrac{dy}{dt} = -2y$

$x(t) = C_1e^{-4t} \qquad y(t) = C_2e^{-2t}$

$10 = x(0) = C_1 \qquad 10 = y(0) = C_2$

$x(t) = 10e^{-4t} \qquad y(t) = 10e^{-2t}$

$x = \dfrac{y^2}{10} \qquad y^2(t) = 100e^{-4t}$

$y^2 = 10x$

74. $T(x, y) = 100 - x^2 - 2y^2, \quad P = (4, 3)$

$\dfrac{dx}{dt} = -2x \qquad \dfrac{dy}{dt} = -4y$

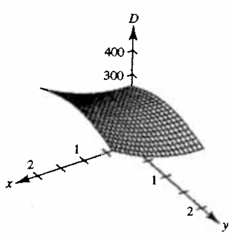
$x(t) = C_1e^{-2t} \qquad y(t) = C_2e^{-4t}$

$4 = x(0) = C_1 \qquad 3 = y(0) = C_2$

$x(t) = 4e^{-2t} \qquad y(t) = 3e^{-4t}$

$\dfrac{3x^2}{16} = e^{-4t} = y \implies u = \dfrac{3}{16}x^2$

75. (a)



(b) The graph of $-D = -250 - 30x^2 - 50\sin(\pi y/2)$ would model the ocean floor.

(c) $D(1, 0.5) = 250 + 30(1) + 50\sin\dfrac{\pi}{4} \approx 315.4$ ft

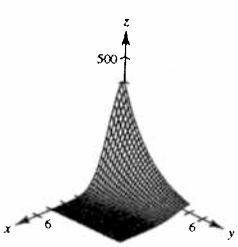
(d) $\dfrac{\partial D}{\partial x} = 60x$ and $\dfrac{\partial D}{\partial x}(1, 0.5) = 60$

(e) $\dfrac{\partial D}{\partial y} = 25\pi\cos\dfrac{\pi y}{2}$ and $\dfrac{\partial D}{\partial y}(1, 0.5) = 25\pi\cos\dfrac{\pi}{4} \approx 55.5$

(f) $\nabla D = 60x\mathbf{i} + 25\pi\cos\left(\dfrac{\pi y}{2}\right)\mathbf{j}$

$\nabla D(1, 0.5) = 60\mathbf{i} + 55.5\mathbf{j}$

76. (a)



(b) $\nabla T(x, y) = 400e^{-(x^2+y)/2}\left[(-x)\mathbf{i} - \frac{1}{2}\mathbf{j}\right]$

$\nabla T(3, 5) = 400e^{-7}\left[-3\mathbf{i} - \frac{1}{2}\mathbf{j}\right]$

There will be no change in directions perpendicular to the gradient: $\pm(\mathbf{i} - 6\mathbf{j})$

(c) The greatest increase is in the direction of the gradient: $-3\mathbf{i} - \frac{1}{2}\mathbf{j}$

77. True

78. False

$D_{\mathbf{u}} f(x, y) = \sqrt{2} > 1$ when

$\mathbf{u} = \left(\cos \frac{\pi}{4}\right)\mathbf{i} + \left(\sin \frac{\pi}{4}\right)\mathbf{j}.$

79. True

80. True

81. Let $f(x, y, z) = e^x \cos y + \frac{z^2}{2} + C$. Then

$\nabla f(x, y, z) = e^x \cos y\mathbf{i} - e^x \sin y\mathbf{j} + z\mathbf{k}.$

Section 13.7 Tangent Planes and Normal Lines

1. $F(x, y, z) = 3x - 5y + 3z - 15 = 0$

$3x - 5y + 3z = 15$ Plane

2. $F(x, y, z) = x^2 + y^2 + z^2 - 25 = 0$

$x^2 + y^2 + z^2 = 25$

Sphere, radius 5, centered at origin.

3. $F(x, y, z) = 4x^2 + 9y^2 - 4z^2 = 0$

$4x^2 + 9y^2 = 4z^2$ Elliptic cone

4. $F(x, y, z) = 16x^2 - 9y^2 + 144z = 0$

$16x^2 - 9y^2 + 144z = 0$ Hyperbolic paraboloid

5. $F(x, y, z) = x + y + z - 4$

$\nabla F = \mathbf{i} + \mathbf{j} + \mathbf{k}$

$\mathbf{n} = \frac{\nabla F}{\|\nabla F\|} = \frac{1}{\sqrt{3}}(\mathbf{i} + \mathbf{j} + \mathbf{k})$

$= \frac{\sqrt{3}}{3}(\mathbf{i} + \mathbf{j} + \mathbf{k})$

6. $F(x, y, z) = x^2 + y^2 + z^2 - 11$

$\nabla F(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$

$\nabla F(3, 1, 1) = 6\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$

$\mathbf{n} = \frac{\nabla F}{\|\nabla F\|} = \frac{1}{\sqrt{44}}(6\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})$

$= \frac{1}{\sqrt{11}}(3\mathbf{i} + \mathbf{j} + \mathbf{k}) = \frac{\sqrt{11}}{11}(3\mathbf{i} + \mathbf{j} + \mathbf{k})$

7. $F(x, y, z) = \sqrt{x^2 + y^2} - z$

$\nabla F(x, y, z) = \frac{x}{\sqrt{x^2 + y^2}}\mathbf{i} + \frac{y}{\sqrt{x^2 + y^2}}\mathbf{j} - \mathbf{k}$

$\nabla F(3, 4, 5) = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j} - \mathbf{k}$

$\mathbf{n} = \frac{\nabla F}{\|\nabla F\|} = \frac{5}{5\sqrt{2}}\left(\frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j} - \mathbf{k}\right)$

$= \frac{1}{5\sqrt{2}}(3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k})$

$= \frac{\sqrt{2}}{10}(3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k})$

8. $F(x, y, z) = x^3 - z$

$\nabla F(x, y, z) = 3x^2\mathbf{i} - \mathbf{k}$

$\nabla F(2, 1, 8) = 12\mathbf{i} - \mathbf{k}$

$\mathbf{n} = \frac{\nabla F}{\|\nabla F\|} = \frac{1}{\sqrt{145}}(12\mathbf{i} - \mathbf{k})$

$= \frac{\sqrt{145}}{145}(12\mathbf{i} - \mathbf{k})$

9. $F(x, y, z) = x^2y^4 - z$

$\nabla F(x, y, z) = 2xy^4\mathbf{i} + 4x^2y^3\mathbf{j} - \mathbf{k}$

$\nabla F(1, 2, 16) = 32\mathbf{i} + 32\mathbf{j} - \mathbf{k}$

$\mathbf{n} = \frac{\nabla F}{\|\nabla F\|} = \frac{1}{\sqrt{2049}}(32\mathbf{i} + 32\mathbf{j} - \mathbf{k})$

$= \frac{\sqrt{2049}}{2049}(32\mathbf{i} + 32\mathbf{j} - \mathbf{k})$

10. $F(x, y, z) = x^2 + 3y + z^3 - 9$

$\nabla F(x, y, z) = 2x\mathbf{i} + 3\mathbf{j} + 3z^2\mathbf{k}$

$\nabla F(2, -1, 2) = 4\mathbf{i} + 3\mathbf{j} + 12\mathbf{k}$

$\mathbf{n} = \frac{\nabla F}{\|\nabla F\|} = \frac{1}{13}(4\mathbf{i} + 3\mathbf{j} + 12\mathbf{k})$

11. $F(x, y, z) = \ln\left(\frac{x}{y-z}\right) = \ln x - \ln(y - z)$

$\nabla F(x, y, z) = \frac{1}{x}\mathbf{i} - \frac{1}{y-z}\mathbf{j} + \frac{1}{y-z}\mathbf{k}$

$\nabla F(1, 4, 3) = \mathbf{i} - \mathbf{j} + \mathbf{k}$

$\mathbf{n} = \frac{\nabla F}{\|\nabla F\|} = \frac{1}{\sqrt{3}}(\mathbf{i} - \mathbf{j} + \mathbf{k})$

$= \frac{\sqrt{3}}{3}(\mathbf{i} - \mathbf{j} + \mathbf{k})$

12. $F(x, y, z) = ze^{x^2-y^2} - 3$

$\nabla F(x, y, z) = 2xze^{x^2-y^2}\mathbf{i} - 2yze^{x^2-y^2}\mathbf{j} + e^{x^2-y^2}\mathbf{k}$

$\nabla F(2, 2, 3) = 12\mathbf{i} - 12\mathbf{j} + \mathbf{k}$

$\mathbf{n} = \frac{\nabla F}{\|\nabla F\|} = \frac{1}{17}(12\mathbf{i} - 12\mathbf{j} + \mathbf{k})$

13. $F(x, y, z) = -x \sin y + z - 4$

$\nabla F(x, y, z) = -\sin y\mathbf{i} - x\cos y\mathbf{j} + \mathbf{k}$

$\nabla F\left(6, \frac{\pi}{6}, 7\right) = -\frac{1}{2}\mathbf{i} - 3\sqrt{3}\mathbf{j} + \mathbf{k}$

$\mathbf{n} = \frac{\nabla F}{\|\nabla F\|} = \frac{2}{\sqrt{113}}\left(-\frac{1}{2}\mathbf{i} - 3\sqrt{3}\mathbf{j} + \mathbf{k}\right)$

$= \frac{1}{\sqrt{113}}(-\mathbf{i} - 6\sqrt{3}\mathbf{j} + 2\mathbf{k})$

$= \frac{\sqrt{113}}{113}(-\mathbf{i} - 6\sqrt{3}\mathbf{j} + 2\mathbf{k})$

14. $F(x, y, z) = \sin(x - y) - z - 2$

$\nabla F(x, y, z) = \cos(x - y)\mathbf{i} - \cos(x - y)\mathbf{j} - \mathbf{k}$

$\nabla F\left(\frac{\pi}{3}, \frac{\pi}{6}, -\frac{3}{2}\right) = \frac{\sqrt{3}}{2}\mathbf{i} - \frac{\sqrt{3}}{2}\mathbf{j} - \mathbf{k}$

$\mathbf{n} = \frac{\nabla F}{\|\nabla F\|} = \frac{2}{\sqrt{10}}\left(\frac{\sqrt{3}}{2}\mathbf{i} - \frac{\sqrt{3}}{2}\mathbf{j} - \mathbf{k}\right)$

$= \frac{1}{\sqrt{10}}(\sqrt{3}\mathbf{i} - \sqrt{3}\mathbf{j} - 2\mathbf{k})$

$= \frac{\sqrt{10}}{10}(\sqrt{3}\mathbf{i} - \sqrt{3}\mathbf{j} - 2\mathbf{k})$

15. $f(x, y) = 25 - x^2 - y^2$, $(3, 1, 15)$

$F(x, y, z) = 25 - x^2 - y^2 - z$

$F_x(x, y, z) = -2x$ $\quad F_y(x, y, z) = -2y$ $\quad F_z(x, y, z) = -1$

$F_x(3, 1, 15) = -6$ $\quad F_y(3, 1, 15) = -2$ $\quad F_z(3, 1, 15) = -1$

$-6(x - 3) - 2(y - 1) - (z - 15) = 0$

$0 = 6x + 2y + z - 35$

$6x + 2y + z = 35$

16. $f(x, y) = \frac{y}{x}$, $(1, 2, 2)$

$F(x, y, z) = \frac{y}{x} - z$

$F_x(x, y, z) = -\frac{y}{x^2}$ $\quad F_y(x, y, z) = \frac{1}{x}$ $\quad F_z(x, y, z) = -1$

$F_x(1, 2, 2) = -2$ $\quad F_y(1, 2, 2) = 1$ $\quad F_z(1, 2, 2) = -1$

$-2(x - 1) + (y - 2) - (z - 2) = 0$

$-2x + y - z + 2 = 0$

$2x - y + z = 2$

17. $f(x, y) = \sqrt{x^2 + y^2}$, $(3, 4, 5)$

$F(x, y, z) = \sqrt{x^2 + y^2} - z$

$F_x(x, y, z) = \dfrac{x}{\sqrt{x^2 + y^2}}$ $\quad F_y(x, y, z) = \dfrac{y}{\sqrt{x^2 + y^2}}$ $\quad F_z(x, y, z) = -1$

$F_x(3, 4, 5) = \dfrac{3}{5}$ $\quad F_y(3, 4, 5) = \dfrac{4}{5}$ $\quad F_z(3, 4, 5) = -1$

$\dfrac{3}{5}(x - 3) + \dfrac{4}{5}(y - 4) - (z - 5) = 0$

$3(x - 3) + 4(y - 4) - 5(z - 5) = 0$

$3x + 4y - 5z = 0$

18. $g(x, y) = \arctan \dfrac{y}{x}$, $(1, 0, 0)$

$G(x, y, z) = \arctan \dfrac{y}{x} - z$

$G_x(x, y, z) = \dfrac{-(y/x^2)}{1 + (y^2/x^2)} = \dfrac{-y}{x^2 + y^2}$ $\quad G_y(x, y, z) = \dfrac{1/x}{1 + (y^2/x^2)} = \dfrac{x}{x^2 + y^2}$ $\quad G_z(x, y, z) = -1$

$G_x(1, 0, 0) = 0$ $\quad G_y(1, 0, 0) = 1$ $\quad G_z(1, 0, 0) = -1$

$y - z = 0$

19. $g(x, y) = x^2 - y^2$, $(5, 4, 9)$

$G(x, y, z) = x^2 - y^2 - z$

$G_x(x, y, z) = 2x$ $\quad G_y(x, y, z) = -2y$ $\quad G_z(x, y, z) = -1$

$G_x(5, 4, 9) = 10$ $\quad G_y(5, 4, 9) = -8$ $\quad G_z(5, 4, 9) = -1$

$10(x - 5) - 8(y - 4) - (z - 9) = 0$

$10x - 8y - z = 9$

20. $f(x, y) = 2 - \frac{2}{3}x - y$, $(3, -1, 1)$

$F(x, y, z) = 2 - \frac{2}{3}x - y - z$

$F_x(x, y, z) = -\frac{2}{3}$, $\quad F_y(x, y, z) = -1$, $\quad F_z(x, y, z) = -1$

$-\frac{2}{3}(x - 3) - (y + 1) - (z - 1) = 0$

$-\frac{2}{3}x - y - z + 2 = 0$

$2x + 3y + 3z = 6$

21. $z = e^x(\sin y + 1)$, $\left(0, \dfrac{\pi}{2}, 2\right)$

$F(x, y, z) = e^x(\sin y + 1) - z$

$F_x(x, y, z) = e^x(\sin y + 1)$ $\quad F_y(x, y, z) = e^x \cos y$ $\quad F_z(x, y, z) = -1$

$F_x\left(0, \dfrac{\pi}{2}, 2\right) = 2$ $\quad F_y\left(0, \dfrac{\pi}{2}, 2\right) = 0$ $\quad F_z\left(0, \dfrac{\pi}{2}, 2\right) = -1$

$2x - z = -2$

22. $z = x^2 - 2xy + y^2$, $(1, 2, 1)$

$F(x, y, z) = x^2 - 2xy + y^2 - z$

$F_x(x, y, z) = 2x - 2y \qquad F_y(x, y, z) = -2x + 2y \qquad F_z(x, y, z) = -1$

$F_x(1, 2, 1) = -2 \qquad F_y(1, 2, 1) = 2 \qquad F_z(1, 2, 1) = -1$

$-2(x - 1) + 2(y - 2) - (z - 1) = 0$

$-2x + 2y - z - 1 = 0$

$2x - 2y + z = -1$

23. $h(x, y) = \ln\sqrt{x^2 + y^2}$, $(3, 4, \ln 5)$

$H(x, y, z) = \ln\sqrt{x^2 + y^2} - z = \frac{1}{2}\ln(x^2 + y^2) - z$

$H_x(x, y, z) = \dfrac{x}{x^2 + y^2} \qquad H_y(x, y, z) = \dfrac{y}{x^2 + y^2} \qquad H_z(x, y, z) = -1$

$H_x(3, 4, \ln 5) = \dfrac{3}{25} \qquad H_y(3, 4, \ln 5) = \dfrac{4}{25} \qquad H_z(3, 4, \ln 5) = -1$

$\frac{3}{25}(x - 3) + \frac{4}{25}(y - 4) - (z - \ln 5) = 0$

$3(x - 3) + 4(y - 4) - 25(z - \ln 5) = 0$

$3x + 4y - 25z = 25(1 - \ln 5)$

24. $h(x, y) = \cos y$, $\left(5, \frac{\pi}{4}, \frac{\sqrt{2}}{2}\right)$

$H(x, y, z) = \cos y - z$

$H_x(x, y, z) = 0 \qquad H_y(x, y, z) = -\sin y \qquad H_z(x, y, z) = -1$

$H_x\left(5, \frac{\pi}{4}, \frac{\sqrt{2}}{2}\right) = 0 \qquad H_y\left(5, \frac{\pi}{4}, \frac{\sqrt{2}}{2}\right) = -\frac{\sqrt{2}}{2} \qquad H_z\left(5, \frac{\pi}{4}, \frac{\sqrt{2}}{2}\right) = -1$

$-\frac{\sqrt{2}}{2}\left(y - \frac{\pi}{4}\right) - \left(z - \frac{\sqrt{2}}{2}\right) = 0$

$-\frac{\sqrt{2}}{2}y - z + \frac{\sqrt{2}\pi}{8} + \frac{\sqrt{2}}{2} = 0$

$4\sqrt{2}y + 8z = \sqrt{2}(\pi + 4)$

25. $x^2 + 4y^2 + z^2 = 36$, $(2, -2, 4)$

$F(x, y, z) = x^2 + 4y^2 + z^2 - 36$

$F_x(x, y, z) = 2x \qquad F_y(x, y, z) = 8y \qquad F_z(x, y, z) = 2z$

$F_x(2, -2, 4) = 4 \qquad F_y(2, -2, 4) = -16 \qquad F_z(2, -2, 4) = 8$

$4(x - 2) - 16(y + 2) + 8(z - 4) = 0$

$(x - 2) - 4(y + 2) + 2(z - 4) = 0$

$x - 4y + 2z = 18$

26. $x^2 + 2z^2 = y^2,\ (1, 3, -2)$

$F(x, y, z) = x^2 - y^2 + 2z^2$

$F_x(x, y, z) = 2x \qquad F_y(x, y, z) = -2y \qquad F_z(x, y, z) = 4z$

$F_x(1, 3, -2) = 2 \qquad F_y(1, 3, -2) = -6 \qquad F_z(1, 3, -2) = -8$

$2(x - 1) - 6(y - 3) - 8(z + 2) = 0$

$(x - 1) - 3(y - 3) - 4(z + 2) = 0$

$x - 3y - 4z = 0$

27. $xy^2 + 3x - z^2 = 4,\ (2, 1, -2)$

$F(x, y, z) = xy^2 + 3x - z^2 - 4$

$F_x(x, y, z) = y^2 + 3 \qquad F_y(x, y, z) = 2xy \qquad F_z(x, y, z) = -2z$

$F_x(2, 1, -2) = 4 \qquad F_y(2, 1, -2) = 4 \qquad F_z(2, 1, -2) = 4$

$4(x - 2) + 4(y - 1) + 4(z + 2) = 0$

$x + y + z = 1$

28. $x = y(2z - 3),\ (4, 4, 2)$

$F(x, y, z) = x - 2yz + 3y$

$F_x(x, y, z) = 1 \qquad F_y(x, y, z) = -2z + 3 \qquad F_z(x, y, z) = -2y$

$F_x(4, 4, 2) = 1 \qquad F_y(4, 4, 2) = -1 \qquad F_z(4, 4, 2) = -8$

$(x - 4) - 1(y - 4) - 8(z - 2) = 0$

$x - y - 8z = -16$

$-x + y + 8z = 16$

29. $x^2 + y^2 + z = 9,\ (1, 2, 4)$

$F(x, y, z) = x^2 + y^2 + z - 9$

$F_x(x, y, z) = 2x \qquad F_y(x, y, z) = 2y \qquad F_z(x, y, z) = 1$

$F_x(1, 2, 4) = 2 \qquad F_y(1, 2, 4) = 4 \qquad F_z(1, 2, 4) = 1$

Direction numbers: 2, 4, 1

Plane: $2(x - 1) + 4(y - 2) + (z - 4) = 0,\ 2x + 4y + z = 14$

Line: $\dfrac{x - 1}{2} = \dfrac{y - 2}{4} = \dfrac{z - 4}{1}$

30. $x^2 + y^2 + z^2 = 9,\ (1, 2, 2)$

$F(x, y, z) = x^2 + y^2 + z^2 - 9$

$F_x(x, y, z) = 2x \qquad F_y(x, y, z) = 2y \qquad F_z(x, y, z) = 2z$

$F_x(1, 2, 2) = 2 \qquad F_y(1, 2, 2) = 4 \qquad F_z(1, 2, 2) = 4$

Direction numbers: 1, 2, 2

Plane: $(x - 1) + 2(y - 2) + 2(z - 2) = 0,\ x + 2y + 2z = 9$

Line: $\dfrac{x - 1}{1} = \dfrac{y - 2}{2} = \dfrac{z - 2}{2}$

31. $xy - z = 0,\ (-2, -3, 6)$

$F(x, y, z) = xy - z$

$F_x(x, y, z) = y \qquad F_y(x, y, z) = x \qquad F_z(x, y, z) = -1$

$F_x(-2, -3, 6) = -3 \qquad F_y(-2, -3, 6) = -2 \qquad F_z(-2, -3, 6) = -1$

Direction numbers: 3, 2, 1

Plane: $3(x + 2) + 2(y + 3) + (z - 6) = 0,\ 3x + 2y + z = -6$

Line: $\dfrac{x+2}{3} = \dfrac{y+3}{2} = \dfrac{z-6}{1}$

32. $x^2 - y^2 + z^2 = 0, (5, 13, -12)$

$F(x, y, z) = x^2 - y^2 + z^2$

$F_x(x, y, z) = 2x \qquad F_y(x, y, z) = -2y \qquad F_z(x, y, z) = 2z$

$F_x(5, 13, -12) = 10 \qquad F_y(5, 13, -12) = -26 \qquad F_z(x, y, z) = -24$

Direction numbers: $5, -13, -12$

Plane

$$5(x - 5) - 13(y - 13) - 12(z + 12) = 0$$

$$5x - 13y - 12z = 0$$

Line: $\dfrac{x-5}{5} = \dfrac{y-13}{-13} = \dfrac{z+12}{-12}$

33. $z = \arctan\dfrac{y}{x},\ \left(1, 1, \dfrac{\pi}{4}\right)$

$F(x, y, z) = \arctan\dfrac{y}{x} - z$

$F_x(x, y, z) = \dfrac{-y}{x^2 + y^2} \qquad F_y(x, y, z) = \dfrac{x}{x^2 + y^2} \qquad F_z(x, y, z) = -1$

