CHAPTER 12
Vector-Valued Functions

Section 12.1 Vector-Valued Functions

1. $\mathbf{r}(t) = 5t\mathbf{i} - 4t\mathbf{j} - \frac{1}{t}\mathbf{k}$

Component functions: $f(t) = 5t$

$g(t) = -4t$

$h(t) = -\frac{1}{t}$

Domain: $(-\infty, 0) \cup (0, \infty)$

2. $\mathbf{r}(t) = \sqrt{4 - t^2}\mathbf{i} + t^2\mathbf{j} - 6t\mathbf{k}$

Component functions: $f(t) = \sqrt{4 - t^2}$

$g(t) = t^2$

$h(t) = -6t$

Domain: $[-2, 2]$

3. $\mathbf{r}(t) = \ln t\mathbf{i} - e^t\mathbf{j} - t\mathbf{k}$

Component functions: $f(t) = \ln t$

$g(t) = -e^t$

$h(t) = -t$

Domain: $(0, \infty)$

4. $\mathbf{r}(t) = \sin t\mathbf{i} + 4\cos t\mathbf{j} + t\mathbf{k}$

Component functions: $f(t) = \sin t$

$g(t) = 4\cos t$

$h(t) = t$

Domain: $(-\infty, \infty)$

5. $\mathbf{r}(t) = \mathbf{F}(t) + \mathbf{G}(t) = (\cos t\mathbf{i} - \sin t\mathbf{j} + \sqrt{t}\mathbf{k}) + (\cos t\mathbf{i} + \sin t\mathbf{j}) = 2\cos t\mathbf{i} + \sqrt{t}\mathbf{k}$

Domain: $[0, \infty)$

6.
$$\begin{aligned}\mathbf{r}(t) = \mathbf{F}(t) - \mathbf{G}(t) &= (\ln t\mathbf{i} + 5t\mathbf{j} - 3t^2\mathbf{k}) - (\mathbf{i} + 4t\mathbf{j} - 3t^2\mathbf{k}) \\ &= (\ln t - 1)\mathbf{i} + (5t - 4t)\mathbf{j} + (-3t^2 + 3t^2)\mathbf{k} \\ &= (\ln t - 1)\mathbf{i} + t\mathbf{j}\end{aligned}$$

Domain: $(0, \infty)$

7.
$$\mathbf{r}(t) = \mathbf{F}(t) \times \mathbf{G}(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \sin t & \cos t & 0 \\ 0 & \sin t & \cos t \end{vmatrix} = \cos^2 t\mathbf{i} - \sin t\cos t\mathbf{j} + \sin^2 t\mathbf{k}$$

Domain: $(-\infty, \infty)$

8.
$$\mathbf{r}(t) = \mathbf{F}(t) \times \mathbf{G}(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t^3 & -t & t \\ \sqrt[3]{t} & \frac{1}{t+1} & t+2 \end{vmatrix} = \left(-t(t+2) - \frac{t}{t+1}\right)\mathbf{i} - \left(t^3(t+2) - t\sqrt[3]{t}\right)\mathbf{j} + \left(\frac{t^3}{t+1} + t\sqrt[3]{t}\right)\mathbf{k}$$

Domain: $(-\infty, -1), (-1, \infty)$

9. $\mathbf{r}(t) = \frac{1}{2}t^2\mathbf{i} - (t - 1)\mathbf{j}$

(a) $\mathbf{r}(1) = \frac{1}{2}\mathbf{i}$

(b) $\mathbf{r}(0) = \mathbf{j}$

(c) $\mathbf{r}(s + 1) = \frac{1}{2}(s + 1)^2\mathbf{i} - (s + 1 - 1)\mathbf{j} = \frac{1}{2}(s + 1)^2\mathbf{i} - s\mathbf{j}$

(d) $\mathbf{r}(2 + \Delta t) - \mathbf{r}(2) = \frac{1}{2}(2 + \Delta t)^2\mathbf{i} - (2 + \Delta t - 1)\mathbf{j} - (2\mathbf{i} - \mathbf{j})$

$$= \left(2 + 2\Delta t + \tfrac{1}{2}(\Delta t)^2\right)\mathbf{i} - (1 + \Delta t)\mathbf{j} - 2\mathbf{i} + \mathbf{j}$$

$$= \left(2\Delta t + \tfrac{1}{2}(\Delta t)^2\right)\mathbf{i} - (\Delta t)\mathbf{j}$$

10. $\mathbf{r}(t) = \cos t\,\mathbf{i} + 2\sin t\,\mathbf{j}$

(a) $\mathbf{r}(0) = \mathbf{i}$

(b) $\mathbf{r}\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}\mathbf{i} + \sqrt{2}\mathbf{j}$

(c) $\mathbf{r}(\theta - \pi) = \cos(\theta - \pi)\mathbf{i} + 2\sin(\theta - \pi)\mathbf{j} = -\cos\theta\,\mathbf{i} - 2\sin\theta\,\mathbf{j}$