$F_x\left(1, 1, \dfrac{\pi}{4}\right) = -\dfrac{1}{2} \qquad F_y\left(1, 1, \dfrac{\pi}{4}\right) = \dfrac{1}{2} \qquad F_z\left(1, 1, \dfrac{\pi}{4}\right) = -1$

Direction numbers: $1, -1, 2$

Plane: $(x - 1) - (y - 1) + 2\left(z - \dfrac{\pi}{4}\right) = 0,\ x - y + 2z = \dfrac{\pi}{2}$

Line: $\dfrac{x-1}{1} = \dfrac{y-1}{-1} = \dfrac{z - (\pi/4)}{2}$

34. $xyz = 10,\ (1, 2, 5)$

$F(x, y, z) = xyz - 10$

$F_x(x, y, z) = yz \qquad F_y(x, y, z) = xz \qquad F_z(x, y, z) = xy$

$F_x(1, 2, 5) = 10 \qquad F_y(1, 2, 5) = 5 \qquad F_z(1, 2, 5) = 2$

Direction numbers: 10, 5, 2

Plane: $10(x - 1) + 5(y - 2) + 2(z - 5) = 0,\ 10x + 5y + 2z = 30$

Line: $\dfrac{x-1}{10} = \dfrac{y-2}{5} = \dfrac{z-5}{2}$

35. $z = f(x, y) = \dfrac{4xy}{(x^2+1)(y^2+1)}, \quad -2 \le x \le 2,\ 0 \le y \le 3$

(a) Let $F(x, y, z) = \dfrac{4xy}{(x^2+1)(y^2+1)} - z$

$$\nabla F(x, y, z) = \frac{4y}{y^2+1}\left(\frac{x^2+1-2x^2}{(x^2+1)^2}\right)\mathbf{i} + \frac{4x}{x^2+1}\left(\frac{y^2+1-2y^2}{(y^2+1)^2}\right)\mathbf{j} - \mathbf{k}$$

$$= \frac{4y(1-x^2)}{(y^2+1)(x^2+1)^2}\mathbf{i} + \frac{4x(1-y^2)}{(x^2+1)(y^2+1)^2}\mathbf{j} - \mathbf{k}$$

$\nabla F(1, 1, 1) = -\mathbf{k}$

Direction numbers: $0, 0, -1$

Line: $x = 1,\ y = 1,\ z = 1 - t$

Tangent plane: $0(x-1) + 0(y-1) - 1(z-1) = 0 \Rightarrow z = 1$

(b) $\nabla F\left(-1, 2, -\dfrac{4}{5}\right) = 0\mathbf{i} + \dfrac{-4(-3)}{(2)(5)^2}\mathbf{j} - \mathbf{k} = \dfrac{6}{25}\mathbf{j} - \mathbf{k}$

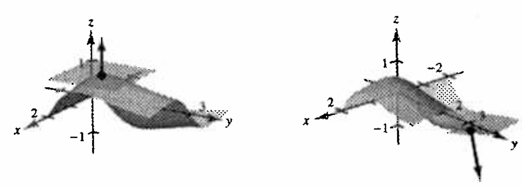
Line: $x = -1,\ y = 2 + \dfrac{6}{25}t,\ z = -\dfrac{4}{5} - t$

Plane: $0(x+1) + \dfrac{6}{25}(y-2) - 1\left(z + \dfrac{4}{5}\right) = 0$

$$6y - 12 - 25z - 20 = 0$$

$$6y - 25z - 32 = 0$$

(c)



(d) At $(1, 1, 1)$, the tangent plane is parallel to the xy-plane, implying that the surface is level there. At $\left(-1, 2, -\frac{4}{5}\right)$, the function does not change in the x-direction.

36. (a) $f(x, y) = \dfrac{\sin y}{x}, \quad -3 \le x \le 3,\ 0 \le y \le 2\pi$

Let $F(x, y, z) = \dfrac{\sin y}{x} - z$

$\nabla F(x, y, z) = \dfrac{-\sin y}{x^2}\mathbf{i} + \dfrac{\cos y}{x}\mathbf{j} - \mathbf{k}$

$\nabla F\left(2, \dfrac{\pi}{2}, \dfrac{1}{2}\right) = -\dfrac{1}{4}\mathbf{i} - \mathbf{k}$

Direction numbers: $-\dfrac{1}{4}, 0, -1$ or $1, 0, 4$

Line: $x = 2 + t,\ y = \dfrac{\pi}{2},\ z = \dfrac{1}{2} + 4t$

Tangent plane: $1(x-2) + 0\left(y - \dfrac{\pi}{2}\right) + 4\left(z - \dfrac{1}{2}\right) = 0 \Rightarrow x + 4z - 4 = 0$

(b) $\nabla F\left(-\dfrac{2}{3}, \dfrac{3\pi}{2}, \dfrac{3}{2}\right) = \dfrac{9}{4}\mathbf{i} - \mathbf{k}$

Direction numbers: $\dfrac{9}{4}, 0, -1$ or $9, 0, -4$

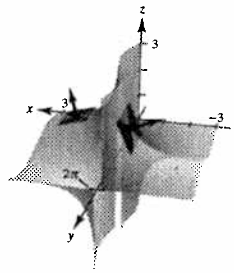
Line: $x = -\dfrac{2}{3} + 9t,\ y = \dfrac{3\pi}{2},\ z = \dfrac{3}{2} - 4t$

Tangent plane: $9\left(x + \dfrac{2}{3}\right) + 0\left(y - \dfrac{3\pi}{2}\right) - 4\left(z - \dfrac{3}{2}\right) = 0 \Rightarrow 9x - 4z + 12 = 0$

—CONTINUED—

36. —CONTINUED—

(c)



(d) At both points the function does not change in the y-direction.

37. See the definition on page 944.

38. $F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$

(Theorem 13.13)

39. For a sphere, the common object is the center of the sphere. For a right circular cylinder, the common object is the axis of the cylinder.

40. Answers will vary.

41. $F(x, y, z) = x^2 + y^2 - 5$ $\qquad$ $G(x, y, z) = x - z$

$\nabla F(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j}$ $\qquad$ $\nabla G(x, y, z) = \mathbf{i} - \mathbf{k}$

$\nabla F(2, 1, 2) = 4\mathbf{i} + 2\mathbf{j}$ $\qquad$ $\nabla G(2, 1, 2) = \mathbf{i} - \mathbf{k}$

(a) $\nabla F \times \nabla G = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 2 & 0 \\ 1 & 0 & -1 \end{vmatrix} = -2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k} = -2(\mathbf{i} - 2\mathbf{j} + \mathbf{k})$

Direction numbers: $1, -2, 1$. $\dfrac{x-2}{1} = \dfrac{y-1}{-2} = \dfrac{z-2}{1}$

(b) $\cos\theta = \dfrac{|\nabla F \cdot \nabla G|}{\|\nabla F\|\,\|\nabla G\|} = \dfrac{4}{\sqrt{20}\sqrt{2}} = \dfrac{2}{\sqrt{10}} = \dfrac{\sqrt{10}}{5}$; not orthogonal

42. $F(x, y, z) = x^2 + y^2 - z$ $\qquad$ $G(x, y, z) = 4 - y - z$

$\nabla F(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k}$ $\qquad$ $\nabla G(x, y, z) = -\mathbf{j} - \mathbf{k}$

$\nabla F(2, -1, 5) = 4\mathbf{i} - 2\mathbf{j} - \mathbf{k}$ $\qquad$ $\nabla G(2, -1, 5) = -\mathbf{j} - \mathbf{k}$

(a) $\nabla F \times \nabla G = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -2 & -1 \\ 0 & -1 & -1 \end{vmatrix} = \mathbf{i} + 4\mathbf{j} - 4\mathbf{k}$

Direction numbers: $1, 4, -4$. $\dfrac{x-2}{1} = \dfrac{y+1}{4} = \dfrac{z-5}{-4}$

(b) $\cos\theta = \dfrac{|\nabla F \cdot \nabla G|}{\|\nabla F\|\,\|\nabla G\|} = \dfrac{3}{\sqrt{21}\sqrt{2}} = \dfrac{3}{\sqrt{42}} = \dfrac{\sqrt{42}}{14}$; not orthogonal

43. $F(x, y, z) = x^2 + z^2 - 25$ $G(x, y, z) = y^2 + z^2 - 25$

$\nabla F = 2x\mathbf{i} + 2z\mathbf{k}$ $\nabla G = 2y\mathbf{j} + 2z\mathbf{k}$

$\nabla F(3, 3, 4) = 6\mathbf{i} + 8\mathbf{k}$ $\nabla G(3, 3, 4) = 6\mathbf{j} + 8\mathbf{k}$

(a) $\nabla F \times \nabla G = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6 & 0 & 8 \\ 0 & 6 & 8 \end{vmatrix} = -48\mathbf{i} - 48\mathbf{j} + 36\mathbf{k} = -12(4\mathbf{i} + 4\mathbf{j} - 3\mathbf{k})$

Direction numbers: $4, 4, -3$. $\dfrac{x-3}{4} = \dfrac{y-3}{4} = \dfrac{z-4}{-3}$

(b) $\cos\theta = \dfrac{|\nabla F \cdot \nabla G|}{\|\nabla F\|\,\|\nabla G\|} = \dfrac{64}{(10)(10)} = \dfrac{16}{25}$; not orthogonal

44. $F(x, y, z) = \sqrt{x^2 + y^2} - z$ $G(x, y, z) = 5x - 2y + 3z = 22$

$\nabla F(x, y, z) = \dfrac{x}{\sqrt{x^2 + y^2}}\mathbf{i} + \dfrac{y}{\sqrt{x^2 + y^2}}\mathbf{j} - \mathbf{k}$ $\nabla G(x, y, z) = 5\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$

$\nabla F(3, 4, 5) = \dfrac{3}{5}\mathbf{i} + \dfrac{4}{5}\mathbf{j} - \mathbf{k}$ $\nabla G(3, 4, 5) = 5\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$

$\nabla F \times \nabla G = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3/5 & 4/5 & -1 \\ 5 & -2 & 3 \end{vmatrix} = \dfrac{2}{5}\mathbf{i} - \dfrac{34}{5}\mathbf{j} - \dfrac{26}{5}\mathbf{k}$

Direction numbers: $1, -17, -13$

$\dfrac{x-3}{1} = \dfrac{y-4}{-17} = \dfrac{z-5}{-13}$ Tangent line

$\cos\theta = \dfrac{|\nabla F \cdot \nabla G|}{\|\nabla F\|\,\|\nabla G\|} = \dfrac{-(8/5)}{\sqrt{2}\sqrt{38}} = \dfrac{-8}{5\sqrt{76}}$ Not orthogonal

45. $F(x, y, z) = x^2 + y^2 + z^2 - 6$ $G(x, y, z) = x - y - z$

$\nabla F(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ $\nabla G(x, y, z) = \mathbf{i} - \mathbf{j} - \mathbf{k}$

$\nabla F(2, 1, 1) = 4\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ $\nabla G(2, 1, 1) = \mathbf{i} - \mathbf{j} - \mathbf{k}$

(a) $\nabla F \times \nabla G = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 2 & 2 \\ 1 & -1 & -1 \end{vmatrix} = 6\mathbf{j} - 6\mathbf{k} = 6(\mathbf{j} - \mathbf{k})$

Direction numbers: $0, 1, -1$. $x = 2, \dfrac{y-1}{1} = \dfrac{z-1}{-1}$

(b) $\cos\theta = \dfrac{|\nabla F \cdot \nabla G|}{\|\nabla F\|\,\|\nabla G\|} = 0$; orthogonal

46. $F(x, y, z) = x^2 + y^2 - z$ $G(x, y, z) = x + y + 6z - 33$

$\nabla F(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k}$ $\nabla G(x, y, z) = \mathbf{i} + \mathbf{j} + 6\mathbf{k}$

$\nabla F(1, 2, 5) = 2\mathbf{i} + 4\mathbf{j} - \mathbf{k}$ $\nabla G(1, 2, 5) = \mathbf{i} + \mathbf{j} + 6\mathbf{k}$

(a) $\nabla F \times \nabla G = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 4 & -1 \\ 1 & 1 & 6 \end{vmatrix} = 25\mathbf{i} - 13\mathbf{j} - 2\mathbf{k}$

Direction numbers: $25, -13, -2$. $\dfrac{x-1}{25} = \dfrac{y-2}{-13} = \dfrac{z-5}{-2}$

(b) $\cos\theta = \dfrac{|\nabla F \cdot \nabla G|}{\|\nabla F\|\,\|\nabla G\|} = 0$; orthogonal

47. $f(x, y) = 6 - x^2 - \frac{y^2}{4}$, $g(x, y) = 2x + y$

(a) $F(x, y, z) = z + x^2 + \frac{y^2}{4} - 6$ $\qquad G(x, y, z) = z - 2x - y$

$\nabla F(x, y, z) = 2x\mathbf{i} + \frac{1}{2}y\mathbf{j} + \mathbf{k}$ $\qquad \nabla G(x, y, z) = -2\mathbf{i} - \mathbf{j} + \mathbf{k}$

$\nabla F(1, 2, 4) = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$ $\qquad \nabla G(1, 2, 4) = -2\mathbf{i} - \mathbf{j} + \mathbf{k}$

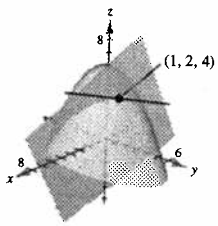
The cross product of these gradients is parallel to the curve of intersection.

$$\nabla F(1, 2, 4) \times \nabla G(1, 2, 4) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 1 \\ -2 & -1 & 1 \end{vmatrix} = 2\mathbf{i} - 4\mathbf{j}$$

Using direction numbers $1, -2, 0$, you get $x = 1 + t$, $y = 2 - 2t$, $z = 4$.

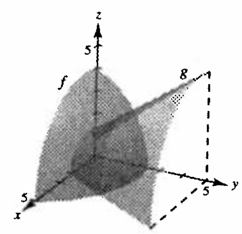
$$\cos\theta = \frac{\nabla F \cdot \nabla G}{\|\nabla F\|\,\|\nabla G\|} = \frac{-4 - 1 + 1}{\sqrt{6}\,\sqrt{6}} = \frac{-4}{6} \implies \theta \approx 48.2°$$

(b)



48. (a) $f(x, y) = \sqrt{16 - x^2 - y^2 + 2x - 4y}$

$g(x, y) = \frac{\sqrt{2}}{2}\sqrt{1 - 3x^2 + y^2 + 6x + 4y}$



(b)

$$f(x, y) = g(x, y)$$

$$16 - x^2 - y^2 + 2x - 4y = \frac{1}{2}(1 - 3x^2 + y^2 + 6x + 4y)$$

$$32 - 2x^2 - 2y^2 + 4x - 8y = 1 - 3x^2 + y^2 + 6x + 4y$$

$$x^2 - 2x + 31 = 3y^2 + 12y$$

$$(x^2 - 2x + 1) + 42 = 3(y^2 + 4y + 4)$$

$$(x - 1)^2 + 42 = 3(y + 2)^2$$

To find points of intersection, let $x = 1$. Then

$$3(y + 2)^2 = 42$$

$$(y + 2)^2 = 14$$

$$y = -2 \pm \sqrt{14}$$

$\nabla f\left(1, -2 + \sqrt{14}\right) = -\sqrt{2}\mathbf{j}$, $\nabla g\left(1, -2 + \sqrt{14}\right) = \left(1/\sqrt{2}\right)\mathbf{j}$. The normals to f and g at this point are $-\sqrt{2}\mathbf{j} - \mathbf{k}$ and $\left(1/\sqrt{2}\right)\mathbf{j} - \mathbf{k}$, which are orthogonal.

Similarly, $\nabla f\left(1, -2 - \sqrt{14}\right) = \sqrt{2}\mathbf{j}$ and $\nabla g\left(1, -2 - \sqrt{14}\right) = \left(-1/\sqrt{2}\right)\mathbf{j}$ and the normals are $\sqrt{2}\mathbf{j} - \mathbf{k}$ and $\left(-1/\sqrt{2}\right)\mathbf{j} - \mathbf{k}$, which are also orthogonal.

(c) No, showing that the surfaces are orthogonal at 2 points does not imply that they are orthogonal at every point of intersection.

49. $F(x, y, z) = 3x^2 + 2y^2 - z - 15,\ (2, 2, 5)$

$\nabla F(x, y, z) = 6x\mathbf{i} + 4y\mathbf{j} - \mathbf{k}$

$\nabla F(2, 2, 5) = 12\mathbf{i} + 8\mathbf{j} - \mathbf{k}$

$$\cos\theta = \frac{|\nabla F(2, 2, 5) \cdot \mathbf{k}|}{\|\nabla F(2, 2, 5)\|} = \frac{1}{\sqrt{209}}$$

$$\theta = \arccos\left(\frac{1}{\sqrt{209}}\right) \approx 86.03°$$

50. $F(x, y, z) = 2xy - z^3,\ (2, 2, 2)$

$\nabla F = 2y\mathbf{i} + 2x\mathbf{j} - 3z^2\mathbf{k}$

$\nabla F(2, 2, 2) = 4\mathbf{i} + 4\mathbf{j} - 12\mathbf{k}$

$$\cos\theta = \frac{|\nabla F(2, 2, 2) \cdot \mathbf{k}|}{\|\nabla F(2, 2, 2)\|} = \frac{|-12|}{\sqrt{176}} = \frac{3\sqrt{11}}{11}$$

$$\theta = \arccos\left(\frac{3\sqrt{11}}{11}\right) \approx 25.24°$$

51. $F(x, y, z) = x^2 - y^2 + z,\ (1, 2, 3)$

$\nabla F(x, y, z) = 2x\mathbf{i} - 2y\mathbf{j} + \mathbf{k}$

$\nabla F(1, 2, 3) = 2\mathbf{i} - 4\mathbf{j} + \mathbf{k}$

$$\cos\theta = \frac{|\nabla F(1, 2, 3) \cdot \mathbf{k}|}{\|\nabla F(1, 2, 3)\|} = \frac{1}{\sqrt{21}}$$

$$\theta = \arccos\frac{1}{\sqrt{21}} \approx 77.40°$$

52. $F(x, y, z) = x^2 + y^2 - 5,\ (2, 1, 3)$

$\nabla F(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j}$

$\nabla F(2, 1, 3) = 4\mathbf{i} + 2\mathbf{j}$

$$\cos\theta = \frac{|\nabla F(2, 1, 3) \cdot \mathbf{k}|}{\|\nabla F(2, 1, 3)\|} = 0$$

$$\theta = \arccos 0 = 90°$$

53. $F(x, y, z) = 3 - x^2 - y^2 + 6y - z$

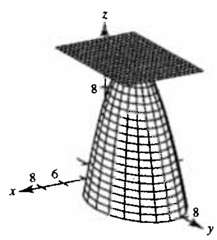
$\nabla F(x, y, z) = -2x\mathbf{i} + (-2y + 6)\mathbf{j} - \mathbf{k}$

$-2x = 0,\ x = 0$

$-2y + 6 = 0,\ y = 3$

$z = 3 - 0^2 - 3^2 + 6(3) = 12$

$(0, 3, 12)$ (vertex of paraboloid)



54. $F(x, y, z) = 3x^2 + 2y^2 - 3x + 4y - z - 5$

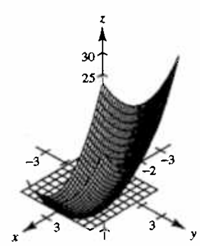
$\nabla F(x, y, z) = (6x - 3)\mathbf{i} + (4y + 4)\mathbf{j} - \mathbf{k}$

$6x - 3 = 0,\ x = \frac{1}{2}$

$4y + 4 = 0,\ y = -1$

$z = 3\left(\frac{1}{2}\right)^2 + 2(-1)^2 - 3\left(\frac{1}{2}\right) + 4(-1) - 5 = -\frac{31}{4}$

$\left(\frac{1}{2}, -1, -\frac{31}{4}\right)$



55. $T(x, y, z) = 400 - 2x^2 - y^2 - 4z^2,\ (4, 3, 10)$

$\frac{dx}{dt} = -4kx$	$\frac{dy}{dt} = -2ky$	$\frac{dz}{dt} = -8kz$
$x(t) = C_1e^{-4kt}$	$y(t) = C_2e^{-2kt}$	$z(t) = C_3e^{-8kt}$
$x(0) = C_1 = 4$	$y(0) = C_2 = 3$	$z(0) = C_3 = 10$
$x = 4e^{-4kt}$	$y = 3e^{-2kt}$	$z = 10e^{-8kt}$

56. $T(x, y, z) = 100 - 3x - y - z^2,\ (2, 2, 5)$

$\frac{dx}{dt} = -3$	$\frac{dy}{dt} = -1$	$\frac{dz}{dt} = -2z$
$x(t) = -3t + C_1$	$y(t) = -t + C_2$	$z(t) = C_3e^{-2t}$
$x(0) = C_1 = 2$	$y(0) = C_2 = 2$	$z(0) = C_3 = 5$
$x = -3t + 2$	$y = -t + 2$	$z = 5e^{-2t}$

57. $F(x, y, z) = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} - 1$

$F_x(x, y, z) = \dfrac{2x}{a^2}$

$F_y(x, y, z) = \dfrac{2y}{b^2}$

$F_z(x, y, z) = \dfrac{2z}{c^2}$

Plane: $\dfrac{2x_0}{a^2}(x - x_0) + \dfrac{2y_0}{b^2}(y - y_0) + \dfrac{2z_0}{c^2}(z - z_0) = 0$

$$\frac{x_0x}{a^2} + \frac{y_0y}{b^2} + \frac{z_0z}{c^2} = \frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} + \frac{z_0^2}{c^2} = 1$$

58. $F(x, y, z) = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} - 1$

$F_x(x, y, z) = \dfrac{2x}{a^2}$

$F_y(x, y, z) = \dfrac{2y}{b^2}$

$F_z(x, y, z) = \dfrac{-2z}{c^2}$

Plane: $\dfrac{2x_0}{a^2}(x - x_0) + \dfrac{2y_0}{b^2}(y - y_0) - \dfrac{2z_0}{c^2}(z - z_0) = 0$

$$\frac{x_0x}{a^2} + \frac{y_0y}{b^2} - \frac{z_0z}{c^2} = \frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} - \frac{z_0^2}{c^2} = 1$$

59. $F(x, y, z) = a^2x^2 + b^2y^2 - z^2$

$F_x(x, y, z) = 2a^2x$

$F_y(x, y, z) = 2b^2y$

$F_z(x, y, z) = -2z$

Plane: $2a^2x_0(x - x_0) + 2b^2y_0(y - y_0) - 2z_0(z - z_0) = 0$

$$a^2x_0x + b^2y_0y - z_0z = a^2x_0^2 + b^2y_0^2 - z_0^2 = 0$$

Hence, the plane passes through the origin.