(d) $\mathbf{r}\left(\frac{\pi}{6} + \Delta t\right) - \mathbf{r}\left(\frac{\pi}{6}\right) = \cos\left(\frac{\pi}{6} + \Delta t\right)\mathbf{i} + 2\sin\left(\frac{\pi}{6} + \Delta t\right)\mathbf{j} - \left(\cos\left(\frac{\pi}{6}\right)\mathbf{i} + 2\sin\frac{\pi}{6}\mathbf{j}\right)$

11. $\mathbf{r}(t) = \ln t\,\mathbf{i} + \frac{1}{t}\mathbf{j} + 3t\mathbf{k}$

(a) $\mathbf{r}(2) = \ln 2\,\mathbf{i} + \frac{1}{2}\mathbf{j} + 6\mathbf{k}$

(b) $\mathbf{r}(-3)$ is not defined. ($\ln(-3)$ does not exist.)

(c) $\mathbf{r}(t - 4) = \ln(t - 4)\mathbf{i} + \frac{1}{t - 4}\mathbf{j} + 3(t - 4)\mathbf{k}$

(d) $\mathbf{r}(1 + \Delta t) - \mathbf{r}(1) = \ln(1 + \Delta t)\mathbf{i} + \frac{1}{1 + \Delta t}\mathbf{j} + 3(1 + \Delta t)\mathbf{k} - (0\mathbf{i} + \mathbf{j} + 3\mathbf{k})$

$$= \ln(1 + \Delta t)\mathbf{i} + \left(\frac{1}{1 + \Delta t} - 1\right)\mathbf{j} + (3\Delta t)\mathbf{k}$$

12. $\mathbf{r}(t) = \sqrt{t}\,\mathbf{i} + t^{3/2}\mathbf{j} + e^{-t/4}\mathbf{k}$

(a) $\mathbf{r}(0) = \mathbf{k}$

(b) $\mathbf{r}(4) = 2\mathbf{i} + 8\mathbf{j} + e^{-1}\mathbf{k}$

(c) $\mathbf{r}(c + 2) = \sqrt{c + 2}\,\mathbf{i} + (c + 2)^{3/2}\mathbf{j} + e^{-[(c+2)/4]}\mathbf{k}$

(d) $\mathbf{r}(9 + \Delta t) - \mathbf{r}(9) = \left(\sqrt{9 + \Delta t}\right)\mathbf{i} + (9 + \Delta t)^{3/2}\mathbf{j} + e^{-[(9+\Delta t)/4]}\mathbf{k} - (3\mathbf{i} + 27\mathbf{j} + e^{-9/4}\mathbf{k})$

$$= \left(\sqrt{9 + \Delta t} - 3\right)\mathbf{i} + ((9 + \Delta t)^{3/2} - 27)\mathbf{j} + (e^{-[(9+\Delta t)/4]} - e^{-9/4})\mathbf{k}$$

13. $\mathbf{r}(t) = \sin 3t\mathbf{i} + \cos 3t\mathbf{j} + t\mathbf{k}$

$\|\mathbf{r}(t)\| = \sqrt{(\sin 3t)^2 + (\cos 3t)^2 + t^2} = \sqrt{1 + t^2}$

14. $\mathbf{r}(t) = \sqrt{t}\,\mathbf{i} + 3t\mathbf{j} - 4t\mathbf{k}$

$\|\mathbf{r}(t)\| = \sqrt{\left(\sqrt{t}\right)^2 + (3t)^2 + (-4t)^2}$

$= \sqrt{t + 9t^2 + 16t^2} = \sqrt{t(1 + 25t)}$

15. $\mathbf{r}(t) \cdot \mathbf{u}(t) = (3t - 1)(t^2) + \left(\frac{1}{4}t^3\right)(-8) + 4(t^3)$

$= 3t^3 - t^2 - 2t^3 + 4t^3 = 5t^3 - t^2$, a scalar.

The dot product is a scalar-valued function.

16. $\mathbf{r}(t) \cdot \mathbf{u}(t) = (3\cos t)(4\sin t) + (2\sin t)(-6\cos t) + (t-2)(t^2) = t^3 - 2t^2$, a scalar.

The dot product is a scalar-valued function.

17. $\mathbf{r}(t) = t\mathbf{i} + 2t\mathbf{j} + t^2\mathbf{k}, -2 \le t \le 2$

$x = t, y = 2t, z = t^2$

Thus, $z = x^2$. Matches (b)

18. $\mathbf{r}(t) = \cos(\pi t)\mathbf{i} + \sin(\pi t)\mathbf{j} + t^2\mathbf{k}, -1 \le t \le 1$

$x = \cos(\pi t), y = \sin(\pi t), z = t^2$

Thus, $x^2 + y^2 = 1$. Matches (c)

19. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + e^{0.75t}\mathbf{k}, -2 \le t \le 2$

$x = t, y = t^2, z = e^{0.75t}$

Thus, $y = x^2$. Matches (d)