60. $z = xf\left(\dfrac{y}{x}\right)$

$F(x, y, z) = xf\left(\dfrac{y}{x}\right) - z$

$F_x(x, y, z) = f\left(\dfrac{y}{x}\right) + xf'\left(\dfrac{y}{x}\right)\left(-\dfrac{y}{x^2}\right) = f\left(\dfrac{y}{x}\right) - \dfrac{y}{x}f'\left(\dfrac{y}{x}\right)$

$F_y(x, y, z) = xf'\left(\dfrac{y}{x}\right)\left(\dfrac{1}{x}\right) = f'\left(\dfrac{y}{x}\right)$

$F_z(x, y, z) = -1$

Tangent plane at (x_0, y_0, z_0):

$$\left[f\left(\frac{y_0}{x_0}\right) - \frac{y_0}{x_0}f'\left(\frac{y_0}{x_0}\right)\right](x - x_0) + f'\left(\frac{y_0}{x_0}\right)(y - y_0) - (z - z_0) = 0$$

$$\left[f\left(\frac{y_0}{x_0}\right) - \frac{y_0}{x_0}f'\left(\frac{y_0}{x_0}\right)\right]x - x_0f\left(\frac{y_0}{x_0}\right) + y_0f'\left(\frac{y_0}{x_0}\right) + yf'\left(\frac{y_0}{x_0}\right) - y_0f'\left(\frac{y_0}{x_0}\right) - z + x_0f\left(\frac{y_0}{x_0}\right) = 0$$

$$\left[f\left(\frac{y_0}{x_0}\right) - \frac{y_0}{x_0}f'\left(\frac{y_0}{x_0}\right)\right]x + f'\left(\frac{y_0}{x_0}\right)y - z = 0$$

Therefore, the plane passes through the origin $(x, y, z) = (0, 0, 0)$.

61. $f(x, y) = e^{x-y}$

$f_x(x, y) = e^{x-y}$, $\quad f_y(x, y) = -e^{x-y}$

$f_{xx}(x, y) = e^{x-y}$, $\quad f_{yy}(x, y) = e^{x-y}$, $\quad f_{xy}(x, y) = -e^{x-y}$

(a) $P_1(x, y) \approx f(0, 0) + f_x(0, 0)x + f_y(0, 0)y = 1 + x - y$

—CONTINUED—

61. —CONTINUED—

(b) $P_2(x, y) \approx f(0, 0) + f_x(0, 0)x + f_y(0,0)y + \frac{1}{2}f_{xx}(0, 0)x^2 + f_{xy}(0, 0)xy + \frac{1}{2}f_{yy}(0, 0)y^2$

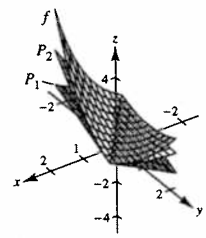
$= 1 + x - y + \frac{1}{2}x^2 - xy + \frac{1}{2}y^2$

(c) If $x = 0$, $P_2(0, y) = 1 - y + \frac{1}{2}y^2$. This is the second-degree Taylor polynomial for e^{-y}.

If $y = 0$, $P_2(x, 0) = 1 + x + \frac{1}{2}x^2$. This is the second-degree Taylor polynomial for e^x.

(d)

x	y	$f(x, y)$	$P_1(x, y)$	$P_2(x, y)$
0	0	1	1	1
0	0.1	0.9048	0.9000	0.9050
0.2	0.1	1.1052	1.1000	1.1050
0.2	0.5	0.7408	0.7000	0.7450
1	0.5	1.6487	1.5000	1.6250

(e) 

62. $f(x, y) = \cos(x + y)$

$f_x(x, y) = -\sin(x + y)$ $\qquad$ $f_y(x, y) = -\sin(x + y)$

$f_{xx}(x, y) = -\cos(x + y)$, $\qquad$ $f_{yy}(x, y) = -\cos(x + y)$, $\qquad$ $f_{xy}(x, y) = -\cos(x + y)$

(a) $P_1(x, y) \approx f(0, 0) + f_x(0, 0)x + f_y(0, 0)y = 1$

(b) $P_2(x, y) \approx f(0, 0) + f_x(0, 0)x + f_y(0,0)y + \frac{1}{2}f_{xx}(0, 0)x^2 + f_{xy}(0, 0)xy + \frac{1}{2}f_{yy}(0, 0)y^2$

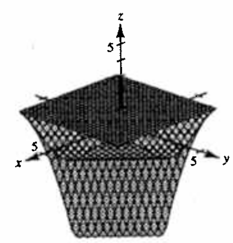
$= 1 - \frac{1}{2}x^2 - xy - \frac{1}{2}y^2$

(c) If $x = 0$, $P_2(0, y) = 1 - \frac{1}{2}y^2$. This is the second-degree Taylor polynomial for $\cos y$.

If $y = 0$, $P_2(x, 0) = 1 - \frac{1}{2}x^2$. This is the second-degree Taylor polynomial for $\cos x$.

(d)

x	y	$f(x, y)$	$P_1(x, y)$	$P_2(x, y)$
0	0	1	1	1
0	0.1	0.9950	1	0.9950
0.2	0.1	0.9553	1	0.9950
0.2	0.5	0.7648	1	0.7550
1	0.5	0.0707	1	−0.1250

(e) 

63. Given $w = F(x, y, z)$ where F is differentiable at

(x_0, y_0, z_0) and $\nabla F(x_0, y_0, z_0) \neq \mathbf{0}$,

the level surface of F at (x_0, y_0, z_0) is of the form $F(x, y, z) = C$ for some constant C. Let

$G(x, y, z) = F(x, y, z) - C = 0.$

Then $\nabla G(x_0, y_0, z_0) = \nabla F(x_0, y_0, z_0)$ where $\nabla G(x_0, y_0, z_0)$ is normal to $F(x, y, z) - C = 0$ at (x_0, y_0, z_0).

Therefore, $\nabla F(x_0, y_0 z_0)$ is normal to the level surface through (x_0, y_0, z_0).

64. Given $z = f(x, y)$, then:

$$F(x, y, z) = f(x, y) - z = 0$$

$$\nabla F(x_0, y_0, z_0) = f_x(x_0, y_0)\mathbf{i} + f_y(x_0, y_0)\mathbf{j} - \mathbf{k}$$

$$\cos\theta = \frac{|\nabla F(x_0, y_0, z_0) \cdot \mathbf{k}|}{\|\nabla F(x_0, y_0, z_0)\| \, \|\mathbf{k}\|}$$

$$= \frac{|-1|}{\sqrt{[f_x(x_0, y_0)]^2 + [f_y(x_0, y_0)]^2 + (-1)^2}}$$

$$= \frac{1}{\sqrt{[f_x(x_0, y_0)]^2 + [f_y(x_0, y_0)]^2 + 1}}$$

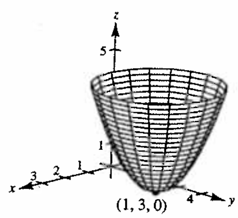
Section 13.8 Extrema of Functions of Two Variables

1. $g(x, y) = (x - 1)^2 + (y - 3)^2 \geq 0$

Relative minimum: $(1, 3, 0)$

$g_x = 2(x - 1) = 0 \implies x = 1$

$g_y = 2(y - 3) = 0 \implies y = 3$

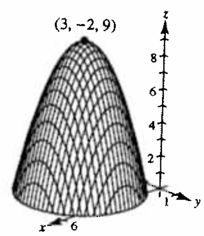


2. $g(x, y) = 9 - (x - 3)^2 - (y + 2)^2 \leq 9$

Relative maximum: $(3, -2, 9)$

$g_x = -2(x - 3) = 0 \implies x = 3$

$g_y = -2(y + 2) = 0 \implies y = -2$



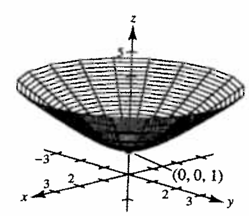
3. $f(x, y) = \sqrt{x^2 + y^2 + 1} \geq 1$

Relative minimum: $(0, 0, 1)$

Check: $f_x = \dfrac{x}{\sqrt{x^2 + y^2 + 1}} = 0 \implies x = 0$

$f_y = \dfrac{y}{\sqrt{x^2 + y^2 + 1}} = 0 \implies y = 0$

$f_{xx} = \dfrac{y^2 + 1}{(x^2 + y^2 + 1)^{3/2}}, \; f_{yy} = \dfrac{x^2 + 1}{(x^2 + y^2 + 1)^{3/2}}, \; f_{xy} = \dfrac{-xy}{(x^2 + y^2 + 1)^{3/2}}$



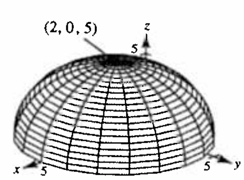
At the critical point $(0, 0)$, $f_{xx} > 0$ and $f_{xx}f_{yy} - (f_{xy})^2 > 0$. Therefore, $(0, 0, 1)$ is a relative minimum.

4. $f(x, y) = \sqrt{25 - (x - 2)^2 - y^2} \leq 5$

Relative maximum: $(2, 0, 5)$

Check: $f_x = -\dfrac{x - 2}{\sqrt{25 - (x - 2)^2 - y^2}} = 0 \implies x = 2$

$f_y = -\dfrac{y}{\sqrt{25 - (x - 2)^2 - y^2}} = 0 \implies y = 0$



$f_{xx} = -\dfrac{25 - y^2}{[25 - (x - 2)^2 - y^2]^{3/2}}, \; f_{yy} = -\dfrac{25 - (x - 2)^2}{[25 - (x - 2)^2 - y^2]^{3/2}}, \; f_{xy} = -\dfrac{y(x - 2)}{[25 - (x - 2)^2 - y^2]^{3/2}}$

At the critical point $(2, 0)$, $f_{xx} < 0$ and $f_{xx}f_{yy} - (f_{xy})^2 > 0$. Therefore, $(2, 0, 5)$ is a relative maximum.

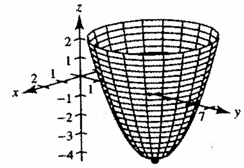
5. $f(x, y) = x^2 + y^2 + 2x - 6y + 6 = (x + 1)^2 + (y - 3)^2 - 4 \geq -4$

Relative minimum: $(-1, 3, -4)$

Check: $f_x = 2x + 2 = 0 \Rightarrow x = -1$

$f_y = 2y - 6 = 0 \Rightarrow y = 3$

$f_{xx} = 2,\ f_{yy} = 2,\ f_{xy} = 0$



At the critical point $(-1, 3)$, $f_{xx} > 0$ and $f_{xx}f_{yy} - (f_{xy})^2 > 0$. Therefore, $(-1, 3, -4)$ is a relative minimum.

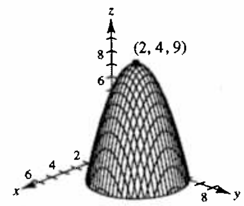
6. $f(x, y) = -x^2 - y^2 + 4x + 8y - 11 = -(x - 2)^2 - (y - 4)^2 + 9 \leq 9$

Relative maximum: $(2, 4, 9)$

Check: $f_x = -2x + 4 = 0 \Rightarrow x = 2$

$f_y = -2y + 8 = 0 \Rightarrow y = 4$

$f_{xx} = -2,\ f_{yy} = -2,\ f_{xy} = 0$



At the critical point $(2, 4)$, $f_{xx} < 0$ and $f_{xx}f_{yy} - (f_{xy})^2 > 0$. Therefore, $(2, 4, 9)$ is a relative maximum.

7. $f(x, y) = 2x^2 + 2xy + y^2 + 2x - 3$

$\left.\begin{aligned} f_x &= 4x + 2y + 2 = 0 \\ f_y &= 2x + 2y = 0 \end{aligned}\right\}$ Solving simultaneously yields $x = -1$ and $y = 1$.

$f_{xx} = 4,\ f_{yy} = 2,\ f_{xy} = 2$

At the critical point $(-1, 1)$, $f_{xx} > 0$ and $f_{xx}f_{yy} - (f_{xy})^2 > 0$. Therefore, $(-1, 1, -4)$ is a relative minimum.

8. $f(x, y) = -x^2 - 5y^2 + 10x - 30y - 62$

$\left.\begin{aligned} f_x &= -2x + 10 = 0 \\ f_y &= -10y - 30 = 0 \end{aligned}\right\}$ $x = 5, y = -3$

$f_{xx} = -2, f_{yy} = -10, f_{xy} = 0$

At the critical point $(5, -3)$, $f_{xx} < 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$.

Therefore, $(5, -3, 8)$ is a relative maximum.

9. $f(x, y) = -5x^2 + 4xy - y^2 + 16x + 10$

$\left.\begin{aligned} f_x &= -10x + 4y + 16 = 0 \\ f_y &= 4x - 2y = 0 \end{aligned}\right\}$ Solving simultaneously yields $x = 8$ and $y = 16$.

$f_{xx} = -10,\ f_{yy} = -2,\ f_{xy} = 4$

At the critical point $(8, 16)$, $f_{xx} < 0$ and $f_{xx}f_{yy} - (f_{xy})^2 > 0$. Therefore, $(8, 16, 74)$ is a relative maximum.

10. $f(x, y) = x^2 + 6xy + 10y^2 - 4y + 4$

$\left.\begin{aligned} f_x &= 2x + 6y = 0 \\ f_y &= 6x + 20y - 4 = 0 \end{aligned}\right\}$ Solving simultaneously yields $x = -6$ and $y = 2$.

$f_{xx} = 2,\ f_{yy} = 20,\ f_{xy} = 6$

At the critical point $(-6, 2)$, $f_{xx} > 0$ and $f_{xx}f_{yy} - (f_{xy})^2 > 0$. Therefore, $(-6, 2, 0)$ is a relative minimum.

11. $f(x, y) = 2x^2 + 3y^2 - 4x - 12y + 13$

$f_x = 4x - 4 = 4(x - 1) = 0$ when $x = 1$.

$f_y = 6y - 12 = 6(y - 2) = 0$ when $y = 2$.

$f_{xx} = 4,\ f_{yy} = 6,\ f_{xy} = 0$

At the critical point $(1, 2)$, $f_{xx} > 0$ and $f_{xx}f_{yy} - (f_{xy})^2 > 0$. Therefore, $(1, 2, -1)$ is a relative minimum.

12. $f(x, y) = -3x^2 - 2y^2 + 3x - 4y + 5$

$f_x = -6x + 3 = 0$ when $x = \frac{1}{2}$.

$f_y = -4y - 4 = 0$ when $y = -1$.

$f_{xx} = -6,\ f_{yy} = -4,\ f_{xy} = 0$

At the critical point $\left(\frac{1}{2}, -1\right)$, $f_{xx} < 0$ and $f_{xx}f_{yy} - (f_{xy})^2 > 0$. Therefore, $\left(\frac{1}{2}, -1, \frac{31}{4}\right)$ is a relative maximum.

13. $f(x, y) = 2\sqrt{x^2 + y^2} + 3$

$$\left.\begin{aligned} f_x &= \frac{2x}{\sqrt{x^2 + y^2}} = 0 \\ f_y &= \frac{2y}{\sqrt{x^2 + y^2}} = 0 \end{aligned}\right\} \quad x = 0, y = 0$$

Since $f(x, y) \geq 3$ for all (x, y), $(0, 0, 3)$ is a relative minimum.

14. $h(x, y) = (x^2 + y^2)^{1/3} + 2$

$$\left.\begin{aligned} h_x &= \frac{2x}{3(x^2 + y^2)^{2/3}} = 0 \\ h_y &= \frac{2y}{3(x^2 + y^2)^{2/3}} = 0 \end{aligned}\right\} \quad x = 0, y = 0$$

Since $h(x, y) \geq 2$ for all (x, y), $(0, 0, 2)$ is a relative minimum.

15. $g(x, y) = 4 - |x| - |y|$

$(0, 0)$ is the only critical point. Since $g(x, y) \leq 4$ for all (x, y), $(0, 0, 4)$ is a relative maximum.

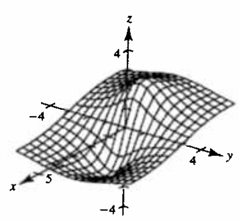
16. $f(x, y) = |x + y| - 2$

Since $f(x, y) \geq -2$ for all (x, y), the relative minima of f consist of all points (x, y) satisfying $x + y = 0$.

17. $z = \dfrac{-4x}{x^2 + y^2 + 1}$

Relative minimum: $(1, 0, -2)$

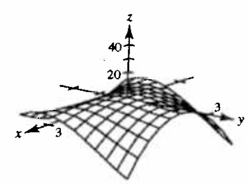
Relative maximum: $(-1, 0, 2)$



18. $f(x, y) = y^3 - 3yx^2 - 3y^2 - 3x^2 + 1$

Relative maximum: $(0, 0, 1)$

Saddle points: $(0, 2, -3)$, $\left(\pm\sqrt{3}, -1, -3\right)$

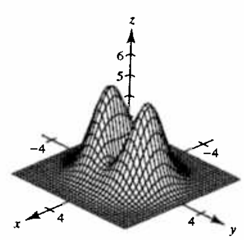


19. $z = (x^2 + 4y^2)e^{1 - x^2 - y^2}$

Relative minimum: $(0, 0, 0)$

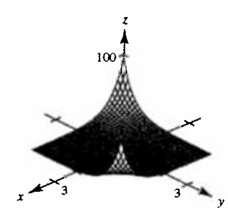
Relative maxima: $(0, \pm 1, 4)$

Saddle points: $(\pm 1, 0, 1)$



20. $z = e^{xy}$

Saddle point: $(0, 0, 1)$



21. $h(x, y) = x^2 - y^2 - 2x - 4y - 4$

$h_x = 2x - 2 = 2(x - 1) = 0$ when $x = 1$.

$h_y = -2y - 4 = -2(y + 2) = 0$ when $y = -2$.

$h_{xx} = 2,\ h_{yy} = -2,\ h_{xy} = 0$

At the critical point $(1, -2)$, $h_{xx}h_{yy} - (h_{xy})^2 < 0$. Therefore, $(1, -2, -1)$ is a saddle point.

22. $g(x, y) = 120x + 120y - xy - x^2 - y^2$

$$\left.\begin{aligned} g_x &= 120 - y - 2x = 0 \\ g_y &= 120 - x - 2y = 0 \end{aligned}\right\} \quad \text{Solving simultaneously yields } x = 40 \text{ and } y = 40.$$

$g_{xx} = -2,\ g_{yy} = -2,\ g_{xy} = -1$

At the critical point $(40, 40)$, $g_{xx} < 0$ and $g_{xx}\,g_{yy} - (g_{xy})^2 > 0$. Therefore, $(40, 40, 4800)$ is a relative maximum.

23. $h(x, y) = x^2 - 3xy - y^2$

$$\left.\begin{aligned} h_x &= 2x - 3y = 0 \\ h_y &= -3x - 2y = 0 \end{aligned}\right\} \quad \text{Solving simultaneously yields } x = 0 \text{ and } y = 0.$$

$h_{xx} = 2,\ h_{yy} = -2,\ h_{xy} = -3$

At the critical point $(0, 0)$, $h_{xx}\,h_{yy} - (h_{xy})^2 < 0$. Therefore, $(0, 0, 0)$ is a saddle point.

24. $g(x, y) = xy$

$$\left.\begin{aligned} g_x &= y \\ g_y &= x \end{aligned}\right\} \quad x = 0 \text{ and } y = 0$$

$g_{xx} = 0,\ g_{yy} = 0,\ g_{xy} = 1$

At the critical point $(0, 0)$, $g_{xx}\,g_{yy} - (g_{xy})^2 < 0$. Therefore, $(0, 0, 0)$ is a saddle point.

25. $f(x, y) = x^3 - 3xy + y^3$

$$\left.\begin{aligned} f_x &= 3(x^2 - y) = 0 \\ f_y &= 3(-x + y^2) = 0 \end{aligned}\right\} \quad \text{Solving by substitution yields two critical points } (0, 0) \text{ and } (1, 1).$$

$f_{xx} = 6x,\ f_{yy} = 6y,\ f_{xy} = -3$

At the critical point $(0, 0)$, $f_{xx}f_{yy} - (f_{xy})^2 < 0$. Therefore, $(0, 0, 0)$ is a saddle point. At the critical point $(1, 1)$, $f_{xx} = 6 > 0$ and $f_{xx}f_{yy} - (f_{xy})^2 > 0$. Therefore, $(1, 1, -1)$ is a relative minimum.

26. $f(x, y) = 2xy - \frac{1}{2}(x^4 + y^2) + 1$

$$\left.\begin{aligned} f_x &= 2y - 2x^3 \\ f_y &= 2x - 2y^3 \end{aligned}\right\} \quad \begin{aligned} &\text{Solving by substitution yields 3 critical points:} \\ &(0, 0), (1, 1), (-1, -1) \end{aligned}$$

$f_{xx} = -6x^2, f_{yy} = -6y^2, f_{xy} = 2$

At $(0, 0), f_{xx}f_{yy} - (f_{xy})^2 < 0 \implies (0, 0, 1)$ saddle point.

At $(1, 1), f_{xx}f_{yy} - (f_{xy})^2 > 0$ and $f_{xx} < 0 \implies (1, 1, 2)$ relative maximum.

At $(-1, -1), f_{xx}f_{yy} - (f_{xy})^2 > 0$ and $f_{xx} < 0 \implies (-1, -1, 2)$ relative maximum.

27. $f(x, y) = e^{-x} \sin y$

$$\left.\begin{aligned} f_x &= -e^{-x} \sin y = 0 \\ f_y &= e^{-x} \cos y = 0 \end{aligned}\right\} \quad \begin{aligned} &\text{Since } e^{-x} > 0 \text{ for all } x \text{ and } \sin y \text{ and } \cos y \text{ are never both zero for a} \\ &\text{given value of } y \text{, there are no critical points.} \end{aligned}$$

28. $f(x, y) = \left(\frac{1}{2} - x^2 + y^2\right)e^{1-x^2-y^2}$

$$\left.\begin{aligned} f_x &= (2x^3 - 2xy^2 - 3x)e^{1-x^2-y^2} = 0 \\ f_y &= (2x^2y - 2y^3 + y)e^{1-x^2-y^2} = 0 \end{aligned}\right\} \text{ Solving yields the critical points } (0, 0), \left(0, \pm\frac{\sqrt{2}}{2}\right), \left(\pm\frac{\sqrt{6}}{2}, 0\right).$$

$$f_{xx} = (-4x^4 + 4x^2y^2 + 12x^2 - 2y^2 - 3)e^{1-x^2-y^2}$$

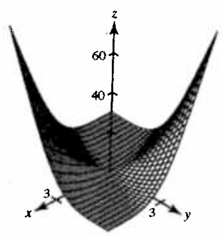
$$f_{yy} = (4y^4 - 4x^2y^2 + 2x^2 - 8y^2 + 1)e^{1-x^2-y^2}$$

$$f_{xy} = (-4x^3y + 4xy^3 + 2xy)e^{1-x^2-y^2}$$

At the critical point $(0, 0)$, $f_{xx}f_{yy} - (f_{xy})^2 < 0$. Therefore, $(0, 0, e/2)$ is a saddle point. At the critical points $\left(0, \pm\sqrt{2}/2\right)$, $f_{xx} < 0$ and $f_{xx}f_{yy} - (f_{xy})^2 > 0$. Therefore, $\left(0, \pm\sqrt{2}/2, \sqrt{e}\right)$ are relative maxima. At the critical points $\left(\pm\sqrt{6}/2, 0\right)$, $f_{xx} > 0$ and $f_{xx}f_{yy} - (f_{xy})^2 > 0$. Therefore, $\left(\pm\sqrt{6}/2, 0, -\sqrt{e}/e\right)$ are relative minima.

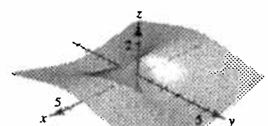
29. $z = \dfrac{(x - y)^4}{x^2 + y^2} \geq 0$. $z = 0$ if $x = y \neq 0$.

Relative minimum at all points (x, x), $x \neq 0$.



30. $z = \dfrac{(x^2 - y^2)^2}{x^2 + y^2} \geq 0$. $z = 0$ if $x^2 = y^2 \neq 0$.

Relative minima at all points (x, x) and $(x, -x)$, $x \neq 0$.



31. $f_{xx}f_{yy} - (f_{xy})^2 = (9)(4) - 6^2 = 0$
Insufficient information.

32. $f_{xx} < 0$ and $f_{xx}f_{yy} - (f_{xy})^2 = (-3)(-8) - 2^2 > 0$
f has a relative maximum at (x_0, y_0).

33. $f_{xx}f_{yy} - (f_{xy})^2 = (-9)(6) - 10^2 < 0$
f has a saddle point at (x_0, y_0).

34. $f_{xx} > 0$ and $f_{xx}f_{yy} - (f_{xy})^2 = (25)(8) - 10^2 > 0$
f has a relative minimum at (x_0, y_0).

35. (a) The function f defined on a region R containing (x_0, y_0) has a relative minimum at (x_0, y_0) if $f(x, y) \geq f(x_0, y_0)$ for all (x, y) in R.

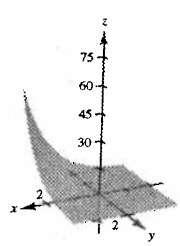
(b) The function f defined on a region R containing (x_0, y_0) has a relative maximum at (x_0, y_0) if $f(x, y) \leq f(x_0, y_0)$ for all (x, y) in R.

(c) A saddle point is a critical point which is not a relative extremum.

(d) See definition page 953.

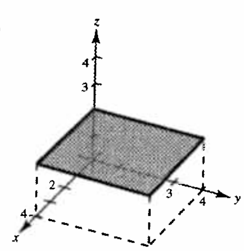
36. See Theorem 13.17.

37.

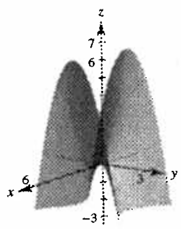


No extrema

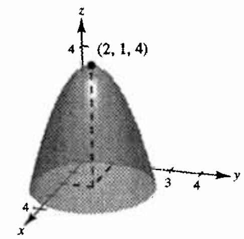
38.



Extrema at all (x, y)

39. 

Saddle point

40. 

Relative maximum

41. In this case, the point A will be a saddle point. The function could be

$$f(x, y) = xy.$$

42. A and B are relative extrema. C and D are saddle points.

43. $d = f_{xx}f_{yy} - f_{xy}^2 = (2)(8) - f_{xy}^2 = 16 - f_{xy}^2 > 0$
$\Longrightarrow f_{xy}^2 < 16 \Longrightarrow -4 < f_{xy} < 4$

44. $d = f_{xx}f_{yy} - f_{xy}^2 < 0$ if f_{xx} and f_{yy} have opposite signs. Hence, $(a, b, f(a, b))$ is a saddle point. For example, consider $f(x, y) = x^2 - y^2$ and $(a, b) = (0, 0)$.

45. $f(x, y) = x^3 + y^3$

$$\left.\begin{aligned} f_x &= 3x^2 = 0 \\ f_y &= 3y^2 = 0 \end{aligned}\right\} \text{ Solving yields } x = y = 0$$

$f_{xx} = 6x,\ f_{yy} = 6y,\ f_{xy} = 0$

At $(0, 0)$, $f_{xx}f_{yy} - (f_{xy})^2 = 0$ and the test fails. $(0, 0, 0)$ is a saddle point.

46. $f(x, y) = x^3 + y^3 - 6x^2 + 9y^2 + 12x + 27y + 19$

$$\left.\begin{aligned} f_x &= 3x^2 - 12x + 12 = 0 \\ f_y &= 3y^2 + 18y + 27 = 0 \end{aligned}\right\} \text{ Solving yields } x = 2 \text{ and } y = -3.$$

$f_{xx} = 6x - 12, f_{yy} = 6y + 18, f_{xy} = 0$

At $(2, -3)$, $f_{xx}f_{yy} - (f_{xy})^2 = 0$ and the test fails. $(1, -2, 0)$ is a saddle point.

47. $f(x, y) = (x - 1)^2(y + 4)^2 \geq 0$

$$\left.\begin{aligned} f_x &= 2(x - 1)(y + 4)^2 = 0 \\ f_y &= 2(x - 1)^2(y + 4) = 0 \end{aligned}\right\} \text{ Solving yields the critical points } (1, a) \text{ and } (b, -4).$$

$f_{xx} = 2(y + 4)^2, f_{yy} = 2(x - 1)^2, f_{xy} = 4(x - 1)(y + 4)$

At both $(1, a)$ and $(b, -4)$, $f_{xx}f_{yy} - (f_{xy})^2 = 0$ and the test fails.

Absolute minima: $(1, a, 0)$ and $(b, -4, 0)$

48. $f(x, y) = \sqrt{(x - 1)^2 + (y + 2)^2} \geq 0$

$$\left.\begin{aligned} f_x &= \frac{x - 1}{\sqrt{(x - 1)^2 + (y + 2)^2}} = 0 \\ f_y &= \frac{y + 2}{\sqrt{(x - 1)^2 + (y + 2)^2}} = 0 \end{aligned}\right\} \text{ Solving yields } x = 1 \text{ and } y = -2.$$

$$f_{xx} = \frac{(y + 2)^2}{[(x - 1)^2 + (y + 2)^2]^{3/2}},\ f_{yy} = \frac{(x - 1)^2}{[(x - 1)^2 + (y + 2)^2]^{3/2}},\ f_{xy} = \frac{(x - 1)(y + 2)}{[(x - 1)^2 + (y + 2)^2]^{3/2}}$$

At $(1, -2)$, $f_{xx}f_{yy} - (f_{xy})^2$ is undefined and the test fails.

Absolute minimum: $(1, -2, 0)$

49. $f(x, y) = x^{2/3} + y^{2/3} \geq 0$

$$\left.\begin{aligned} f_x &= \frac{2}{3\sqrt[3]{x}} \\ f_y &= \frac{2}{3\sqrt[3]{y}} \end{aligned}\right\} \text{ } f_x \text{ and } f_y \text{ are undefined at } x = 0, y = 0. \text{ The critical point is } (0, 0).$$

$$f_{xx} = -\frac{2}{9x\sqrt[3]{x}}, f_{yy} = -\frac{2}{9y\sqrt[3]{y}}, f_{xy} = 0$$

At $(0, 0)$, $f_{xx}f_{yy} - (f_{xy})^2$ is undefined and the test fails.

Absolute minimum: 0 at $(0, 0)$

50. $f(x, y) = (x^2 + y^2)^{2/3} \geq 0$

$$\left.\begin{aligned} f_x &= \frac{4x}{3(x^2 + y^2)^{1/3}} \\ f_y &= \frac{4y}{3(x^2 + y^2)^{1/3}} \end{aligned}\right\} \text{ } f_x \text{ and } f_y \text{ are undefined at } x = 0, y = 0. \text{ The critical point is } (0, 0).$$

$$f_{xx} = \frac{4(x^2 + 3y^2)}{9(x^2 + y^2)^{4/3}}, \ f_{yy} = \frac{4(3x^2 + y^2)}{9(x^2 + y^2)^{4/3}}, \ f_{xy} = \frac{-8xy}{9(x^2 + y^2)^{4/3}}$$

At $(0, 0)$, $f_{xx}f_{yy} - (f_{xy})^2$ is undefined and the test fails.

Absolute minimum: $(0, 0, 0)$

51. $f(x, y, z) = x^2 + (y - 3)^2 + (z + 1)^2 \geq 0$

$$\left.\begin{aligned} f_x &= 2x = 0 \\ f_y &= 2(y - 3) = 0 \\ f_z &= 2(z + 1) = 0 \end{aligned}\right\} \text{ Solving yields the critical point } (0, 3, -1).$$

Absolute minimum: 0 at $(0, 3, -1)$

52. $f(x, y, z) = 4 - [x(y - 1)(z + 2)]^2 \leq 4$

$$\left.\begin{aligned} f_x &= -2x(y - 1)^2(z + 2)^2 = 0 \\ f_y &= -2x^2(y - 1)(z + 2)^2 = 0 \\ f_z &= -2x(y - 1)^2(z + 2) = 0 \end{aligned}\right\}$$

Solving yields the critical points $(0, a, b)$, $(c, 1, d)$, $(e, f, -2)$. These points are all absolute maxima.

53. $f(x, y) = 12 - 3x - 2y$ has no critical points. On the line $y = x + 1$, $0 \leq x \leq 1$,

$$f(x, y) = f(x) = 12 - 3x - 2(x + 1) = -5x + 10$$

and the maximum is 10, the minimum is 5. On the line $y = -2x + 4$, $1 \leq x \leq 2$,

$$f(x, y) = f(x) = 12 - 3x - 2(-2x + 4) = x + 4$$

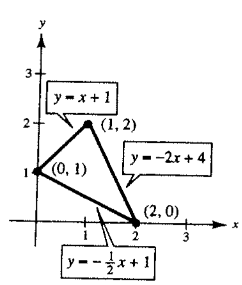
and the maximum is 6, the minimum is 5. On the line $y = -\frac{1}{2}x + 1$, $0 \leq x \leq 2$,

$$f(x, y) = f(x) = 12 - 3x - 2\left(-\tfrac{1}{2}x + 1\right) = -2x + 10$$

and the maximum is 10, the minimum is 6.

Absolute maximum: 10 at $(0, 1)$

Absolute minimum: 5 at $(1, 2)$



54. $f(x, y) = (2x - y)^2$

$f_x = 4(2x - y) = 0 \Rightarrow 2x = y$

$f_y = -2(2x - y) = 0 \Rightarrow 2x = y$

On the line $y = x + 1, 0 \le x \le 1$,

$f(x, y) = f(x) = (2x - (x + 1))^2 = (x - 1)^2$

and the maximum is 1, the minimum is 0. On the line $y = -\frac{1}{2}x + 1, 0 \le x \le 2$,

$f(x, y) = f(x) = \left(2x - \left(-\frac{1}{2}x + 1\right)\right)^2 = \left(\frac{5}{2}x - 1\right)^2$

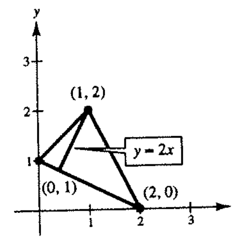
and the maximum is 16, the minimum is 0. On the line $y = -2x + 4, 1 \le x \le 2$,

$f(x, y) = f(x) = (2x - (-2x + 4))^2 = (4x - 4)^2$

and the maximum is 16, the minimum is 0.

Absolute maximum: 16 at (2, 0)

Absolute minimum: 0 at (1, 2) and along the line $y = 2x$.



55. $f(x, y) = 3x^2 + 2y^2 - 4y$

$\left.\begin{array}{l} f_x = 6x = 0 \quad \Rightarrow x = 0 \\ f_y = 4y - 4 = 0 \Rightarrow y = 1 \end{array}\right\} \quad f(0, 1) = -2$

On the line $y = 4, -2 \le x \le 2$,

$f(x, y) = f(x) = 3x^2 + 32 - 16 = 3x^2 + 16$

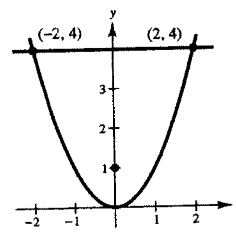
and the maximum is 28, the minimum is 16. On the curve $y = x^2, -2 \le x \le 2$,

$f(x, y) = f(x) = 3x^2 + 2(x^2)^2 - 4x^2 = 2x^4 - x^2 = x^2(2x^2 - 1)$

and the maximum is 28, the minimum is $-\frac{1}{8}$.

Absolute maximum: 28 at $(\pm 2, 4)$

Absolute minimum: -2 at (0, 1)



56. $f(x, y) = 2x - 2xy + y^2$

$\left.\begin{array}{l} f_x = 2 - 2y = 0 \quad \Rightarrow y = 1 \\ f_y = 2y - 2x = 0 \Rightarrow y = x \Rightarrow x = 1 \end{array}\right\} f(1, 1) = 1$

On the line $y = 1, -1 \le x \le 1$,

$f(x, y) = f(x) = 2x - 2x + 1 = 1$.

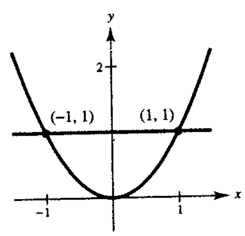
On the curve $y = x^2, -1 \le x \le 1$

$f(x, y) = f(x) = 2x - 2x(x^2) + (x^2)^2 = x^4 - 2x^3 + 2x$

and the maximum is 1, the minimum is $-\frac{11}{16}$.

Absolute maximum: 1 at (1, 1) and on $y = 1$

Absolute minimum: $-\frac{11}{16} = -0.6875$ at $\left(-\frac{1}{2}, \frac{1}{4}\right)$



57. $f(x, y) = x^2 + xy, R = \{(x, y): |x| \le 2, |y| \le 1\}$

$\left.\begin{array}{l} f_x = 2x + y = 0 \\ f_y = x = 0 \end{array}\right\} x = y = 0$

$f(0, 0) = 0$

Along $y = 1, -2 \le x \le 2, f = x^2 + x, f' = 2x + 1 = 0 \Rightarrow x = -\frac{1}{2}$.

Thus, $f(-2, 1) = 2, f\left(-\frac{1}{2}, 1\right) = -\frac{1}{4}$ and $f(2, 1) = 6$.

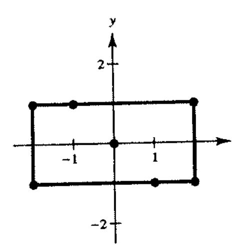
Along $y = -1, -2 \le x \le 2, f = x^2 - x, f' = 2x - 1 = 0 \Rightarrow x = \frac{1}{2}$.

Thus, $f(-2, -1) = 6, f\left(\frac{1}{2}, -1\right) = -\frac{1}{4}, f(2, -1) = 2$.

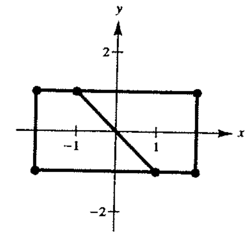
Along $x = 2, -1 \le y \le 1, f = 4 + 2y \Rightarrow f' = 2 \ne 0$.

Along $x = -2, -1 \le y \le 1, f = 4 - 2y \Rightarrow f' = -2 \ne 0$.

Thus, the maxima are $f(2, 1) = 6$ and $f(-2, -1) = 6$ and the minima are $f\left(-\frac{1}{2}, 1\right) = -\frac{1}{4}$ and $f\left(\frac{1}{2}, -1\right) = -\frac{1}{4}$.



58. $f(x, y) = x^2 + 2xy + y^2, R = \{(x, y): |x| \le 2, |y| \le 1\}$



$$\left.\begin{aligned} f_x &= 2x + 2y = 0 \\ f_y &= 2x + 2y = 0 \end{aligned}\right\} y = -x$$

$f(x, -x) = x^2 - 2x^2 + x^2 = 0$

Along $y = 1, -2 \le x \le 2,$

$f = x^2 + 2x + 1, f' = 2x + 2 = 0 \Rightarrow x = -1, f(-2, 1) = 1, f(-1, 1) = 0, f(2, 1) = 9.$

Along $y = -1, -2 \le x \le 2,$

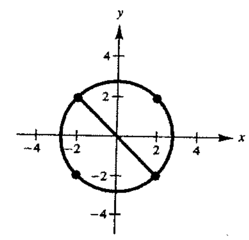
$f = x^2 - 2x + 1, f' = 2x - 2 = 0 \Rightarrow x = 1, f(-2, -1) = 9, f(1, -1) = 0, f(2, -1) = 1.$

Along $x = 2, -1 \le y \le 1, f = 4 + 4y + y^2, f' = 2y + 4 \ne 0.$

Along $x = -2, -1 \le y \le 1, f = 4 - 4y + y^2, f' = 2y - 4 \ne 0.$

Thus, the maxima are $f(-2, -1) = 9$ and $f(2, 1) = 9$, and the minima are $f(x, -x) = 0, -1 \le x \le 1.$

59. $f(x, y) = x^2 + 2xy + y^2, R = \{(x, y): x^2 + y^2 \le 8\}$



$$\left.\begin{aligned} f_x &= 2x + 2y = 0 \\ f_y &= 2x + 2y = 0 \end{aligned}\right\} y = -x$$

$f(x, -x) = x^2 - 2x^2 + x^2 = 0$

On the boundary $x^2 + y^2 = 8$, we have $y^2 = 8 - x^2$ and $y = \pm\sqrt{8 - x^2}$. Thus,

$$f = x^2 \pm 2x\sqrt{8 - x^2} + (8 - x^2) = 8 \pm 2x\sqrt{8 - x^2}$$

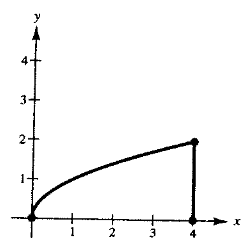
$$f' = \pm[(8 - x^2)^{-1/2}(-2x^2) + 2(8 - x^2)^{1/2}] = \pm\frac{16 - 4x^2}{\sqrt{8 - x^2}}.$$

Then, $f' = 0$ implies $16 = 4x^2$ or $x = \pm 2$.

$$f(2, 2) = f(-2, -2) = 16 \quad \text{and} \quad f(2, -2) = f(-2, 2) = 0$$

Thus, the maxima are $f(2, 2) = 16$ and $f(-2, -2) = 16$, and the minima are $f(x, -x) = 0, |x| \le 2.$

60. $f(x, y) = x^2 - 4xy + 5, R = \{(x, y): 0 \le x \le 4, 0 \le y \le \sqrt{x}\}$



$$\left.\begin{aligned} f_x &= 2x - 4y = 0 \\ f_y &= -4x = 0 \end{aligned}\right\} x = y = 0$$

$f(0, 0) = 5$

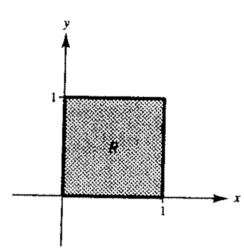
Along $y = 0, 0 \le x \le 4, f = x^2 + 5$ and $f(4, 0) = 21.$

Along $x = 4, 0 \le y \le 2, f = 16 - 16y + 5, f' = -16 \ne 0$ and $f(4, 2) = -11.$

Along $y = \sqrt{x}, 0 \le x \le 4, f = x^2 - 4x^{3/2} + 5, f' = 2x - 6x^{1/2} \ne 0$ on $[0, 4]$.

Thus, the maximum is $f(4, 0) = 21$ and the minimum is $f(4, 2) = -11.$

61. $f(x, y) = \dfrac{4xy}{(x^2 + 1)(y^2 + 1)}, R = \{(x, y): 0 \le x \le 1, 0 \le y \le 1\}$



$$f_x = \frac{4(1 - x^2)y}{(y^2 + 1)(x^2 + 1)^2} = 0 \Rightarrow x = 1 \text{ or } y = 0$$

$$f_y = \frac{4(1 - y^2)x}{(x^2 + 1)(y^2 + 1)^2} \Rightarrow x = 0 \text{ or } y = 1$$

For $x = 0, y = 0$, also, and $f(0, 0) = 0.$

For $x = 1, y = 1, f(1, 1) = 1.$

The absolute maximum is $1 = f(1, 1).$

The absolute minimum is $0 = f(0, 0)$. (In fact, $f(0, y) = f(x, 0) = 0$.)

62. $f(x, y) = \dfrac{4xy}{(x^2+1)(y^2+1)}, R = \{(x, y): x \geq 0, y \geq 0, x^2 + y^2 \leq 1\}$

$$f_x = \frac{4(1-x^2)y}{(y^2+1)(x^2+1)^2} = 0 \Rightarrow x = 1 \text{ or } y = 0$$

$$f_y = \frac{4(1-y^2)x}{(x^2+1)(y^2+1)^2} = 0 \Rightarrow y = 1 \text{ or } x = 0$$

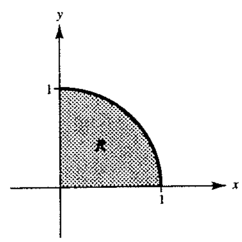
For $x = 0, y = 0$, also, and $f(0, 0) = 0$.

For $x = 1$ and $y = 1$, the point $(1, 1)$ is outside R.

For $x^2 + y^2 = 1, f(x, y) = f\left(x, \sqrt{1-x^2}\right) = \dfrac{4x\sqrt{1-x^2}}{2+x^2-x^4}$, and the maximum occurs at $x = \dfrac{\sqrt{2}}{2}, y = \dfrac{\sqrt{2}}{2}$.