20. $\mathbf{r}(t) = t\mathbf{i} + \ln t\mathbf{j} + \frac{2t}{3}\mathbf{k}, 0.1 \le t \le 5$

$x = t,\ y = \ln t,\ z = \frac{2t}{3}$

Thus, $z = \frac{2}{3}x$ and $y = \ln x$. Matches (a)

21. (a) View from the negative x-axis: $(-20, 0, 0)$

(b) View from above the first octant: $(10, 20, 10)$

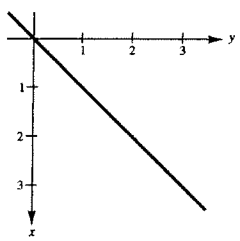
(c) View from the z-axis: $(0, 0, 20)$

(d) View from the positive x-axis: $(20, 0, 0)$

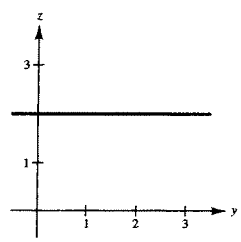
22. $\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + 2\mathbf{k}$

$x = t, y = t, z = 2 \implies x = y$

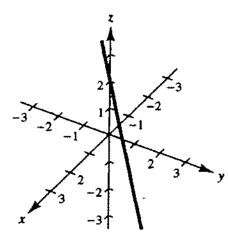
(a) $(0, 0, 20)$



(b) $(10, 0, 0)$



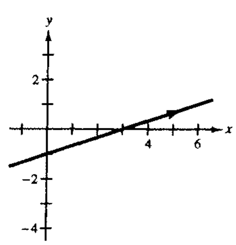
(c) $(5, 5, 5)$



23. $x = 3t$

$y = t - 1$

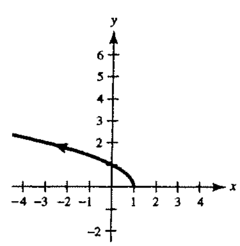
$y = \frac{x}{3} - 1$



24. $x = 1 - t, y = \sqrt{t}$

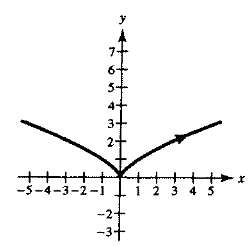
$y = \sqrt{1-x}$

Domain: $t \ge 0$

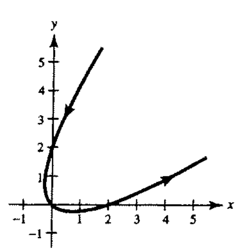


25. $x = t^3, y = t^2$

$y = x^{2/3}$

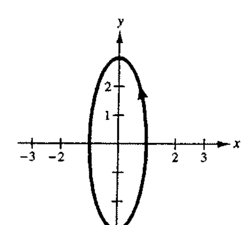


26. $x = t^2 + t, y = t^2 - t$



27. $x = \cos\theta, y = 3\sin\theta$

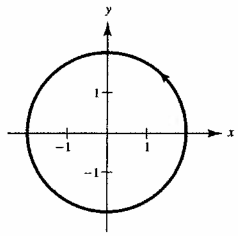
$x^2 + \frac{y^2}{9} = 1$ Ellipse



28. $x = 2\cos t$

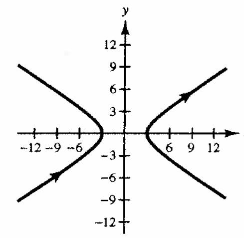
$y = 2\sin t$

$x^2 + y^2 = 4$



29. $x = 3\sec\theta, y = 2\tan\theta$

$\frac{x^2}{9} = \frac{y^2}{4} + 1$ Hyperbola

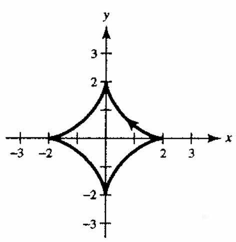


30. $x = 2\cos^3 t, y = 2\sin^3 t$

$$\left(\frac{x}{2}\right)^{2/3} + \left(\frac{y}{2}\right)^{2/3} = \cos^2 t + \sin^2 t$$

$$= 1$$

$$x^{2/3} + y^{2/3} = 2^{2/3}$$



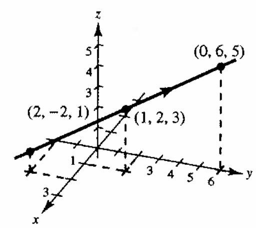
31. $x = -t + 1$

$y = 4t + 2$

$z = 2t + 3$

Line passing through the points:

$(0, 6, 5), (1, 2, 3)$



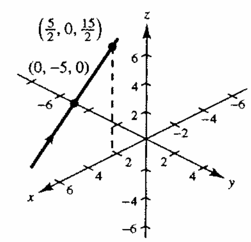
32. $x = t$

$y = 2t - 5$

$y = 3t$

Line passing through the points:

$(0, -5, 0), \left(\frac{5}{2}, 0, \frac{15}{2}\right)$

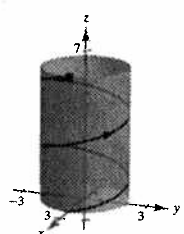


33. $x = 2\cos t,\ y = 2\sin t,\ z = t$

$\frac{x^2}{4} + \frac{y^2}{4} = 1$

$z = t$

Circular helix

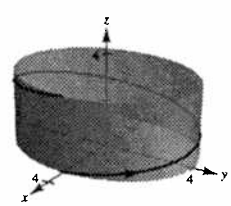


34. $x = 3\cos t,\ y = 4\sin t,\ z = \frac{t}{2}$

$\frac{x^2}{9} + \frac{y^2}{16} = 1$

$z = \frac{t}{2}$

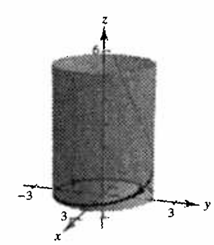
Elliptic helix



35. $x = 2\sin t,\ y = 2\cos t,\ z = e^{-t}$

$x^2 + y^2 = 4$

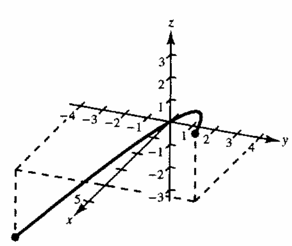
$z = e^{-t}$



36. $x = t^2, y = 2t, z = \frac{3}{2}t$

$x = \frac{y^2}{4}, z = \frac{3}{4}y$

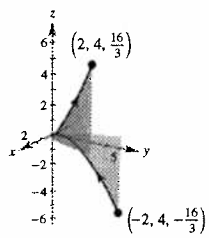
t	-2	-1	0	1	2
x	4	1	0	1	4
y	-4	-2	0	2	4
z	-3	$-\frac{3}{2}$	0	$\frac{3}{2}$	3



37. $x = t,\ y = t^2,\ z = \frac{2}{3}t^3$

$y = x^2,\ z = \frac{2}{3}x^3$

t	-2	-1	0	1	2
x	-2	-1	0	1	2
y	4	1	0	1	4
z	$-\frac{16}{3}$	$-\frac{2}{3}$	0	$\frac{2}{3}$	$\frac{16}{3}$



38. $x = \cos t + t\sin t$

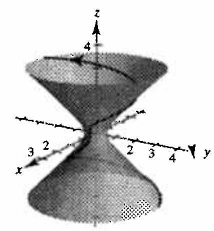
$y = \sin t - t\cos t$

$z = t$

$x^2 + y^2 = 1 + t^2 = 1 + z^2$ or $x^2 + y^2 - z^2 = 1$

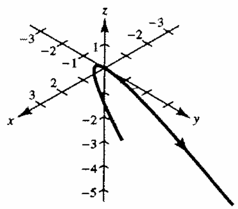
$z = t$

Helix along a hyperboloid of one sheet



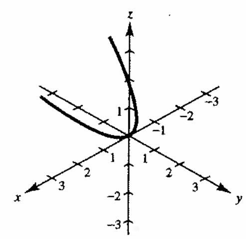
39. $\mathbf{r}(t) = -\frac{1}{2}t^2\mathbf{i} + t\mathbf{j} - \frac{\sqrt{3}}{2}t^2\mathbf{k}$

Parabola



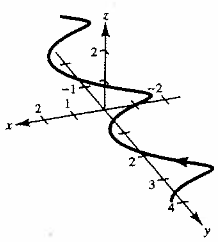
40. $\mathbf{r}(t) = t\mathbf{i} - \frac{\sqrt{3}}{2}t^2\mathbf{j} + \frac{1}{2}t^2\mathbf{k}$

Parabola



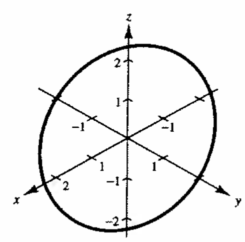
41. $\mathbf{r}(t) = \sin t\mathbf{i} + \left(\frac{\sqrt{3}}{2}\cos t - \frac{1}{2}t\right)\mathbf{j} + \left(\frac{1}{2}\cos t + \frac{\sqrt{3}}{2}\right)\mathbf{k}$

Helix

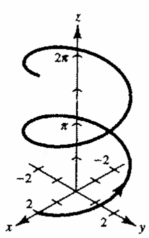


42. $\mathbf{r}(t) = -\sqrt{2}\sin t\mathbf{i} + 2\cos t\mathbf{j} + \sqrt{2}\sin t\mathbf{k}$

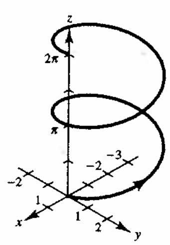
Ellipse



43.