Absolute maximum is $\dfrac{8}{9} = f\left(\dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{2}}{2}\right)$.

The absolute minimum is $0 = f(0, 0)$. (In fact, $f(0, y) = f(x, 0) = 0$)



63. False

Let $f(x, y) = 1 - |x| - |y|$.

$(0, 0, 1)$ is a relative maximum, but $f_x(0, 0)$ and $f_y(0, 0)$ do not exist.

64. False

Let $f(x, y) = x^4 - 2x^2 + y^2$.

Relative minima: $(\pm 1, 0, -1)$

Saddle point: $(0, 0, 0)$

Section 13.9 Applications of Extrema of Functions of Two Variables

1. A point on the plane is given by $(x, y, 12 - 2x - 3y)$. The square of the distance from the origin to this point is

$$S = x^2 + y^2 + (12 - 2x - 3y)^2$$

$$S_x = 2x + 2(12 - 2x - 3y)(-2)$$

$$S_y = 2y + 2(12 - 2x - 3y)(-3).$$

From the equations $S_x = 0$ and $S_y = 0$, we obtain the system

$$5x + 6y = 24$$

$$3x + 5y = 18.$$

Solving simultaneously, we have $x = \frac{12}{7}, y = \frac{18}{7}$ $z = 12 - \frac{24}{7} - \frac{54}{7} = \frac{6}{7}$. Therefore, the distance from the origin to $\left(\frac{12}{7}, \frac{18}{7}, \frac{6}{7}\right)$ is

$$\sqrt{\left(\frac{12}{7}\right)^2 + \left(\frac{18}{7}\right)^2 + \left(\frac{6}{7}\right)^2} = \frac{6\sqrt{14}}{7}.$$

2. A point on the plane is given by $(x, y, 12 - 2x - 3y)$. The square of the distance from $(1, 2, 3)$ to a point on the plane is given by

$$S = (x-1)^2 + (y-2)^2 + (9 - 2x - 3y)^2$$

$$S_x = 2(x-1) + 2(9 - 2x - 3y)(-2)$$

$$S_y = 2(y-2) + 2(9 - 2x - 3y)(-3).$$

From the equations $S_x = 0$ and $S_y = 0$, we obtain the system

$$5x + 6y = 19$$

$$6x + 10y = 29.$$

Solving simultaneously, we have $x = \frac{16}{14}, y = \frac{31}{14}, z = \frac{43}{14}$ and the distance is

$$\sqrt{\left(\frac{16}{14} - 1\right)^2 + \left(\frac{31}{14} - 2\right)^2 + \left(\frac{43}{14} - 3\right)^2} = \frac{1}{\sqrt{14}}.$$

3. A point on the paraboloid is given by $(x, y, x^2 + y^2)$. The square of the distance from $(5, 5, 0)$ to a point on the paraboloid is given by

$$S = (x - 5)^2 + (y - 5)^2 + (x^2 + y^2)^2$$
$$S_x = 2(x - 5) + 4x(x^2 + y^2) = 0$$
$$S_y = 2(y - 5) + 4y(x^2 + y^2) = 0.$$

From the equations $S_x = 0$ and $S_y = 0$, we obtain the system

$$2x^3 + 2xy^2 + x - 5 = 0$$
$$2y^3 + 2x^2y + y - 5 = 0.$$

Multiply the first equation by y and the second equation by x, and subtract to obtain $x = y$. Then, we have $x = 1$, $y = 1$, $z = 2$ and the distance is

$$\sqrt{(1 - 5)^2 + (1 - 5)^2 + (2 - 0)^2} = 6.$$

4. A point on the paraboloid is given by $(x, y, x^2 + y^2)$. The square of the distance from $(5, 0, 0)$ to a point on the paraboloid is given by

$$S = (x - 5)^2 + y^2 + (x^2 + y^2)^2$$
$$S_x = 2(x - 5) + 4x(x^2 + y^2) = 0$$
$$S_y = 2y + 4y(x^2 + y^2) = 0.$$

From the equations $S_x = 0$ and $S_y = 0$, we obtain the system

$$2x^3 + 2xy^2 + x - 5 = 0$$
$$2y^3 + 2x^2y + y = 0.$$

Solving as in Exercise 3, we have $x \approx 1.235$, $y = 0$, $z \approx 1.525$ and the distance is

$$\sqrt{(1.235 - 5)^2 + (1.525)^2} \approx 4.06.$$

5. Let x, y and z be the numbers. Since $x + y + z = 30$, $z = 30 - x - y$.

$$P = xyz = 30xy - x^2y - xy^2$$
$$\left.\begin{aligned} P_x &= 30y - 2xy - y^2 = y(30 - 2x - y) = 0 \\ P_y &= 30x - x^2 - 2xy = x(30 - x - 2y) = 0 \end{aligned}\right\} \begin{aligned} 2x + y &= 30 \\ x + 2y &= 30 \end{aligned}$$

Solving simultaneously yields $x = 10$, $y = 10$, and $z = 10$.

6. Since $x + y + z = 32$, $z = 32 - x - y$. Therefore,

$$P = xy^2z = 32xy^2 - x^2y^2 - xy^3$$
$$P_x = 32y^2 - 2xy^2 - y^3 = y^2(32 - 2x - y) = 0$$
$$P_y = 64xy - 2x^2y - 3xy^2 = y(64x - 2x^2 - 3xy) = 0.$$

Ignoring the solution $y = 0$ and substituting $y = 32 - 2x$ into $P_y = 0$, we have

$$64x - 2x^2 - 3x(32 - 2x) = 0$$
$$4x(x - 8) = 0.$$

Therefore, $x = 8$, $y = 16$, and $z = 8$.

7. Let x, y, and z be the numbers and let $S = x^2 + y^2 + z^2$. Since $x + y + z = 30$, we have

$$S = x^2 + y^2 + (30 - x - y)^2$$
$$\left.\begin{aligned} S_x &= 2x + 2(30 - x - y)(-1) = 0 \\ S_y &= 2y + 2(30 - x - y)(-1) = 0 \end{aligned}\right\} \begin{aligned} 2x + y &= 30 \\ x + 2y &= 30. \end{aligned}$$

Solving simultaneously yields $x = 10$, $y = 10$, and $z = 10$.

8. Let x, y, and z be the numbers and let $S = x^2 + y^2 + z^2$. Since $x + y + z = 1$, we have

$$S = x^2 + y^2 + (1 - x - y)^2$$
$$\left.\begin{aligned} S_x &= 2x - 2(1 - x - y) = 0 \\ S_y &= 2y - 2(1 - x - y) = 0 \end{aligned}\right\} \begin{aligned} 2x + y &= 1 \\ x + 2y &= 1. \end{aligned}$$

Solving simultaneously yields $x = \frac{1}{3}$, $y = \frac{1}{3}$, and $z = \frac{1}{3}$.

9. Let x, y, and z be the length, width, and height, respectively. Then the sum of the length and girth is given by $x + (2y + 2z) = 108$ or $x = 108 - 2y - 2z$. The volume is given by

$$V = xyz = 108zy - 2zy^2 - 2yz^2$$
$$V_y = 108z - 4yz - 2z^2 = z(108 - 4y - 2z) = 0$$
$$V_z = 108y - 2y^2 - 4yz = y(108 - 2y - 4z) = 0.$$

Solving the system $4y + 2z = 108$ and $2y + 4z = 108$, we obtain the solution $x = 36$ inches, $y = 18$ inches, and $z = 18$ inches.

10. Let x, y, and z be the length, width, and height, respectively. Then $C_0 = 1.5xy + 2yz + 2xz$ and $z = \dfrac{C_0 - 1.5xy}{2(x+y)}$. The volume is given by

$$V = xyz = \frac{C_0xy - 1.5x^2y^2}{2(x+y)}$$

$$V_x = \frac{y^2(2C_0 - 3x^2 - 6xy)}{4(x+y)^2}$$

$$V_y = \frac{x^2(2C_0 - 3y^2 - 6xy)}{4(x+y)^2}.$$

In solving the system $V_x = 0$ and $V_y = 0$, we note by the symmetry of the equations that $y = x$. Substituting $y = x$ into $V_x = 0$ yields

$$\frac{x^2(2C_0 - 9x^2)}{16x^2} = 0,\ 2C_0 = 9x^2,\ x = \frac{1}{3}\sqrt{2C_0},\ y = \frac{1}{3}\sqrt{2C_0},\ \text{and } z = \frac{1}{4}\sqrt{2C_0}.$$

11. Let $a + b + c = k$. Then

$$V = \frac{4\pi abc}{3} = \frac{4}{3}\pi ab(k - a - b)$$

$$= \frac{4}{3}\pi(kab - a^2b - ab^2)$$

$$\left.\begin{aligned} V_a &= \frac{4\pi}{3}(kb - 2ab - b^2) = 0 \\ V_b &= \frac{4\pi}{3}(ka - a^2 - 2ab) = 0 \end{aligned}\right\} \begin{aligned} kb - 2ab - b^2 &= 0 \\ ka - a^2 - 2ab &= 0. \end{aligned}$$

Solving this system simultaneously yields $a = b$ and substitution yields $b = k/3$. Therefore, the solution is $a = b = c = k/3$.

12. Consider the sphere given by $x^2 + y^2 + z^2 = r^2$ and let a vertex of the rectangular box be $\left(x, y, \sqrt{r^2 - x^2 - y^2}\right)$. Then the volume is given by

$$V = (2x)(2y)\left(2\sqrt{r^2 - x^2 - y^2}\right) = 8xy\sqrt{r^2 - x^2 - y^2}$$

$$V_x = 8\left(xy\frac{-x}{\sqrt{r^2 - x^2 - y^2}} + y\sqrt{r^2 - x^2 - y^2}\right) = \frac{8y}{\sqrt{r^2 - x^2 - y^2}}(r^2 - 2x^2 - y^2) = 0$$

$$V_y = 8\left(xy\frac{-y}{\sqrt{r^2 - x^2 - y^2}} + x\sqrt{r^2 - x^2 - y^2}\right) = \frac{8x}{\sqrt{r^2 - x^2 - y^2}}(r^2 - x^2 - 2y^2) = 0.$$

Solving the system

$$2x^2 + y^2 = r^2$$

$$x^2 + 2y^2 = r^2$$

yields the solution $x = y = z = r/\sqrt{3}$.

13. Let x, y, and z be the length, width, and height, respectively and let V_0 be the given volume.

Then $V_0 = xyz$ and $z = V_0/xy$. The surface area is

$$S = 2xy + 2yz + 2xz = 2\left(xy + \frac{V_0}{x} + \frac{V_0}{y}\right)$$

$$\left.\begin{aligned} S_x &= 2\left(y - \frac{V_0}{x^2}\right) = 0 \\ S_y &= 2\left(x - \frac{V_0}{y^2}\right) = 0 \end{aligned}\right\} \begin{aligned} x^2y - V_0 &= 0 \\ xy^2 - V_0 &= 0. \end{aligned}$$

Solving simultaneously yields $x = \sqrt[3]{V_0}$, $y = \sqrt[3]{V_0}$, and $z = \sqrt[3]{V_0}$.

14. Let x, y, and z be the length, width, and height, respectively.
Then the sum of the two perimeters of the two cross sections is given by

$$(2x + 2z) + (2y + 2z) = 144 \text{ or } x = 72 - y - 2z.$$

The volume is given by

$$V = xyz = 72yz - y^2z - 2yz^2$$
$$V_y = 72z - 2yz - 2z^2 = z(72 - 2y - 2z) = 0$$
$$V_z = 72y - y^2 - 4yz = y(72 - y - 4z) = 0.$$

Solving the system $2y + 2z = 72$ and $y + 4z = 72$, we obtain the solution

$$x = 24 \text{ inches}, y = 24 \text{ inches, and } z = 18 \text{ inches}.$$

15. $A = \frac{1}{2}[(30 - 2x) + (30 - 2x) + 2x\cos\theta]x\sin\theta$

$$= 30x\sin\theta - 2x^2\sin\theta + x^2\sin\theta\cos\theta$$

$$\frac{\partial A}{\partial x} = 30\sin\theta - 4x\sin\theta + 2x\sin\theta\cos\theta = 0$$

$$\frac{\partial A}{\partial \theta} = 30\cos\theta - 2x^2\cos\theta + x^2(2\cos^2\theta - 1) = 0$$

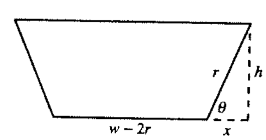
From $\frac{\partial A}{\partial x} = 0$ we have $15 - 2x + x\cos\theta = 0 \implies \cos\theta = \frac{2x - 15}{x}$.

From $\frac{\partial A}{\partial \theta} = 0$ we obtain

$$30x\left(\frac{2x - 15}{x}\right) - 2x^2\left(\frac{2x - 15}{x}\right) + x^2\left(2\left(\frac{2x - 15}{x}\right)^2 - 1\right) = 0$$
$$30(2x - 15) - 2x(2x - 15) + 2(2x - 15)^2 - x^2 = 0$$
$$3x^2 - 30x = 0$$
$$x = 10.$$

Then $\cos\theta = \frac{1}{2} \implies \theta = 60°$.

16. Let h be the height of the trough and r the length of the slanted sides. We observe that the area of a trapezoidal cross section is given by



$$A = h\left[\frac{(w - 2r) + [(w - 2r) + 2x]}{2}\right] = (w - 2r + x)h$$

where $x = r\cos\theta$ and $h = r\sin\theta$. Substituting these expressions for x and h, we have

$$A(r, \theta) = (w - 2r + r\cos\theta)(r\sin\theta) = wr\sin\theta - 2r^2\sin\theta + r^2\sin\theta\cos\theta.$$

Now

$$A_r(r, \theta) = w\sin\theta - 4r\sin\theta + 2r\sin\theta\cos\theta = \sin\theta(w - 4r + 2r\cos\theta) = 0 \implies w = r(4 - 2\cos\theta)$$
$$A_\theta(r, \theta) = wr\cos\theta - 2r^2\cos\theta + r^2\cos 2\theta = 0.$$

Substituting the expression for w from $A_r(r, \theta) = 0$ into the equation $A_\theta(r, \theta) = 0$, we have

$$r^2(4 - 2\cos\theta)\cos\theta - 2r^2\cos\theta + r^2(2\cos^2\theta - 1) = 0$$
$$r^2(2\cos\theta - 1) = 0 \text{ or } \cos\theta = \frac{1}{2}.$$

Therefore, the first partial derivatives are zero when $\theta = \pi/3$ and $r = w/3$. (Ignore the solution $r = \theta = 0$.) Thus, the trapezoid of maximum area occurs when each edge of width $w/3$ is turned up 60° from the horizontal.

17. $R(x_1, x_2) = -5x_1^2 - 8x_2^2 - 2x_1x_2 + 42x_1 + 102x_2$

$R_{x_1} = -10x_1 - 2x_2 + 42 = 0,\ 5x_1 + x_2 = 21$

$R_{x_2} = -16x_2 - 2x_1 + 102 = 0, x_1 + 8x_2 = 51$

Solving this system yields $x_1 = 3$ and $x_2 = 6$.

$R_{x_1x_1} = -10$

$R_{x_1x_2} = -2$

$R_{x_2x_2} = -16$

$R_{x_1x_1} < 0$ and $R_{x_1x_1}R_{x_2x_2} - (R_{x_1x_2})^2 > 0$

Thus, revenue is maximized when $x_1 = 3$ and $x_2 = 6$.

18. $R = 515p_1 + 805p_2 + 1.5p_1p_2 - 1.5p_1^2 - p_2^2$

$R_{p_1} = 515 + 1.5p_2 - 3p_1 = 0$

$R_{p_2} = 805 + 1.5p_1 - 2p_2 = 0$

$3p_1 - 1.5p_2 = 515$

$-1.5p_1 + 2p_2 = 805$

$-3p_1 + 4p_2 = 1610$

$2.5p_2 = 2125$

$p_2 = 850$

$p_1 = 596\frac{2}{3}$

19. $P(x_1, x_2) = 15(x_1 + x_2) - C_1 - C_2$

$= 15x_1 + 15x_2 - (0.02x_1^2 + 4x_1 + 500) - (0.05x_2^2 + 4x_2 + 275)$

$= -0.02x_1^2 - 0.05x_2^2 + 11x_1 + 11x_2 - 775$

$P_{x_1} = -0.04x_1 + 11 = 0, x_1 = 275$

$P_{x_2} = -0.10x_2 + 11 = 0, x_2 = 110$

$P_{x_1x_1} = -0.04$

$P_{x_1x_2} = 0$

$P_{x_2x_2} = -0.10$

$P_{x_1x_1} < 0$ and $P_{x_1x_1}P_{x_2x_2} - (P_{x_1x_2})^2 > 0$

Therefore, profit is maximized when $x_1 = 275$ and $x_2 = 110$.

20. $P(p, q, r) = 2pq + 2pr + 2qr.$

$p + q + r = 1$ implies that $r = 1 - p - q$.

$P(p, q) = 2pq + 2p(1 - p - q) + 2q(1 - p - q)$

$= 2pq + 2p - 2p^2 - 2pq + 2q - 2pq - 2q^2$

$= -2pq + 2p + 2q - 2p^2 - 2q^2$

$\frac{\partial P}{\partial p} = -2q + 2 - 4p;\ \frac{\partial P}{\partial q} = -2p + 2 - 4q$

Solving $\frac{\partial P}{\partial p} = \frac{\partial P}{\partial q} = 0$ gives

$q + 2p = 1$

$p + 2q = 1$

and hence $p = q = \frac{1}{3}$ and

$$P\left(\frac{1}{3}, \frac{1}{3}\right) = -2\left(\frac{1}{9}\right) + 2\left(\frac{1}{3}\right) + 2\left(\frac{1}{3}\right) - 2\left(\frac{1}{9}\right) - 2\left(\frac{1}{9}\right)$$

$$= \frac{6}{9} = \frac{2}{3}.$$

21. The distance from P to Q is $\sqrt{x^2+4}$. The distance from Q to R is $\sqrt{(y-x)^2+1}$. The distance from R to S is $10-y$.

$$C = 3k\sqrt{x^2+4} + 2k\sqrt{(y-x)^2+1} + k(10-y)$$

$$C_x = 3k\left(\frac{x}{\sqrt{x^2+4}}\right) + 2k\left(\frac{-(y-x)}{\sqrt{(y-x)^2+1}}\right) = 0$$

$$C_y = 2k\left(\frac{y-x}{\sqrt{(y-x)^2+1}}\right) - k = 0 \implies \frac{y-x}{\sqrt{(y-x)^2+1}} = \frac{1}{2}$$

$$3k\left(\frac{x}{\sqrt{x^2+4}}\right) + 2k\left(-\frac{1}{2}\right) = 0$$

$$\frac{x}{\sqrt{x^2+4}} = \frac{1}{3}$$

$$3x = \sqrt{x^2+4}$$

$$9x^2 = x^2+4$$

$$x^2 = \frac{1}{2}$$

$$x = \frac{\sqrt{2}}{2}$$

$$2(y-x) = \sqrt{(y-x)^2+1}$$

$$4(y-x)^2 = (y-x)^2+1$$

$$(y-x)^2 = \frac{1}{3}$$

$$y = \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{2}} = \frac{2\sqrt{3}+3\sqrt{2}}{6}$$

Therefore, $x = \dfrac{\sqrt{2}}{2} \approx 0.707$ km and $y = \dfrac{2\sqrt{3}+3\sqrt{2}}{6} \approx 1.284$ kms.

22. $S = d_1 + d_2 + d_3 = \sqrt{(0-0)^2+(y-0)^2} + \sqrt{(0-2)^2+(y-2)^2} + \sqrt{(0+2)^2+(y-2)^2}$

$$= y + 2\sqrt{4+(y-2)^2}$$

$$\frac{dS}{dy} = 1 + \frac{2(y-2)}{\sqrt{4+(y-2)^2}} = 0 \text{ when } y = 2 - \frac{2\sqrt{3}}{3} = \frac{6-2\sqrt{3}}{3}.$$

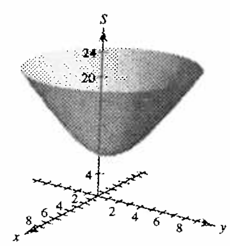
The sum of the distance is minimized when $y = \dfrac{2(3-\sqrt{3})}{3} \approx 0.845$.

23. (a) $S(x, y) = d_1 + d_2 + d_3$

$$= \sqrt{(x-0)^2+(y-0)^2} + \sqrt{(x+2)^2+(y-2)^2} + \sqrt{(x-4)^2+(y-2)^2}$$

$$= \sqrt{x^2+y^2} + \sqrt{(x+2)^2+(y-2)^2} + \sqrt{(x-4)^2+(y-2)^2}$$

From the graph we see that the surface has a minimum.



(b) $S_x(x, y) = \dfrac{x}{\sqrt{x^2+y^2}} + \dfrac{x+2}{\sqrt{(x+2)^2+(y-2)^2}} + \dfrac{x-4}{\sqrt{(x-4)^2+(y-2)^2}}$

$S_y(x, y) = \dfrac{y}{\sqrt{x^2+y^2}} + \dfrac{y-2}{\sqrt{(x+2)^2+(y-2)^2}} + \dfrac{y-2}{\sqrt{(x-4)^2+(y-2)^2}}$

(c) $-\nabla S(1, 1) = -S_x(1, 1)\mathbf{i} - S_y(1, 1)\mathbf{j} = -\dfrac{1}{\sqrt{2}}\mathbf{i} - \left(\dfrac{1}{\sqrt{2}} - \dfrac{2}{\sqrt{10}}\right)\mathbf{j}$

$$\tan\theta = \frac{(2/\sqrt{10}) - (1/\sqrt{2})}{-1/\sqrt{2}} = 1 - \frac{2}{\sqrt{5}} \implies \theta \approx 186.027°$$

—CONTINUED—

23. —CONTINUED—

(d) $(x_2, y_2) = (x_1 - S_x(x_1, y_1)t, y_1 - S_y(x_1, y_1)t) = \left(1 - \frac{1}{\sqrt{2}}t, 1 + \left(\frac{2}{\sqrt{10}} - \frac{1}{\sqrt{2}}\right)t\right)$

$$S\left(1 - \frac{1}{\sqrt{2}}t, 1 + \left(\frac{2}{\sqrt{10}} - \frac{1}{\sqrt{2}}\right)t\right) = \sqrt{2 + \left(\frac{2\sqrt{10}}{5} - 2\sqrt{2}\right)t + \left(1 - \frac{2\sqrt{5}}{5} + \frac{2}{5}\right)t^2}$$

$$+ \sqrt{10 - \left(\frac{2\sqrt{10}}{5} + 2\sqrt{2}\right)t + \left(1 - \frac{2\sqrt{5}}{5} + \frac{2}{5}\right)t^2}$$

$$+ \sqrt{10 - \left(\frac{2\sqrt{10}}{5} - 4\sqrt{2}\right)t + \left(1 - \frac{2\sqrt{5}}{5} + \frac{2}{5}\right)t^2}$$

Using a computer algebra system, we find that the minimum occurs when $t \approx 1.344$. Thus, $(x_2, y_2) \approx (0.05, 0.90)$.