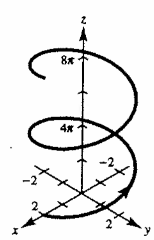


(a)



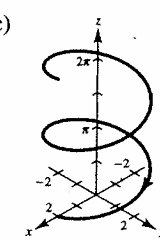
The helix is translated 2 units back on the x-axis.

(b)



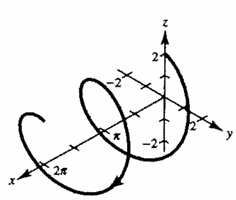
The height of the helix increases at a faster rate.

(c)



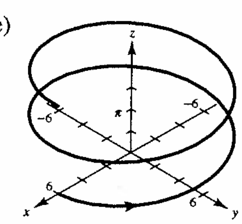
The orientation of the helix is reversed.

(d)



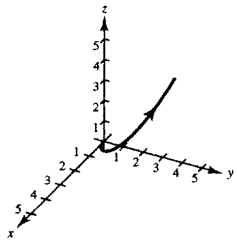
The axis of the helix is the x-axis.

(e)

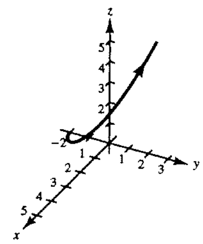


The radius of the helix is increased from 2 to 6.

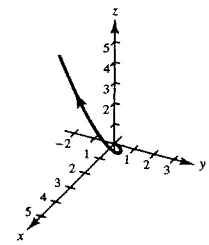
44. $r(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{1}{2}t^3\mathbf{h}$



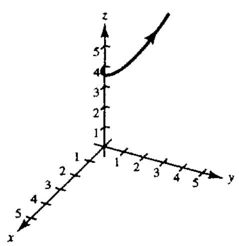
(a) $\mathbf{u}(t) = \mathbf{r}(t) - 2\mathbf{j}$ is a translation 2 units to the left along the y-axis.



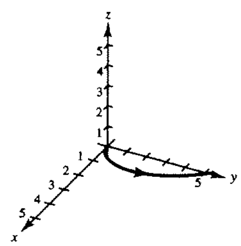
(b) $\mathbf{u}(t) = t^2\mathbf{i} + t\mathbf{j} + \frac{1}{2}t^3\mathbf{k}$ has the roles of x and y interchanged. The graph is a reflection in the plane $x = y$.



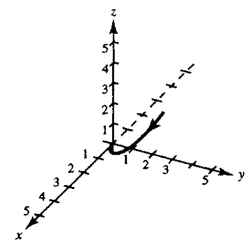
(c) $\mathbf{u}(t) = \mathbf{r}(t) + 4\mathbf{k}$ is an upward shift 4 units.



(d) $\mathbf{u}(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{1}{8}t^3\mathbf{k}$ shrinks the z-value by a factor of 4. The curve rises more slowly.



(e) $\mathbf{u}(t) = \mathbf{r}(-t)$ reverses the orientation.



45. $y = 4 - x$

Let $x = t$, then $y = 4 - t$.

$\mathbf{r}(t) = t\mathbf{i} + (4 - t)\mathbf{j}$

46. $2x - 3y + 5 = 0$

Let $x = t$, then $y = \frac{1}{3}(2t + 5)$.

$\mathbf{r}(t) = t\mathbf{i} + \frac{1}{3}(2\mathbf{t} + 5)\mathbf{j}$

47. $y = (x - 2)^2$

Let $x = t$, then $y = (t - 2)^2$.

$\mathbf{r}(t) = t\mathbf{i} + (t - 2)^2\mathbf{j}$

48. $y = 4 - x^2$

Let $x = t$, then $y = 4 - t^2$.

$\mathbf{r}(t) = t\mathbf{i} + (4 - t^2)\mathbf{j}$

49. $x^2 + y^2 = 25$

Let $x = 5\cos t$, then $y = 5\sin t$.

$\mathbf{r}(t) = 5\cos t\mathbf{i} + 5\sin t\mathbf{j}$

50. $(x - 2)^2 + y^2 = 4$

Let $x - 2 = 2\cos t$, $y = 2\sin t$.

$\mathbf{r}(t) = (2 + 2\cos t)\mathbf{i} + 2\sin t\,\mathbf{j}$

51. $\frac{x^2}{16} - \frac{y^2}{4} = 1$

Let $x = 4\sec t$, $y = 2\tan t$.

$\mathbf{r}(t) = 4\sec t\mathbf{i} + 2\tan t\mathbf{j}$

52. $\frac{x^2}{16} + \frac{y^2}{9} = 1$

Let $x = 4\cos t$, $y = 3\sin t$.

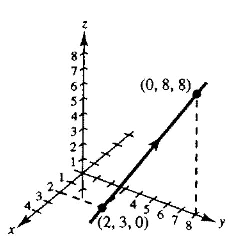
$\mathbf{r}(t) = 4\cos t\mathbf{i} + 3\sin t\mathbf{j}$

53. The parametric equations for the line are

$x = 2 - 2t,\ y = 3 + 5t,\ z = 8t.$

One possible answer is

$\mathbf{r}(t) = (2 - 2t)\mathbf{i} + (3 + 5t)\mathbf{j} + 8t\mathbf{k}.$



54. One possible answer is

$$\mathbf{r}(t) = 1.5\cos t\mathbf{i} + 1.5\sin t\mathbf{j} + \frac{1}{\pi}t\mathbf{k},\ 0 \le t \le 2\pi$$

Note that $\mathbf{r}(2\pi) = 1.5\mathbf{i} + 2\mathbf{k}$.

55. $\mathbf{r}_1(t) = t\mathbf{i}, \quad 0 \le t \le 4 \quad (\mathbf{r}_1(0) = \mathbf{0}, \mathbf{r}_1(4) = 4\mathbf{i})$

$\mathbf{r}_2(t) = (4 - 4t)\mathbf{i} + 6t\mathbf{j}, \quad 0 \le t \le 1 \quad (\mathbf{r}_2(0) = 4\mathbf{i}, \mathbf{r}_2(1) = 6\mathbf{j})$

$\mathbf{r}_3(t) = (6 - t)\mathbf{j}, \quad 0 \le t \le 6 \quad (\mathbf{r}_3(0) = 6\mathbf{j}, \mathbf{r}_3(6) = \mathbf{0})$

(Other answers possible)

56. $\mathbf{r}_1(t) = t\mathbf{i}, \quad 0 \le t \le 10 \quad (\mathbf{r}_1(0) = \mathbf{0}, \mathbf{r}_1(10) = 10\mathbf{i})$

$\mathbf{r}_2(t) = 10(\cos t\mathbf{i} + \sin t\mathbf{j}), \quad 0 \le t \le \frac{\pi}{4} \quad \left(\mathbf{r}_2(0) = 10\mathbf{i}, \mathbf{r}_2\left(\frac{\pi}{4}\right) = 5\sqrt{2}\mathbf{i} + 5\sqrt{2}\mathbf{j}\right)$

$\mathbf{r}_3(t) = 5\sqrt{2}(1 - t)\mathbf{i} + 5\sqrt{2}(1 - t)\mathbf{j}, \quad 0 \le t \le 1 \quad \left(\mathbf{r}_3(0) = 5\sqrt{2}\mathbf{i} + 5\sqrt{2}\mathbf{j}, \mathbf{r}_3(1)\right) = \mathbf{0}$

(Other answers possible)

57. $\mathbf{r}_1(t) = t\mathbf{i} + t^2\mathbf{j}, \quad 0 \le t \le 2 \quad (y = x^2)$

$\mathbf{r}_2(t) = (2 - t)\mathbf{i} + 4\mathbf{j}, \quad 0 \le t \le 2$

$\mathbf{r}_3(t) = (4 - t)\mathbf{j}, \quad 0 \le t \le 4$

(Other answers possible)

58. $\mathbf{r}_1(t) = t\mathbf{i} + \sqrt{t}\mathbf{j}, \quad 0 \le t \le 1 \quad (y = \sqrt{x})$

$\mathbf{r}_2(t) = (1 - t)\mathbf{i} + (1 - t)\mathbf{j}, \quad 0 \le t \le 1 \quad (y = x)$

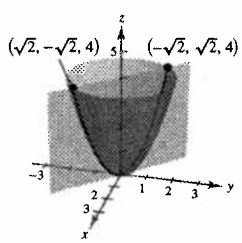
(Other answers possible)

59. $z = x^2 + y^2, \; x + y = 0$

Let $x = t$, then $y = -x = -t$ and $z = x^2 + y^2 = 2t^2$. Therefore,

$$x = t, \; y = -t, \; z = 2t^2.$$

$\mathbf{r}(t) = t\mathbf{i} - t\mathbf{j} + 2t^2\mathbf{k}$

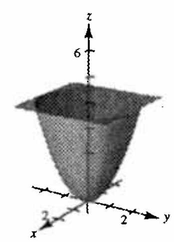


60. $z = x^2 + y^2, \; z = 4$

Therefore, $x^2 + y^2 = 4$ or

$$x = 2\cos t, \; y = 2\sin t, \; z = 4.$$

$\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j} + 4\mathbf{k}$



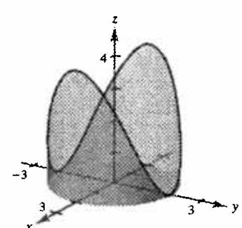
61. $x^2 + y^2 = 4, \; z = x^2$

$x = 2\sin t, \; y = 2\cos t$

$z = x^2 = 4\sin^2 t$

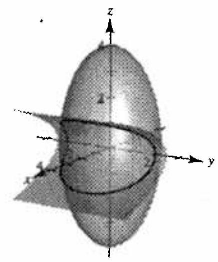
t	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
x	0	1	$\sqrt{2}$	2	$\sqrt{2}$	0
y	2	$\sqrt{3}$	$\sqrt{2}$	0	$-\sqrt{2}$	-2
z	0	1	2	4	2	0