(e) $(x_3, y_3) = (x_2 - S_x(x_2, y_2)t, y_2 - S_y(x_2, y_2)t) \approx (0.05 + 0.03t, 0.90 - 0.26t)$

$$S(0.05 + 0.03t, 0.90 - 0.26t) = \sqrt{(0.05 + 0.03t)^2 + (0.90 - 0.26t)^2} + \sqrt{(2.05 + 0.03t)^2 + (-1.10 - 0.26t)^2}$$
$$+ \sqrt{(-3.95 + 0.03t)^2 + (-1.10 - 0.26t)^2}$$

Using a computer algebra system, we find that the minimum occurs when $t \approx 1.78$. Thus $(x_3, y_3) \approx (0.10, 0.44)$.

$(x_4, y_4) = (x_3 - S_x(x_3, y_3)t, y_3 - S_y(x_3, y_3)t) \approx (0.10 - 0.09t, 0.44 - 0.01t)$

$$S(0.10 - 0.09t, 0.45 - 0.01t) = \sqrt{(0.10 - 0.09t)^2 + (0.45 - 0.01t)^2} + \sqrt{(2.10 - 0.09t)^2 + (-1.55 - 0.01t)^2}$$
$$+ \sqrt{(-3.90 - 0.09t)^2 + (-1.55 - 0.01t)^2}$$

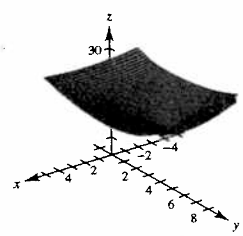
Using a computer algebra system, we find that the minimum occurs when $t \approx 0.44$. Thus, $(x_4, y_4) \approx (0.06, 0.44)$.

Note: The minimum occurs at $(x, y) = (0.0555, 0.3992)$

(f) $-\nabla S(x, y)$ points in the direction that S *decreases* most rapidly. You would use $\nabla S(x, y)$ for maximization problems.

24. (a) $S = \sqrt{(x + 4)^2 + y^2} + \sqrt{(x - 1)^2 + (y - 6)^2} + \sqrt{(x - 12)^2 + (y - 2)^2}$

The surface appears to have a minimum near $(x, y) = (1, 5)$.



(b) $$S_x = \frac{x + 4}{\sqrt{(x + 4)^2 + y^2}} + \frac{x - 1}{\sqrt{(x - 1)^2 + (y - 6)^2}} + \frac{x - 12}{\sqrt{(x - 12)^2 + (y - 2)^2}}$$

$$S_y = \frac{y}{\sqrt{(x + 4)^2 + y^2}} + \frac{y - 6}{\sqrt{(x - 1)^2 + (y - 6)^2}} + \frac{y - 2}{\sqrt{(x - 12)^2 + (y - 2)^2}}$$

(c) Let $(x_1, y_1) = (1, 5)$. Then

$-\nabla S(1, 5) = 0.258\mathbf{i} + 0.03\mathbf{j}$.

Direction $\approx 6.6°$

(d) $t \approx 0.94$ $x_2 \approx 1.24$ $y_2 \approx 5.03$

(e) $t \approx 3.56$, $x_3 \approx 1.24$, $y_3 \approx 5.06$,

$t \approx 1.04$, $x_4 \approx 1.23$, $y_4 \approx 5.06$

Note: Minimum occurs at $(x, y) = (1.2335, 5.0694)$

(f) $-\nabla S(x, y)$ points in the direction that S *decreases* most rapidly.

25. Write the equation to be maximized or minimized as a function of two variables. Set the partial derivatives equal to zero (or undefined) to obtain the critical points. Use the Second Partials Test to test for relative extrema using the critical points. Check the boundary points, too.

26. See pages 962 and 963.

27. (a)

x	y	xy	x^2
-2	0	0	4
0	1	0	0
2	3	6	4
$\sum x_i = 0$	$\sum y_i = 4$	$\sum x_i y_i = 6$	$\sum x_i^2 = 8$

$$a = \frac{3(6) - 0(4)}{3(8) - 0^2} = \frac{3}{4},\ b = \frac{1}{3}\left[4 - \frac{3}{4}(0)\right] = \frac{4}{3},$$

$$y = \frac{3}{4}x + \frac{4}{3}$$

(b) $S = \left(-\frac{3}{2} + \frac{4}{3} - 0\right)^2 + \left(\frac{4}{3} - 1\right)^2 + \left(\frac{3}{2} + \frac{4}{3} - 3\right)^2$

$= \frac{1}{6}$

28. (a)

x	y	xy	x^2
-3	0	0	9
-1	1	-1	1
1	1	1	1
3	2	6	9
$\sum x_i = 0$	$\sum y_i = 4$	$\sum x_i y_i = 6$	$\sum x_i^2 = 20$

$$a = \frac{4(6) - 0(4)}{4(20) - (0)^2} = \frac{3}{10},\ b = \frac{1}{4}\left[4 - \frac{3}{10}(0)\right] = 1,$$

$$y = \frac{3}{10}x + 1$$

(b) $S = \left(\frac{1}{10} - 0\right)^2 + \left(\frac{7}{10} - 1\right)^2 + \left(\frac{13}{10} - 1\right)^2 + \left(\frac{19}{10} - 2\right)^2$

$= \frac{1}{5}$

29. (a)

x	y	xy	x^2
0	4	0	0
1	3	3	1
1	1	1	1
2	0	0	4
$\sum x_i = 4$	$\sum y_i = 8$	$\sum x_i y_i = 4$	$\sum x_i^2 = 6$

$$a = \frac{4(4) - 4(8)}{4(6) - 4^2} = -2,\ b = \frac{1}{4}[8 + 2(4)] = 4,$$

$$y = -2x + 4$$

(b) $S = (4 - 4)^2 + (2 - 3)^2 + (2 - 1)^2 + (0 - 0)^2 = 2$

30. (a)

x	y	xy	x^2
3	0	0	9
1	0	0	1
2	0	0	4
3	1	3	9
4	1	4	16
4	2	8	16
5	2	10	25
6	2	12	36
$\sum x_i = 28$	$\sum y_i = 8$	$\sum x_i y_i = 37$	$\sum x_i^2 = 116$

$$a = \frac{8(37) - (28)(8)}{8(116) - (28)^2} = \frac{72}{144} = \frac{1}{2},\ b = \frac{1}{8}\left[8 - \frac{1}{2}(28)\right] = -\frac{3}{4},\ y = \frac{1}{2}x - \frac{3}{4}$$

(b) $S = \left(\frac{3}{4} - 0\right)^2 + \left(-\frac{1}{4} - 0\right)^2 + \left(\frac{1}{4} - 0\right)^2 + \left(\frac{3}{4} - 1\right)^2 + \left(\frac{5}{4} - 1\right)^2 + \left(\frac{5}{4} - 2\right)^2 + \left(\frac{7}{4} - 2\right)^2 + \left(\frac{9}{4} - 2\right)^2 = \frac{3}{2}$

31. $(0, 0), (1, 1), (3, 4), (4, 2), (5, 5)$

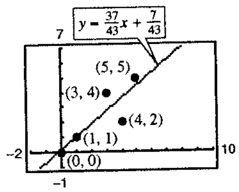
$\sum x_i = 13, \quad \sum y_i = 12,$

$\sum x_i y_i = 46, \quad \sum x_i^2 = 51$

$$a = \frac{5(46) - 13(12)}{5(51) - (13)^2} = \frac{74}{86} = \frac{37}{43}$$

$$b = \frac{1}{5}\left[12 - \frac{37}{43}(13)\right] = \frac{7}{43}$$

$$y = \frac{37}{43}x + \frac{7}{43}$$



32. $(1, 0), (3, 3), (5, 6)$

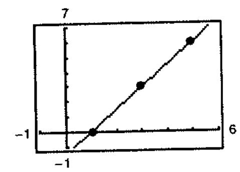
$\sum x_i = 9, \quad \sum y_i = 9,$

$\sum x_i y_i = 39, \quad \sum x_i^2 = 35$

$$a = \frac{3(39) - 9(9)}{3(35) - (9)^2} = \frac{36}{24} = \frac{3}{2}$$

$$b = \frac{1}{3}\left[9 - \frac{3}{2}(9)\right] = -\frac{9}{6} = -\frac{3}{2}$$

$$y = \frac{3}{2}x - \frac{3}{2}$$



33. $(0, 6), (4, 3), (5, 0), (8, -4), (10, -5)$

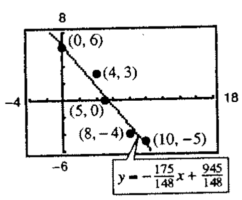
$\sum x_i = 27, \quad \sum y_i = 0,$

$\sum x_i y_i = -70, \quad \sum x_i^2 = 205$

$$a = \frac{5(-70) - (27)(0)}{5(205) - (27)^2} = \frac{-350}{296} = -\frac{175}{148}$$

$$b = \frac{1}{5}\left[0 - \left(-\frac{175}{148}\right)(27)\right] = \frac{945}{148}$$

$$y = -\frac{175}{148}x + \frac{945}{148}$$



34. $(6, 4), (1, 2), (3, 3), (8, 6), (11, 8), (13, 8); n = 6$

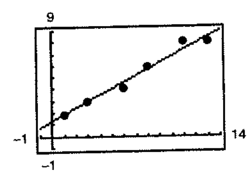
$\sum x_i = 42 \quad \sum y_i = 31$

$\sum x_i y_i = 275 \quad \sum x_i^2 = 400$

$$a = \frac{6(275) - (42)(31)}{6(400) - (42)^2} = \frac{29}{53} \approx 0.5472$$

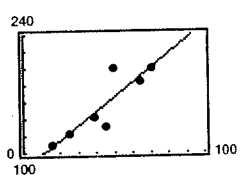
$$b = \frac{1}{6}\left(31 - \frac{29}{53}42\right) = \frac{425}{318} \approx 1.3365$$

$$y = \frac{29}{53}x + \frac{425}{318}$$



35. (a) $y = 1.7236x + 79.7334$

(b)



(c) For each one-year increase in age, the pressure changes by 1.7236 (slope of line).

36. (a) $(1.00, 450), (1.25, 375), (1.50, 330)$

$\sum x_i = 3.75, \sum y_i = 1{,}155, \sum x_i^2 = 4.8125,$

$\sum x_i y_i = 1{,}413.75$

$$a = \frac{3(1{,}413.75) - (3.75)(1{,}155)}{3(4.8125) - (3.75)^2} = -240$$

$$b = \frac{1}{3}[1{,}155 - (-240)(3.75)] = 685$$

$$y = -240x + 685$$

(b) When $x = 1.40$, $y = -240(1.40) + 685 = 349$.

37. $(1.0, 32), (1.5, 41), (2.0, 48), (2.5, 53)$

$\sum x_i = 7, \sum y_i = 174, \sum x_i y_i = 322, \sum x_i^2 = 13.5$

$a = 14, b = 19, y = 14x + 19$

When $x = 1.6$, $y = 41.4$ bushels per acre.

38. (a) $y = 1.85x - 48.3$

(b) For each 1 point increase in the percent (x), y increases by about 1.85 (million).

39. $S(a, b, c) = \sum_{i=1}^{n} (y_i - ax_i^2 - bx_i - c)^2$

$$\frac{\partial S}{\partial a} = \sum_{i=1}^{n} -2x_i^2(y_i - ax_i^2 - bx_i - c) = 0$$

$$\frac{\partial S}{\partial b} = \sum_{i=1}^{n} -2x_i(y_i - ax_i^2 - bx_i - c) = 0$$

$$\frac{\partial S}{\partial c} = -2\sum_{i=1}^{n} (y_i - ax_i^2 - bx_i - c) = 0$$

$$a\sum_{i=1}^{n} x_i^4 + b\sum_{i=1}^{n} x_i^3 + c\sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} x_i^2 y_i$$

$$a\sum_{i=1}^{n} x_i^3 + b\sum_{i=1}^{n} x_i^2 + c\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} x_i y_i$$

$$a\sum_{i=1}^{n} x_i^2 + b\sum_{i=1}^{n} x_i + cn = \sum_{i=1}^{n} y_i$$

40. $S(a, b) = \sum_{i=1}^{n} (ax_i + b - y_i)^2$

$$S_a(a, b) = 2a\sum_{i=1}^{n} x_i^2 + 2b\sum_{i=1}^{n} x_i - 2\sum_{i=1}^{n} x_i y_i$$

$$S_b(a, b) = 2a\sum_{i=1}^{n} x_i + 2nb - 2\sum_{i=1}^{n} y_i$$

$$S_{aa}(a, b) = 2\sum_{i=1}^{n} x_i^2$$

$$S_{bb}(a, b) = 2n$$

$$S_{ab}(a, b) = 2\sum_{i=1}^{n} x_i$$

$S_{aa}(a, b) > 0$ as long as $x_i \neq 0$ for all i. (**Note:** If $x_i = 0$ for all i, then $x = 0$ is the least squares regression line.)

$$d = S_{aa}S_{bb} - S_{ab}^2 = 4n\sum_{i=1}^{n} x_i^2 - 4\left(\sum_{i=1}^{n} x_i\right)^2 = 4\left[n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2\right] \geq 0 \text{ since } n\sum_{i=1}^{n} x_i^2 \geq \left(\sum_{i=1}^{n} x_i\right)^2.$$

As long as $d \neq 0$, the given values for a and b yield a minimum.

41. $(-2, 0), (-1, 0), (0, 1), (1, 2), (2, 5)$

$\sum x_i = 0$

$\sum y_i = 8$

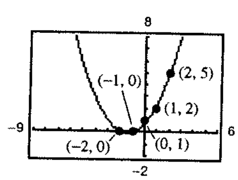
$\sum x_i^2 = 10$

$\sum x_i^3 = 0$

$\sum x_i^4 = 34$

$\sum x_i y_i = 12$

$\sum x_i^2 y_i = 22$



$34a + 10c = 22,\ 10b = 12,\ 10a + 5c = 8$

$a = \frac{3}{7},\ b = \frac{6}{5},\ c = \frac{26}{35},\ y = \frac{3}{7}x^2 + \frac{6}{5}x + \frac{26}{35}$

42. $(-4, 5),\ (-2, 6),\ (2, 6),\ (4, 2)$

$\sum x_i = 0$

$\sum y_i = 19$

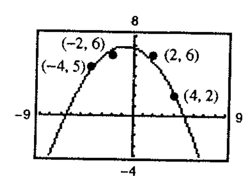
$\sum x_i^2 = 40$

$\sum x_i^3 = 0$

$\sum x_i^4 = 544$

$\sum x_i y_i = -12$

$\sum x_i^2 y_i = 160$



$544a + 40c = 160,\ 40b = -12,\ 40a + 4c = 19$

$a = -\frac{5}{24},\ b = -\frac{3}{10},\ c = \frac{41}{6},\ y = -\frac{5}{24}x^2 - \frac{3}{10}x + \frac{41}{6}$

43. (0, 0), (2, 2), (3, 6), (4, 12)

$\sum x_i = 9$

$\sum y_i = 20$

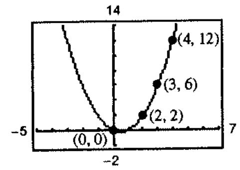
$\sum x_i^2 = 29$

$\sum x_i^3 = 99$

$\sum x_i^4 = 353$

$\sum x_i y_i = 70$

$\sum x_i^2 y_i = 254$



$$353a + 99b + 29c = 254$$
$$99a + 29b + 9c = 70$$
$$29a + 9b + 4c = 20$$

$a = 1,\ b = -1,\ c = 0,\ y = x^2 - x$

44. (0, 10), (1, 9), (2, 6), (3, 0)

$\sum x_i = 6$

$\sum y_i = 25$

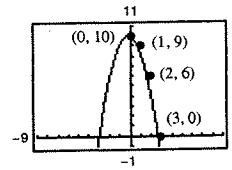
$\sum x_i^2 = 14$

$\sum x_i^3 = 36$

$\sum x_i^4 = 98$

$\sum x_i y_i = 21$

$\sum x_i^2 y_i = 33$



$$98a + 36b + 14c = 33$$
$$36a + 14b + 6c = 21$$
$$14a + 6b + 4c = 25$$

$a = -\frac{5}{4},\ b = \frac{9}{20},\ c = \frac{199}{20},\ y = -\frac{5}{4}x^2 + \frac{9}{20}x + \frac{199}{20}$

45. (a) (0, 0), (2, 15), (4, 30), (6, 50), (8, 65), (10, 70)

$\sum x_i = 30$

$\sum y_i = 230$

$\sum x_i^2 = 220$

$\sum x_i^3 = 1800$

$\sum x_i^4 = 15{,}664$

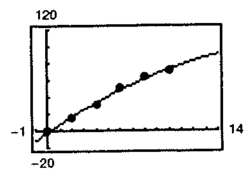
$\sum x_i y_i = 1670$

$\sum x_i^2 y_i = 13{,}500$

$$15{,}664a + 1800b + 220c = 13{,}500$$
$$1800a + 220b + 30c = 1670$$
$$220a + 30b + 6c = 230$$

$y = -\frac{25}{112}x^2 + \frac{541}{56}x - \frac{25}{14} \approx -0.22x^2 + 9.66x - 1.79$

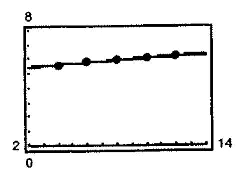
(b)



46. (a) $y = 0.075x + 5.32$ $(x = 4$ is 1994).

(b) $y = -0.002x^2 + 0.10x + 5.22$

(c)



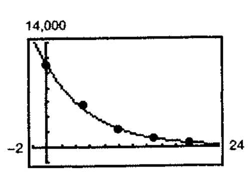
(d) For 2010 ($x = 20$), the linear model gives 6.82 billion and the quadratic model gives 6.42 billion. The quadratic model is less because of the negative x^2-term.

47. (a) $\ln P = -0.1499h + 9.3018$

(b) $\ln P = -0.1499h + 9.3018$

$P = e^{-0.1499h + 9.3018} = 10{,}957.7e^{-0.1499h}$

(c)

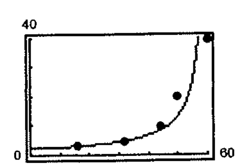


(d) Same answers

48. (a) $\frac{1}{y} = ax + b = -0.0074x + 0.445$

$$y = \frac{1}{-0.0074x + 0.445}$$

(b)



(c) No. For $x = 70$, $y \approx -14$, which is nonsense. $y = 1000$ which seems inaccurate.

Section 13.10 Lagrange Multipliers

1. Maximize $f(x, y) = xy$.

Constraint: $x + y = 10$

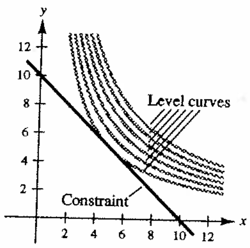
$\nabla f = \lambda \nabla g$

$y\mathbf{i} + x\mathbf{j} = \lambda(\mathbf{i} + \mathbf{j})$

$$\left.\begin{array}{l} y = \lambda \\ x = \lambda \end{array}\right\} x = y$$

$x + y = 10 \implies x = y = 5$

$f(5, 5) = 25$



3. Minimize $f(x, y) = x^2 + y^2$.

Constraint: $x + y = 4$

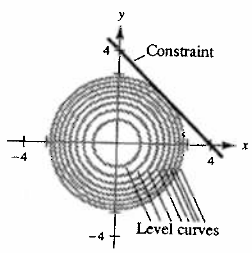
$\nabla f = \lambda \nabla g$

$2x\mathbf{i} + 2y\mathbf{j} = \lambda\mathbf{i} + \lambda\mathbf{j}$

$$\left.\begin{array}{l} 2x = \lambda \\ 2y = \lambda \end{array}\right\} x = y$$

$x + y = 4 \implies x = y = 2$

$f(2, 2) = 8$



5. Minimize $f(x, y) = x^2 - y^2$.

Constraint: $x - 2y = -6$

$\nabla f = \lambda \nabla g$

$2x\mathbf{i} - 2y\mathbf{j} = \lambda\mathbf{i} - 2\lambda\mathbf{j}$

$2x = \lambda \implies x = \dfrac{\lambda}{2}$

$-2y = -2\lambda \implies y = \lambda$

$x - 2y = -6 \implies -\dfrac{3}{2}\lambda = -6$

$\lambda = 4,\ x = 2,\ y = 4$

$f(2, 4) = -12$

2. Maximize $f(x, y) = xy$.

Constraint: $2x + y = 4$

$\nabla f = \lambda \nabla g$

$y\mathbf{i} + x\mathbf{j} = 2\lambda\mathbf{i} + \lambda\mathbf{j}$

$y = 2\lambda$

$x = \lambda$

$2x + y = 4 \implies 4\lambda = 4$

$\lambda = 1,\ x = 1,\ y = 2$

$f(1, 2) = 2$

4. Minimize $f(x, y) = x^2 + y^2$.

Constraint: $2x + 4y = 5$

$\nabla f = \lambda \nabla g$

$2x\mathbf{i} + 2y\mathbf{j} = 2\lambda\mathbf{i} + 4\lambda\mathbf{j}$

$2x = 2\lambda \implies x = \lambda$

$2y = 4\lambda \implies y = 2\lambda$

$2x + 4y = 5 \implies 10\lambda = 5$

$\lambda = \frac{1}{2},\ x = \frac{1}{2},\ y = 1$

$f\left(\frac{1}{2}, 1\right) = \frac{5}{4}$

6. Maximize $f(x, y) = x^2 - y^2$.

Constraint: $2y - x^2 = 0$

$\nabla f = \lambda \nabla g$

$2x\mathbf{i} - 2y\mathbf{j} = -2x\lambda\mathbf{i} + 2\lambda\mathbf{j}$

$2x = -2x\lambda \implies x = 0$ or $\lambda = -1$

If $x = 0$, then $y = 0$ and $f(0, 0) = 0$.

If $\lambda = -1$,

$-2y = 2\lambda = -2 \implies y = 1 \implies x^2 = 2 \implies x = \sqrt{2}$.

$f\left(\sqrt{2}, 1\right) = 2 - 1 = 1$, Maximum

7. Maximize $f(x, y) = 2x + 2xy + y$.

Constraint: $2x + y = 100$

$\nabla f = \lambda \nabla g$

$(2 + 2y)\mathbf{i} + (2x + 1)\mathbf{j} = 2\lambda\mathbf{i} + \lambda\mathbf{j}$

$$\left.\begin{aligned} 2 + 2y = 2\lambda &\implies y = \lambda - 1 \\ 2x + 1 = \lambda &\implies x = \frac{\lambda - 1}{2} \end{aligned}\right\} y = 2x$$

$$2x + y = 100 \implies 4x = 100$$

$$x = 25,\ y = 50$$

$f(25, 50) = 2600$

8. Minimize $f(x, y) = 3x + y + 10$.

Constraint: $x^2y = 6$

$\nabla f = \lambda \nabla g$

$3\mathbf{i} + \mathbf{j} = 2xy\lambda\mathbf{i} + x^2\lambda\mathbf{j}$

$$\left.\begin{aligned} 3 = 2xy\lambda &\implies \lambda = \frac{3}{2xy} \\ 1 = x^2\lambda &\implies \lambda = \frac{1}{x^2} \end{aligned}\right\} 3x^2 = 2xy \implies y = \frac{3x}{2} \quad (x \neq 0)$$

$$x^2y = 6 \implies x^2\left(\frac{3x}{2}\right) = 6$$

$$x^3 = 4$$

$$x = \sqrt[3]{4},\ y = \frac{3\sqrt[3]{4}}{2}$$

$$f\left(\sqrt[3]{4}, \frac{3\sqrt[3]{4}}{2}\right) = \frac{9\sqrt[3]{4} + 20}{2}$$

9. **Note:** $f(x, y) = \sqrt{6 - x^2 - y^2}$ is maximum when $g(x, y)$ is maximum.

Maximize $g(x, y) = 6 - x^2 - y^2$.

Constraint: $x + y = 2$

$$\left.\begin{aligned} -2x &= \lambda \\ -2y &= \lambda \end{aligned}\right\} x = y$$

$x + y = 2 \implies x = y = 1$

$f(1, 1) = \sqrt{g(1, 1)} = 2$

10. **Note:** $f(x, y) = \sqrt{x^2 + y^2}$ is minimum when $g(x, y)$ is minimum.

Minimize $g(x, y) = x^2 + y^2$.

Constraint: $2x + 4y = 15$

$$\left.\begin{aligned} 2x &= 2\lambda \\ 2y &= 4\lambda \end{aligned}\right\} y = 2x$$

$$2x + 4y = 15 \implies 10x = 15$$

$$x = \frac{3}{2},\ y = 3$$

$$f\left(\frac{3}{2}, 3\right) = \sqrt{g\left(\frac{3}{2}, 3\right)} = \frac{3\sqrt{5}}{2}$$

11. Maximize $f(x, y) = e^{xy}$.

Constraint: $x^2 + y^2 = 8$

$$\left.\begin{aligned} ye^{xy} &= 2x\lambda \\ xe^{xy} &= 2y\lambda \end{aligned}\right\} x = y$$

$$x^2 + y^2 = 8 \implies 2x^2 = 8$$

$$x = y = 2$$

$f(2, 2) = e^4$

12. Minimize $f(x, y) = 2x + y$.

Constraint: $xy = 32$

$$\left.\begin{aligned} 2 &= y\lambda \\ 1 &= x\lambda \end{aligned}\right\} y = 2x$$

$$xy = 32 \implies 2x^2 = 32$$

$$x = 4, y = 8$$

$f(4, 8) = 16$

13. Maximize or minimize $f(x, y) = x^2 + 3xy + y^2$.

Constraint: $x^2 + y^2 \le 1$

Case 1: On the circle $x^2 + y^2 = 1$

$$\left.\begin{aligned} 2x + 3y &= 2x\lambda \\ 3x + 2y &= 2y\lambda \end{aligned}\right\} x^2 = y^2$$

$$x^2 + y^2 = 1 \Rightarrow x = \pm\frac{\sqrt{2}}{2}, y = \pm\frac{\sqrt{2}}{2}$$

Maxima: $f\left(\pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{2}}{2}\right) = \frac{5}{2}$

Minima: $f\left(\pm\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = -\frac{1}{2}$

Case 2: Inside the circle

$$\left.\begin{aligned} f_x &= 2x + 3y = 0 \\ f_y &= 3x + 2y = 0 \end{aligned}\right\} x = y = 0$$

$f_{xx} = 2,\ f_{yy} = 2,\ f_{xy} = 3,\ f_{xx}f_{yy} - (f_{xy})^2 \le 0$

Saddle point: $f(0, 0) = 0$

By combining these two cases, we have a maximum of $\frac{5}{2}$ at $\left(\pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{2}}{2}\right)$ and a minimum of $-\frac{1}{2}$ at $\left(\pm\frac{\sqrt{2}}{2}, \mp\frac{\sqrt{2}}{2}\right)$.

14. Maximize or minimize $f(x, y) = e^{-xy/4}$.

Constraint: $x^2 + y^2 \le 1$

Case 1: On the circle $x^2 + y^2 = 1$

$$\left.\begin{aligned} -(y/4)e^{-xy/4} &= 2x\lambda \\ -(x/4)e^{-xy/4} &= 2y\lambda \end{aligned}\right\} \Rightarrow x^2 = y^2$$

$$x^2 + y^2 = 1 \Rightarrow x = \pm\frac{\sqrt{2}}{2}$$

Maxima: $f\left(\pm\frac{\sqrt{2}}{2}, \mp\frac{\sqrt{2}}{2}\right) = e^{1/8} \approx 1.1331$

Minima: $f\left(\pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{2}}{2}\right) = e^{-1/8} \approx 0.8825$

Case 2: Inside the circle

$$\left.\begin{aligned} f_x &= -(y/4)e^{-xy/4} = 0 \\ f_y &= -(x/4)e^{-xy/4} = 0 \end{aligned}\right\} \Rightarrow x = y = 0$$

$$f_{xx} = \frac{y^2}{16}e^{-xy/4},\ f_{yy} = \frac{x^2}{16}e^{-xy/4},\ f_{xy} = e^{-xy}\left[\frac{1}{16}xy - \frac{1}{4}\right)$$

At $(0, 0)$, $f_{xx}f_{yy} - (f_{xy})^2 < 0$.

Saddle point: $f(0, 0) = 1$

Combining the two cases, we have a maximum of $e^{1/8}$ at $\left(\pm\frac{\sqrt{2}}{2}, \mp\frac{\sqrt{2}}{2}\right)$ and a minimum of $e^{-1/8}$ at $\left(\pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{2}}{2}\right)$.

15. Minimize $f(x, y, z) = x^2 + y^2 + z^2$.

Constraint: $x + y + z = 6$

$$\left.\begin{aligned} 2x &= \lambda \\ 2y &= \lambda \\ 2z &= \lambda \end{aligned}\right\} x = y = z$$

$x + y + z = 6 \Rightarrow x = y = z = 2$

$f(2, 2, 2) = 12$

16. Maximize $f(x, y, z) = xyz$.

Constraint: $x + y + z = 6$

$$\left.\begin{aligned} yz &= \lambda \\ xz &= \lambda \\ xy &= \lambda \end{aligned}\right\} x = y = z$$

$x + y + z = 6 \Rightarrow x = y = z = 2$

$f(2, 2, 2) = 8$

17. Minimize $f(x, y, z) = x^2 + y^2 + z^2$.

Constraint: $x + y + z = 1$

$$\left.\begin{aligned} 2x &= \lambda \\ 2y &= \lambda \\ 2z &= \lambda \end{aligned}\right\} x = y = z$$

$x + y + z = 1 \Rightarrow x = y = z = \frac{1}{3}$

$f\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) = \frac{1}{3}$

18. Minimize $x^2 - 10x + y^2 - 14y + 70$.

Constraint: $x + y = 10$

$$\left.\begin{aligned} 2x - 10 &= \lambda \\ 2y - 14 &= \lambda \\ x + y &= 10 \end{aligned}\right\} \begin{aligned} x &= (1/2)(\lambda + 10) \\ y &= (1/2)(\lambda + 14) \end{aligned}$$

$$\begin{aligned} x + y &= \frac{1}{2}(\lambda + 10) + \frac{1}{2}(\lambda + 14) \\ &= \lambda + 12 = 10 \Rightarrow \lambda = -2 \end{aligned}$$

Then $x = 3, y = 5$.

$f(4, 6) = 16 - 40 + 36 - 84 + 70 = -2$

19. Maximize $f(x, y, z) = xyz$.

Constraints: $x + y + z = 32$

$x - y + z = 0$

$\nabla f = \lambda\nabla g + \mu\nabla h$

$yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k} = \lambda(\mathbf{i} + \mathbf{j} + \mathbf{k}) + \mu(\mathbf{i} - \mathbf{j} + \mathbf{k})$

$$\left.\begin{aligned} yz &= \lambda + \mu \\ xz &= \lambda - \mu \\ xy &= \lambda + \mu \end{aligned}\right\} yz = xy \Rightarrow x = z$$

$$\left.\begin{aligned} x + y + z &= 32 \\ x - y + z &= 0 \end{aligned}\right\} 2x + 2z = 32 \Rightarrow x = z = 8$$

$y = 16$

$f(8, 16, 8) = 1024$

20. Minimize $f(x, y, z) = x^2 + y^2 + z^2$.

Constraints: $x + 2z = 6$

$x + y = 12$

$\nabla f = \lambda\nabla g + \mu\nabla h$

$2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} = \lambda(\mathbf{i} + 2\mathbf{k}) + \mu(\mathbf{i} + \mathbf{j})$

$$\left.\begin{aligned} 2x &= \lambda + \mu \\ 2y &= \mu \\ 2z &= 2\lambda \end{aligned}\right\} 2x = 2y + z$$

$$x + 2z = 6 \implies z = \frac{6 - x}{2} = 3 - \frac{x}{2}$$

$$x + y = 12 \implies y = 12 - x$$

$$2x = 2(12 - x) + \left(3 - \frac{x}{2}\right) \implies \frac{9}{2}x = 27 \implies x = 6$$

$x = 6, z = 0$

$f(6, 6, 0) = 72$

21. Maximize $f(x, y, z) = xy + yz$.

Constraints: $x + 2y = 6$

$x - 3z = 0$

$\nabla f = \lambda\nabla g + \mu\nabla h$

$y\mathbf{i} + (x + z)\mathbf{j} + y\mathbf{k} = \lambda(\mathbf{i} + 2\mathbf{j}) + \mu(\mathbf{i} - 3\mathbf{k})$

$$\left.\begin{aligned} y &= \lambda + \mu \\ x + z &= 2\lambda \\ y &= -3\mu \end{aligned}\right\} y = \frac{3}{4}\lambda \implies x + z = \frac{8}{3}y$$

$$x + 2y = 6 \implies y = 3 - \frac{x}{2}$$

$$x - 3z = 0 \implies z = \frac{x}{3}$$

$$x + \frac{x}{3} = \frac{8}{3}\left(3 - \frac{x}{2}\right)$$

$$x = 3, y = \frac{3}{2}, z = 1$$

$$f\left(3, \frac{3}{2}, 1\right) = 6$$

22. Maximize $f(x, y, z) = xyz$.

Constraints: $x^2 + z^2 = 5$

$x - 2y = 0$

$\nabla f = \lambda\nabla g + \mu\nabla h$

$yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k} = \lambda(2x\mathbf{i} + 2z\mathbf{k}) + \mu(\mathbf{i} - 2\mathbf{j})$

$$yz = 2x\lambda + \mu$$

$$xz = -2\mu \implies \mu = -\frac{xy}{2}$$

$$xy = 2z\lambda \implies \lambda = \frac{xy}{2z}$$

$$x^2 + z^2 = 5 \implies z = \sqrt{5 - x^2}$$

$$x - 2y = 0 \implies y = \frac{x}{2}$$

$$yz = 2x\left(\frac{xy}{2z}\right) - \frac{xz}{2}$$

$$\frac{x\sqrt{5 - x^2}}{2} = \frac{x^3}{2\sqrt{5 - x^2}} - \frac{x\sqrt{5 - x^2}}{2}$$

$$x\sqrt{5 - x^2} = \frac{x^3}{2\sqrt{5 - x^2}}$$

$$2x(5 - x^2) = x^3$$

$$0 = 3x^3 - 10x = x(3x^2 - 10)$$

$$x = 0 \text{ or } x = \sqrt{\frac{10}{3}}, y = \frac{1}{2}\sqrt{\frac{10}{3}}, z = \sqrt{\frac{5}{3}}$$

$$f\left(\sqrt{\frac{10}{3}}, \frac{1}{2}\sqrt{\frac{10}{3}}, \sqrt{\frac{5}{3}}\right) = \frac{5\sqrt{15}}{9}$$

Note: $f(0, 0, \sqrt{5}) = 0$ does not yield a maximum.

23. Minimize the square of the distance $f(x, y) = x^2 + y^2$ subject to the constraint $2x + 3y = -1$.

$$\left.\begin{aligned} 2x &= 2\lambda \\ 2y &= 3\lambda \end{aligned}\right\} y = \frac{3x}{2}$$

$$2x + 3y = -1 \implies x = -\frac{2}{13}, y = -\frac{3}{13}$$

The point on the line is $\left(-\frac{2}{13}, -\frac{3}{13}\right)$ and the desired distance is

$$d = \sqrt{\left(-\frac{2}{13}\right)^2 + \left(-\frac{3}{13}\right)^2} = \frac{\sqrt{13}}{13}.$$

24. Minimize the square of the distance $f(x, y) = x^2 + (y - 10)^2$ subject to the constraint $(x - 4)^2 + y^2 = 4$.

$$\left.\begin{array}{r} 2x = 2(x-4)\lambda \\ 2(y-10) = 2y\lambda \end{array}\right\} \frac{x}{x-4} = \frac{y-10}{y} \implies y = -\frac{5}{2}x + 10$$

$$(x-4)^2 + y^2 = 4 \implies (x^2 - 8x + 16) + \left(\frac{25}{4}x^2 - 50x + 100\right) = 4$$

$$\frac{29}{4}x^2 - 58x + 112 = 0$$

Using a graphing utility, we obtain $x \approx 3.2572$ and $x \approx 4.7428$ or, by the Quadratic Formula,

$$x = \frac{58 \pm \sqrt{58^2 - 4(29/4)(112)}}{2(29/4)} = \frac{58 \pm 2\sqrt{29}}{29/2} = 4 \pm \frac{4\sqrt{29}}{29}.$$

Using the smaller value, we have $x = 4\left(1 - \frac{\sqrt{29}}{29}\right)$ and $y = \frac{10\sqrt{29}}{29} \approx 1.8570.$

The point on the circle is $\left[4\left(1 - \frac{\sqrt{29}}{29}\right), \frac{10\sqrt{29}}{29}\right]$

and the desired distance is $d = \sqrt{16\left(1 - \frac{\sqrt{29}}{29}\right)^2 + \left(\frac{10\sqrt{29}}{29} - 10\right)^2} \approx 8.77.$

The larger x-value does not yield a minimum.

25. Minimize the square of the distance

$$f(x, y, z) = (x-2)^2 + (y-1)^2 + (z-1)^2$$

subject to the constraint $x + y + z = 1$.

$$\left.\begin{array}{r} 2(x-2) = \lambda \\ 2(y-1) = \lambda \\ 2(z-1) = \lambda \end{array}\right\} y = z \text{ and } y = x - 1$$

$$x + y + z = 1 \implies x + 2(x-1) = 1$$

$$x = 1, y = z = 0$$

The point on the plane is $(1, 0, 0)$ and the desired distance is

$$d = \sqrt{(1-2)^2 + (0-1)^2 + (0-1)^2} = \sqrt{3}.$$

26. Minimize the square of the distance

$$f(x, y, z) = (x-4)^2 + y^2 + z^2$$

subject to the constraint $\sqrt{x^2 + y^2} - z = 0$.

$$\left.\begin{array}{r} 2(x-4) = \frac{x}{\sqrt{x^2+y^2}}\lambda = \frac{x}{z}\lambda \\ 2y = \frac{y}{\sqrt{x^2+y^2}}\lambda = \frac{y}{z}\lambda \\ 2z = -\lambda \end{array}\right\} \begin{array}{r} 2(x-4) = -2x \\ 2y = -2y \end{array}$$

$$\sqrt{x^2 + y^2} - z = 0,\ x = 2,\ y = 0,\ z = 2$$

The point on the plane is $(2, 0, 2)$ and the desired distance is

$$d = \sqrt{(2-4)^2 + 0^2 + 2^2} = 2\sqrt{2}.$$

27. Maximize $f(x, y, z) = z$ subject to the constraints $x^2 + y^2 + z^2 = 36$ and $2x + y - z = 2$.

$$\left.\begin{array}{r} 0 = 2x\lambda + 2\mu \\ 0 = 2y\lambda + \mu \\ 1 = 2z\lambda - \mu \end{array}\right\} x = 2y$$

$$x^2 + y^2 + z^2 = 36$$

$$2x + y - z = 2 \implies z = 2x + y - 2 = 5y - 2$$

$$(2y)^2 + y^2 + (5y-2)^2 = 36$$

$$30y^2 - 20y - 32 = 0$$

$$15y^2 - 10y - 16 = 0$$

$$y = \frac{5 \pm \sqrt{265}}{15}$$

Choosing the positive value for y we have the point

$$\left(\frac{10 + 2\sqrt{265}}{15}, \frac{5 + \sqrt{265}}{15}, \frac{-1 + \sqrt{265}}{3}\right).$$

28. Maximize $f(x, y, z) = z$ subject to the constraints $x^2 + y^2 - z^2 = 0$ and $x + 2z = 4$.

$$0 = 2x\lambda + \mu$$

$$0 = 2y\lambda \implies y = 0$$

$$1 = -2z\lambda + 2\mu$$

$$x^2 + y^2 - z^2 = 0$$

$$x + 2z = 4 \implies x = 4 - 2z$$

$$(4 - 2z)^2 + 0^2 - z^2 = 0$$

$$3z^2 - 16z + 16 = 0$$

$$(3z - 4)(z - 4) = 0$$

$$z = \frac{4}{3} \text{ or } z = 4$$

The maximum value of f occurs when $z = 4$ at the point of $(-4, 0, 4)$.

29. Optimization problems that have restrictions or constraints on the values that can be used to produce the optimal solution are called contrained optimization problems.

30. See explanation at the bottom of page 969.

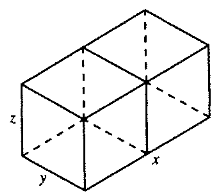
31. Maximize $V(x, y, z) = xyz$ subject to the constraint $x + 2y + 2z = 108$.

$$\left.\begin{aligned} yz &= \lambda \\ xz &= 2\lambda \\ xy &= 2\lambda \end{aligned}\right\} y = z \text{ and } x = 2y$$

$$x + 2y + 2z = 108 \implies 6y = 108, y = 18$$

$$x = 36, y = z = 18$$

Volume is maximum when the dimensions are $36 \times 18 \times 18$ inches.



32. Maximize $V(x, y, z) = xyz$ subject to the constraint $1.5xy + 2xz + 2yz = C$.

$$\left.\begin{aligned} yz &= (1.5y + 2z)\lambda \\ xz &= (1.5x + 2z)\lambda \\ xy &= (2x + 2y)\lambda \end{aligned}\right\} x = y \text{ and } z = \frac{3}{4}x$$

$$1.5xy + 2xz + 2yz = C \implies 1.5x^2 + \frac{3}{2}x^2 + \frac{3}{2}x^2 = C$$

$$x = \frac{\sqrt{2C}}{3}$$

Volume is maximum when

$$x = y = \frac{\sqrt{2C}}{3} \quad \text{and} \quad z = \frac{\sqrt{2C}}{4}.$$

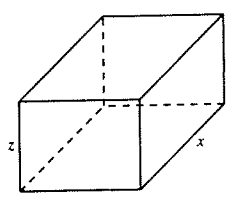
33. Minimize $C(x, y, z) = 5xy + 3(2xz + 2yz + xy)$ subject to the constraint $xyz = 480$.

$$\left.\begin{aligned} 8y + 6z &= yz\lambda \\ 8x + 6z &= xz\lambda \\ 6x + 6y &= xy\lambda \end{aligned}\right\} x = y, 4y = 3z$$

$$xyz = 480 \implies \tfrac{4}{3}y^3 = 480$$

$$x = y = \sqrt[3]{360}, z = \tfrac{4}{3}\sqrt[3]{360}$$

Dimensions: $\sqrt[3]{360} \times \sqrt[3]{360} \times \frac{4}{3}\sqrt[3]{360}$ feet



34. Minimize $A(\pi, r) = 2\pi rh + 2\pi r^2$ subject to the constraint $\pi r^2 h = V_0$.

$$\left.\begin{aligned} 2\pi h + 4\pi r &= 2\pi rh\lambda \\ 2\pi r &= \pi r^2\lambda \end{aligned}\right\} h = 2r$$

$$\pi r^2 h = V_0 \implies 2\pi r^3 = V_0$$

Dimensions: $r = \sqrt[3]{\dfrac{V_0}{2\pi}}$ and $h = 2\sqrt[3]{\dfrac{V_0}{2\pi}}$

35. Maximize $V(x, y, z) = (2x)(2y)(2z) = 8xyz$ subject to the constraint $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$.

$$\left.\begin{aligned} 8yz &= \frac{2x}{a^2}\lambda \\ 8xz &= \frac{2y}{b^2}\lambda \\ 8xy &= \frac{2z}{c^2}\lambda \end{aligned}\right\} \frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{z^2}{c^2}$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \implies \frac{3x^2}{a^2} = 1, \frac{3y^2}{b^2} = 1, \frac{3z^2}{c^2} = 1$$

$$x = \frac{a}{\sqrt{3}}, y = \frac{b}{\sqrt{3}}, z = \frac{c}{\sqrt{3}}$$

Therefore, the dimensions of the box are $\dfrac{2\sqrt{3}a}{3} \times \dfrac{2\sqrt{3}b}{3} \times \dfrac{2\sqrt{3}c}{3}$.