$\mathbf{r}(t) = 2\sin t\mathbf{i} + 2\cos t\mathbf{j} + 4\sin^2 t\mathbf{k}$



62. $4x^2 + 4y^2 + z^2 = 16, x = z^2$

If $z = t$, then $x = t^2$ and $y = \frac{1}{2}\sqrt{16 - 4t^4 - t^2}$.



t	-1.3	-1.2	-1	0	1	1.2
x	1.69	1.44	1	0	1	1.44
y	0.85	1.25	1.66	2	1.66	1.25
z	-1.3	-1.2	-1	0	1	1.2

$\mathbf{r}(t) = t^2\mathbf{i} + \frac{1}{2}\sqrt{16 - 4t^4 - t^2}\mathbf{j} + t\mathbf{k}$

63. $x^2 + y^2 + z^2 = 4,\ x + z = 2$

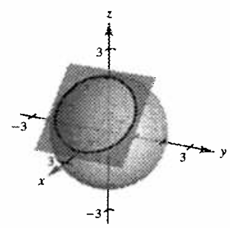
Let $x = 1 + \sin t$, then $z = 2 - x = 1 - \sin t$ and $x^2 + y^2 + z^2 = 4$.

$$(1 + \sin t)^2 + y^2 + (1 - \sin t)^2 = 2 + 2\sin^2 t + y^2 = 4$$

$y^2 = 2\cos^2 t, \quad y = \pm\sqrt{2}\cos t$

$x = 1 + \sin t,\ y = \pm\sqrt{2}\cos t$

$z = 1 - \sin t$



t	$-\frac{\pi}{2}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{2}$
x	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2
y	0	$\pm\frac{\sqrt{6}}{2}$	$\pm\sqrt{2}$	$\pm\frac{\sqrt{6}}{2}$	0
z	2	$\frac{3}{2}$	1	$\frac{1}{2}$	0

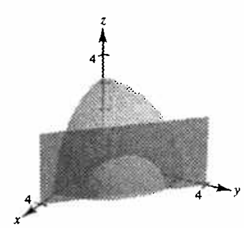
$\mathbf{r}(t) = (1 + \sin t)\mathbf{i} + \sqrt{2}\cos t\mathbf{j} - (1 - \sin t)\mathbf{k}$ and

$\mathbf{r}(t) = (1 + \sin t)\mathbf{i} - \sqrt{2}\cos t\mathbf{j} + (1 - \sin t)\mathbf{k}$

64. $x^2 + y^2 + z^2 = 10,\ x + y = 4$

Let $x = 2 + \sin t$, then $y = 2 - \sin t$ and $z = \sqrt{2(1 - \sin^2 t)} = \sqrt{2}\cos t$.

t	$-\frac{\pi}{2}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{2}$	π
x	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3	2
y	3	$\frac{5}{2}$	2	$\frac{3}{2}$	1	2
z	0	$\frac{\sqrt{6}}{2}$	$\sqrt{2}$	$\frac{\sqrt{6}}{2}$	0	$-\sqrt{2}$



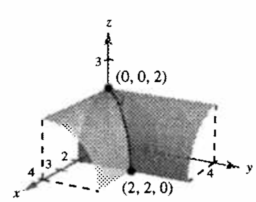
$\mathbf{r}(t) = (2 + \sin t)\mathbf{i} + (2 - \sin t)\mathbf{j} + \sqrt{2}\cos t\mathbf{k}$

65. $x^2 + z^2 = 4,\ y^2 + z^2 = 4$

Subtracting, we have $x^2 - y^2 = 0$ or $y = \pm x$.

Therefore, in the first octant, if we let $x = t$, then $x = t,\ y = t,\ z = \sqrt{4 - t^2}$.

$\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + \sqrt{4 - t^2}\mathbf{k}$



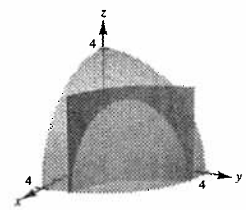
66. $x^2 + y^2 + z^2 = 16,\ xy = 4$ (first octant)

Let $x = t$, then

$$y = \frac{4}{t} \quad \text{and} \quad x^2 + y^2 + z^2 = t^2 + \frac{16}{t^2} + z^2 = 16.$$

$$z = \frac{1}{t}\sqrt{-t^4 + 16t^2 - 16}$$

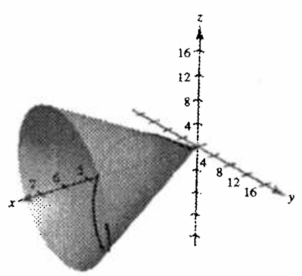
$$\left(\sqrt{8 - 4\sqrt{3}} \le t \le \sqrt{8 + 4\sqrt{3}}\right)$$