36. (a) Maximize $P(x, y, z) = xyz$ subject to the constraint

$$x + y + z = S.$$

$$\left.\begin{aligned} yz &= \lambda \\ xz &= \lambda \\ xy &= \lambda \end{aligned}\right\} x = y = z$$

$$x + y + z = S \implies x = y = z = \frac{S}{3}$$

Therefore,

$$xyz \le \left(\frac{S}{3}\right)\left(\frac{S}{3}\right)\left(\frac{S}{3}\right),\ x, y, z > 0$$

$$xyz \le \frac{S^3}{27}$$

$$\sqrt[3]{xyz} \le \frac{S}{3}$$

$$\sqrt[3]{xyz} \le \frac{x + y + z}{3}.$$

(b) Maximize $P = x_1x_2x_3 \cdots x_n$ subject to the constraint

$$\sum_{i=1}^{n} x_i = S.$$

$$\left.\begin{aligned} x_2x_3 \cdots x_n &= \lambda \\ x_1x_3 \cdots x_n &= \lambda \\ x_1x_2 \cdots x_n &= \lambda \\ &\vdots \\ x_1x_2x_3 \cdots x_{n-1} &= \lambda \end{aligned}\right\} x_1 = x_2 = x_3 = \cdots = x_n$$

$$\sum_{i=1}^{n} x_i = S \implies x_1 = x_2 = x_3 = \cdots = x_n = \frac{S}{n}$$

Therefore,

$$x_1x_2x_3 \cdots x_n \le \left(\frac{S}{n}\right)\left(\frac{S}{n}\right)\left(\frac{S}{n}\right)\cdots\left(\frac{S}{n}\right),\ x_i \ge 0$$

$$x_1x_2x_3 \cdots x_n \le \left(\frac{S}{n}\right)^n$$

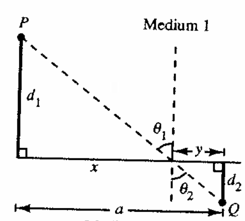
$$\sqrt[n]{x_1x_2x_3 \cdots x_n} \le \frac{S}{n}$$

$$\sqrt[n]{x_1x_2x_3 \cdots x_n} \le \frac{x_1 + x_2 + x_3 + \cdots + x_n}{n}.$$

37. Using the formula Time $= \dfrac{\text{Distance}}{\text{Rate}}$, minimize $T(x, y) = \dfrac{\sqrt{d_1^2 + x^2}}{v_1} + \dfrac{\sqrt{d_2^2 + y^2}}{v_2}$ subject to the constraint $x + y = a$.

$$\left.\begin{aligned} \frac{x}{v_1\sqrt{d_1^2 + x^2}} &= \lambda \\ \frac{y}{v_2\sqrt{d_2^2 + y^2}} &= \lambda \end{aligned}\right\} \frac{x}{v_1\sqrt{d_1^2 + x^2}} = \frac{y}{v_2\sqrt{d_2^2 + y^2}}$$

$$x + y = a$$



Since $\sin\theta_1 = \dfrac{x}{\sqrt{d_1^2 + x^2}}$ and $\sin\theta_2 = \dfrac{y}{\sqrt{d_2^2 + y^2}}$, we have

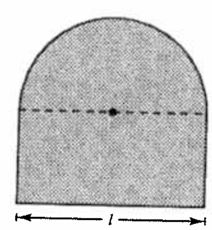
$$\frac{x/\sqrt{d_1^2 + x^2}}{v_1} = \frac{y/\sqrt{d_2^2 + y^2}}{v_2} \quad \text{or} \quad \frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2}.$$

38. Case 1: Minimize $P(l, h) = 2h + l + \left(\dfrac{\pi l}{2}\right)$ subject to the constraint $lh + \left(\dfrac{\pi l^2}{8}\right) = A$.

$$1 + \frac{\pi}{2} = \left(h + \frac{\pi l}{4}\right)\lambda$$

$$2 = l\lambda \implies \lambda = \frac{2}{l},\ 1 + \frac{\pi}{2} = \frac{2h}{l} + \frac{\pi}{2}$$

$$l = 2h$$



Case 2: Minimize $A(l, h) = lh + \left(\dfrac{\pi l^2}{8}\right)$ subject to the constraint $2h + l + \left(\dfrac{\pi l}{2}\right) = P$.

$$h + \frac{\pi l}{4} = \left(\perp + \frac{\pi}{2}\right)\lambda$$

$$l = 2\lambda \implies \lambda = \frac{l}{2},\ h + \frac{\pi l}{4} = \frac{l}{2} + \frac{\pi l}{4}$$

$$h = \frac{l}{2} \text{ or } l = 2h$$

39. Maximize $P(p, q, r) = 2pq + 2pr + 2qr$.

Constraint: $p + q + r = 1$

$\nabla P = \lambda \nabla g$

$$\left.\begin{aligned} 2q + 2r &= \lambda \\ 2p + 2r &= \lambda \\ 2p + 2q &= \lambda \end{aligned}\right\} \Rightarrow 3\lambda = 4(p + q + r) = 4(1) \Rightarrow \lambda = \tfrac{4}{3}$$

$p + q + r = 1$

$$\left.\begin{aligned} q + r &= \tfrac{2}{3} \\ p + q + r &= 1 \end{aligned}\right\} \Rightarrow p = \tfrac{1}{3}, q = \tfrac{1}{3}, r = \tfrac{1}{3}$$

$P\left(\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}\right) = 2\left(\tfrac{1}{3}\right)\left(\tfrac{1}{3}\right) + 2\left(\tfrac{1}{3}\right)\left(\tfrac{1}{3}\right) + 2\left(\tfrac{1}{3}\right)\left(\tfrac{1}{3}\right) = \tfrac{2}{3}$.

40. Maximize $T(x, y, z) = 100 + x^2 + y^2$ subject to the constraints $x^2 + y^2 + z^2 = 50$ and $x - z = 0$.

$$\left.\begin{aligned} 2x &= 2x\lambda + \mu \\ 2y &= 2y\lambda \\ 0 &= 2z\lambda - \mu \end{aligned}\right\}$$

If $y \neq 0$, then $\lambda = 1$ and $\mu = 0$, $z = 0$.

Thus, $x = z = 0$ and $y = \sqrt{50}$.

$T\left(0, \sqrt{50}, 0\right) = 100 + 50 = 150$

If $y = 0$, then $x^2 + z^2 = 2x^2 = 50$ and $x = z = \sqrt{50}/2$.

$$T\left(\frac{\sqrt{50}}{2}, 0, \frac{\sqrt{50}}{2}\right) = 100 + \frac{50}{4} = 112.5$$

Therefore, the maximum temperature is 150.

41. Maximize $P(x, y) = 100x^{0.25}y^{0.75}$ subject to the constraint $48x + 36y = 100{,}000$.

$$25x^{-0.75}y^{0.75} = 48\lambda \Rightarrow \left(\frac{y}{x}\right)^{0.75} = \frac{48\lambda}{25}$$

$$75x^{0.25}y^{-0.25} = 36\lambda \Rightarrow \left(\frac{x}{y}\right)^{0.25} = \frac{36\lambda}{75}$$

$$\left(\frac{y}{x}\right)^{0.75}\left(\frac{y}{x}\right)^{0.25} = \left(\frac{48\lambda}{25}\right)\left(\frac{75}{36\lambda}\right)$$

$$\frac{y}{x} = 4$$

$$y = 4x$$

$48x + 36y = 100{,}000 \Rightarrow 192x = 100{,}000$

$$x = \frac{3125}{6}, y = \frac{6250}{3}$$

Therefore, $P\left(\frac{3125}{6}, \frac{6250}{3}\right) \approx 147{,}314$.

42. Maximize $P(x, y) = 100x^{0.4}y^{0.6}$

Constraint: $48x + 36y = 100{,}000$.

$$40x^{-0.6}y^{0.6} = 48\lambda \Rightarrow \left(\frac{y}{x}\right)^{0.6} = \frac{48\lambda}{40}$$

$$60x^{0.4}y^{-0.4} = 36\lambda \Rightarrow \left(\frac{x}{y}\right)^{0.4} = \frac{36\lambda}{60}$$

$$\left(\frac{y}{x}\right)^{0.6}\left(\frac{y}{x}\right)^{0.4} = \left(\frac{48\lambda}{40}\right)\left(\frac{60}{36\lambda}\right)$$

$$\frac{y}{x} = 2 \Rightarrow y = 2x$$

$$48x + 36y(2x) = 100{,}000 \Rightarrow x = \frac{2500}{3}, y = \frac{5000}{3}$$

$P\left(\frac{2500}{3}, \frac{5000}{3}\right) \approx \$126{,}309.71$.

43. Minimize $C(x, y) = 48x + 36y$ subject to the constraint $100x^{0.25}y^{0.75} = 20{,}000$.

$$48 = 25x^{-0.75}y^{0.75}\lambda \Rightarrow \left(\frac{y}{x}\right)^{0.75} = \frac{48}{25\lambda}$$

$$36 = 75x^{0.25}y^{-0.25}\lambda \Rightarrow \left(\frac{x}{y}\right)^{0.25} = \frac{36}{75\lambda}$$

$$\left(\frac{y}{x}\right)^{0.75}\left(\frac{y}{x}\right)^{0.25} = \left(\frac{48}{25\lambda}\right)\left(\frac{75\lambda}{36}\right)$$

$$\frac{y}{x} = 4 \Rightarrow y = 4x$$

$100x^{0.25}y^{0.75} = 20{,}000 \Rightarrow x^{0.25}(4x)^{0.75} = 200$

$$x = \frac{200}{4^{0.75}} = \frac{200}{2\sqrt{2}} = 50\sqrt{2}$$

$$y = 4x = 200\sqrt{2}$$

Therefore, $C\left(50\sqrt{2}, 200\sqrt{2}\right) \approx \$13{,}576.45$.

44. Minimize $C(x, y) = 48x + 36y$ subject to the constraint $100x^{0.6}y^{0.4} = 20{,}000$.

$$48 = 60x^{-0.4}y^{0.4}\lambda \implies \left(\frac{y}{x}\right)^{0.4} = \frac{48}{60\lambda}$$

$$36 = 40x^{0.6}y^{-0.6}\lambda \implies \left(\frac{x}{y}\right)^{0.6} = \frac{36}{40\lambda}$$

$$\left(\frac{y}{x}\right)^{0.4}\left(\frac{y}{x}\right)^{0.6} = \left(\frac{48}{60\lambda}\right)\left(\frac{40\lambda}{36}\right)$$

$$\frac{y}{x} = \frac{8}{9} \implies y = \frac{8}{9}x$$

$$100x^{0.6}y^{0.4} = 20{,}000 \implies x^{0.6}\left(\frac{8}{9}x\right)^{0.4} = 200$$

$$x = \frac{200}{(8/9)^{0.4}} \approx 209.65$$

$$y = \frac{8}{9}\left[\frac{200}{(8/9)^{0.4}}\right] \approx 186.35$$

Therefore, $C(209.65, 186.35) = \$16{,}771.94$.

45. (a) Maximize $g(\alpha, \beta, \gamma) = \cos\alpha\cos\beta\cos\gamma$ subject to the constraint $\alpha + \beta + \gamma = \pi$.

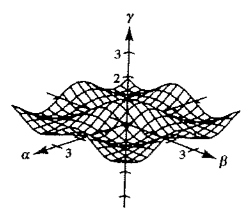
$$\left.\begin{aligned} -\sin\alpha\cos\beta\cos\gamma &= \lambda \\ -\cos\alpha\sin\beta\cos\gamma &= \lambda \\ -\cos\alpha\cos\beta\sin\gamma &= \lambda \end{aligned}\right\} \tan\alpha = \tan\beta = \tan\gamma \implies \alpha = \beta = \gamma$$

$$\alpha + \beta + \gamma = \pi \implies \alpha = \beta = \gamma = \frac{\pi}{3}$$

$$g\left(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}\right) = \frac{1}{8}$$

(b) $\alpha + \beta + \gamma = \pi \implies \gamma = \pi - (\alpha + \beta)$

$$\begin{aligned} g(\alpha + \beta) &= \cos\alpha\cos\beta\cos(\pi - (\alpha + \beta)) \\ &= \cos\alpha\cos\beta[\cos\pi\cos(\alpha + \beta) + \sin\pi\sin(\alpha + \beta)] \\ &= -\cos\alpha\cos\beta\cos(\alpha + \beta) \end{aligned}$$



46. Let r = radius of cylinder, and h = height of cylinder = height of cone.

$S = 2\pi rh + 2\pi r\sqrt{h^2 + r^2}$ = constant surface area

$V = \pi r^2 h + \dfrac{2\pi r^2 h}{3} = \dfrac{5\pi r^2 h}{3}$ volume

We maximize $f(r, h) = r^2 h$ subject to $g(r, h) = rh + r\sqrt{h^2 + r^2} = C$.

$(C - rh)^2 = r^2(h^2 + r^2)$

$C^2 - 2Crh = r^4$

$$h = \frac{C^2 - r^4}{2Cr}$$

$$f(r, h) = F(r) = r^2\left[\frac{C^2 - r^4}{2Cr}\right] = \frac{Cr}{2} - \frac{r^5}{2C}$$

$$F'(r) = \frac{C}{2} - \frac{5r^4}{2C} = 0$$

$$C^2 = 5r^4$$

$$r^2 = \frac{C}{\sqrt{5}}$$

—CONTINUED—

46. —CONTINUED—

$$F''(r) = \frac{-10r^3}{C}$$

$$h = \frac{C^2 - r^4}{2Cr} = \frac{C^2 - C^2/5}{2C(C^2/5)^{1/4}}$$

$$= \frac{(4/5)C}{2(C^2/5)^{1/4}}$$

$$= \frac{2}{5}\frac{C}{r}$$

$$= \frac{2}{5r}(\sqrt{5}r^2)$$

$$= \frac{2\sqrt{5}}{5}r$$

Hence, $\dfrac{h}{r} = \dfrac{2\sqrt{5}}{5}$.

By the Second Derivative Test, this is a maximum.

Review Exercises for Chapter 13

1. No, it is not the graph of a function.

2. Yes, it is the graph of a function.

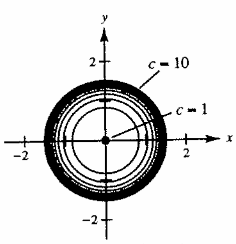
3. $f(x, y) = e^{x^2+y^2}$

The level curves are of the form

$$c = e^{x^2+y^2}$$

$$\ln c = x^2 + y^2.$$

The level curves are circles centered at the origin.



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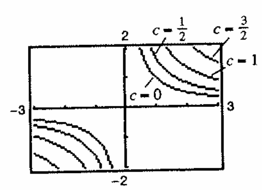
4. $f(x, y) = \ln xy$

The level curves are of the form

$$c = \ln xy$$

$$e^c = xy.$$

The level curves are hyperbolas.



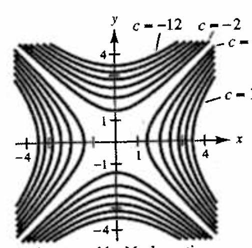
5. $f(x, y) = x^2 - y^2$

The level curves are of the form

$$c = x^2 - y^2$$

$$1 = \frac{x^2}{c} - \frac{y^2}{c}.$$

The level curves are hyperbolas.



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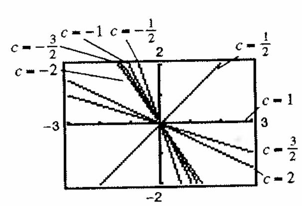
6. $f(x, y) = \dfrac{x}{x+y}$

The level curves are of the form

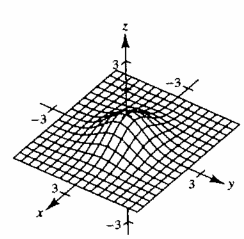
$$c = \frac{x}{x+y}$$

$$y = \left(\frac{1-c}{c}\right)x.$$

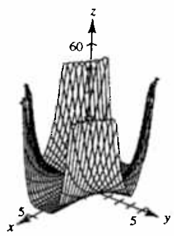
The level curves are passing through the origin with slope $\dfrac{1-c}{c}$.



7. $f(x, y) = e^{-(x^2+y^2)}$



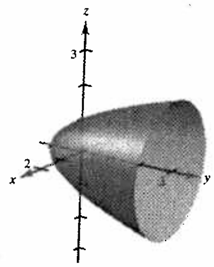
8. $g(x, y) = |y|^{1+|x|}$



9. $f(x, y, z) = x^2 - y + z^2 = 1$

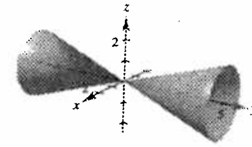
$y = x^2 + z^2 - 1$

Elliptic paraboloid



10. $f(x, y, z) = 9x^2 - y^2 + 9z^2 = 0$

Elliptic cone



11. $\lim_{(x, y)\to(1, 1)} \dfrac{xy}{x^2 + y^2} = \dfrac{1}{2}$

Continuous except at $(0, 0)$.

12. $\lim_{(x, y)\to(1, 1)} \dfrac{xy}{x^2 - y^2}$

Does not exist.

Continuous except when $y = \pm x$.

13. $\lim_{(x, y)\to(0, 0)} \dfrac{-4x^2y}{x^4 + y^2}$

For $y = x^2$, $\dfrac{-4x^2y}{x^4 + y^2} = \dfrac{-4x^4}{x^4 + x^4} = -2$, for $x \neq 0$

For $y = 0$, $\dfrac{-4x^2y}{x^4 + y^2} = 0$, for $x \neq 0$

Thus, the limit does not exist. Continuous except at $(0, 0)$.

14. $\lim_{(x, y)\to(0, 0)} \dfrac{y + xe^{-y^2}}{1 + x^2} = \dfrac{0 + 0}{1 + 0} = 0$

Continuous everywhere.

15. $f(x, y) = e^x \cos y$

$f_x = e^x \cos y$

$f_y = -e^x \sin y$

16. $f(x, y) = \dfrac{xy}{x + y}$

$f_x = \dfrac{y(x + y) - xy}{(x + y)^2} = \dfrac{y^2}{(x + y)^2}$

$f_y = \dfrac{x^2}{(x + y)^2}$

17. $z = xe^y + ye^x$

$\dfrac{\partial z}{\partial x} = e^y + ye^x$

$\dfrac{\partial z}{\partial y} = xe^y + e^x$

18. $z = \ln(x^2 + y^2 + 1)$

$\dfrac{\partial z}{\partial x} = \dfrac{2x}{x^2 + y^2 + 1}$

$\dfrac{\partial z}{\partial y} = \dfrac{2y}{x^2 + y^2 + 1}$

19. $g(x, y) = \dfrac{xy}{x^2 + y^2}$

$g_x = \dfrac{y(x^2 + y^2) - xy(2x)}{(x^2 + y^2)^2} = \dfrac{y(y^2 - x^2)}{(x^2 + y^2)^2}$

$g_y = \dfrac{x(x^2 - y^2)}{(x^2 + y^2)^2}$

20. $w = \sqrt{x^2 + y^2 + z^2}$

$\dfrac{\partial w}{\partial x} = \dfrac{1}{2}(x^2 + y^2 + z^2)^{-1/2}(2x) = \dfrac{x}{\sqrt{x^2 + y^2 + z^2}}$

$\dfrac{\partial w}{\partial y} = \dfrac{y}{\sqrt{x^2 + y^2 + z^2}}$

$\dfrac{\partial w}{\partial z} = \dfrac{z}{\sqrt{x^2 + y^2 + z^2}}$

21. $f(x, y, z) = z \arctan \dfrac{y}{x}$

$f_x = \dfrac{z}{1 + (y^2/x^2)}\left(-\dfrac{y}{x^2}\right) = \dfrac{-yz}{x^2 + y^2}$

$f_y = \dfrac{z}{1 + (y^2/x^2)}\left(\dfrac{1}{x}\right) = \dfrac{xz}{x^2 + y^2}$

$f_z = \arctan \dfrac{y}{x}$

22. $f(x, y, z) = \dfrac{1}{\sqrt{1 - x^2 - y^2 - z^2}}$

$$f_x = -\frac{1}{2}(1 - x^2 - y^2 - z^2)^{-3/2}(-2x)$$

$$= \frac{x}{(1 - x^2 - y^2 - z^2)^{3/2}}$$

$$f_y = \frac{y}{(1 - x^2 - y^2 - z^2)^{3/2}}$$

$$f_z = \frac{z}{(1 - x^2 - y^2 - z^2)^{3/2}}$$

23. $u(x, t) = ce^{-n^2t}\sin(nx)$

$$\frac{\partial u}{\partial x} = cne^{-n^2t}\cos(nx)$$

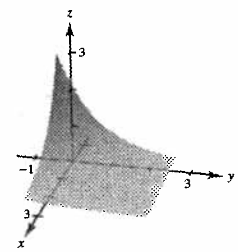
$$\frac{\partial u}{\partial t} = -cn^2e^{-n^2t}\sin(nx)$$

24. $u(x, t) = c(\sin akx)\cos kt$

$$\frac{\partial u}{\partial x} = akc(\cos akx)\cos kt$$

$$\frac{\partial u}{\partial t} = -kc(\sin akx)\sin kt$$

25.



26. $z = x^2\ln(y + 1)$

$\dfrac{\partial z}{\partial x} = 2x\ln(y + 1)$. At $(2, 0, 0)$, $\dfrac{\partial z}{\partial x} = 0$.

Slope in x-direction

$\dfrac{\partial z}{\partial y} = \dfrac{x^2}{1 + y}$. At $(2, 0, 0)$, $\dfrac{\partial z}{\partial y} = 4$.

Slope in y-direction

27. $f(x, y) = 3x^2 - xy + 2y^3$

$$f_x = 6x - y$$

$$f_y = -x + 6y^2$$

$$f_{xx} = 6$$

$$f_{yy} = 12y$$

$$f_{xy} = -1$$

$$f_{yx} = -1$$

28. $h(x, y) = \dfrac{x}{x + y}$

$$h_x = \frac{y}{(x + y)^2}$$

$$h_y = \frac{-x}{(x + y)^2}$$

$$h_{xx} = \frac{-2y}{(x + y)^3}$$

$$h_{yy} = \frac{2x}{(x + y)^3}$$

$$h_{xy} = \frac{(x + y)^2 - 2y(x + y)}{(x + y)^4} = \frac{x - y}{(x + y)^3}$$

$$h_{yx} = \frac{-(x + y)^2 + 2y(x + y)}{(x + y)^4} = \frac{x - y}{(x + y)^3}$$

29. $h(x, y) = x\sin y + y\cos x$

$$h_x = \sin y - y\sin x$$

$$h_y = x\cos y + \cos x$$

$$h_{xx} = -y\cos x$$

$$h_{yy} = -x\sin y$$

$$h_{xy} = \cos y - \sin x$$

$$h_{yx} = \cos y - \sin x$$