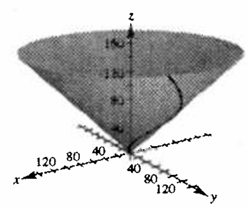
t	$\sqrt{8 + 4\sqrt{3}}$	1.5	2	2.5	3.0	3.5	$\sqrt{8 + 4\sqrt{3}}$
x	1.0	1.5	2	2.5	3.0	3.5	3.9
y	3.9	2.7	2	1.6	1.3	1.1	1.0
z	0	2.6	2.8	2.7	2.3	1.6	0

$$\mathbf{r}(t) = t\mathbf{i} + \frac{4}{t}\mathbf{j} + \frac{1}{t}\sqrt{-t^4 + 16t^2 - 16}\mathbf{k}$$

67. $y^2 + z^2 = (2t\cos t)^2 + (2t\sin t)^2 = 4t^2 = 4x^2$



68. $x^2 + y^2 = (e^{-t}\cos t)^2 + (e^{-t}\sin t)^2 = e^{-2t} = z^2$



69. $\lim_{t\to 2}\left[t\mathbf{i} + \frac{t^2 - 4}{t^2 - 2t}\mathbf{j} + \frac{1}{t}\mathbf{k}\right] = 2\mathbf{i} + 2\mathbf{j} + \frac{1}{2}\mathbf{k}$

since

$$\lim_{t\to 2}\frac{t^2 - 4}{t^2 - 2t} = \lim_{t\to 2}\frac{2t}{2t - 2} = 2. \quad \text{(L'Hôpital's Rule)}$$

70. $\lim_{t\to 0}\left[e^t\mathbf{i} + \frac{\sin t}{t}\mathbf{j} + e^{-t}\mathbf{k}\right] = \mathbf{i} + \mathbf{j} + \mathbf{k}$

since

$$\lim_{t\to 0}\frac{\sin t}{t} = \lim_{t\to 0}\frac{\cos t}{1} = 1 \quad \text{(L'Hôpital's Rule)}$$

71. $\lim_{t\to 0}\left[t^2\mathbf{i} + 3t\mathbf{j} + \frac{1 - \cos t}{t}\mathbf{k}\right] = \mathbf{0}$

since

$$\lim_{t\to 0}\frac{1 - \cos t}{t} = \lim_{t\to 0}\frac{\sin t}{1} = 0. \quad \text{(L'Hôpital's Rule)}$$

72. $\lim_{t\to 1}\left[\sqrt{t}\mathbf{i} + \frac{\ln t}{t^2 - 1}\mathbf{j} + 2t^2\mathbf{k}\right] = \mathbf{i} + \frac{1}{2}\mathbf{j} + 2\mathbf{k}$

since

$$\lim_{t\to 1}\frac{\ln t}{t^2 - 1} = \lim_{t\to 1}\frac{1/t}{2t} = \frac{1}{2}. \quad \text{(L'Hôpital's Rule)}$$

73. $\lim_{t\to 0}\left[\frac{1}{t}\mathbf{i} + \cos t\mathbf{j} + \sin t\mathbf{k}\right]$

does not exist since $\lim_{t\to 0}\frac{1}{t}$ does not exist.

74. $\lim_{t\to\infty}\left[e^{-t}\mathbf{i} + \frac{1}{t}\mathbf{j} + \frac{t}{t^2 + 1}\mathbf{k}\right] = \mathbf{0}$

since

$$\lim_{t\to\infty} e^{-t} = 0, \quad \lim_{t\to\infty}\frac{1}{t} = 0, \text{ and } \lim_{t\to\infty}\frac{t}{t^2 + 1} = 0.$$

75. $\mathbf{r}(t) = t\mathbf{i} + \frac{1}{t}\mathbf{j}$

Continuous on $(-\infty, 0)$, $(0, \infty)$

76. $\mathbf{r}(t) = \sqrt{t}\mathbf{i} + \sqrt{t-1}\mathbf{j}$

Continuous on $[1, \infty)$

77. $\mathbf{r}(t) = t\mathbf{i} + \arcsin t\mathbf{j} + (t-1)\mathbf{k}$

Continuous on $[-1, 1]$

78. $\mathbf{r}(t) = \langle 2e^{-t}, e^{-t}, \ln(t-1)\rangle$

Continuous on $t - 1 > 0$ or $t > 1$: $(1, \infty)$.

79. $\mathbf{r}(t) = \langle e^{-t}, t^2, \tan t\rangle$

Discontinuous at $t = \frac{\pi}{2} + n\pi$

Continuous on $\left(-\frac{\pi}{2} + n\pi, \frac{\pi}{2} + n\pi\right)$

80. $\mathbf{r}(t) = \langle 8, \sqrt{t}, \sqrt[3]{t}\rangle$

Continuous on $[0, \infty)$

81. See the definition on page 832.

82. No. The graph is the same because $\mathbf{r}(t) = \mathbf{u}(t+2)$. For example, if $\mathbf{r}(0)$ is on the graph of $\mathbf{r}$, then $\mathbf{u}(2)$ is the same point.

83. $\mathbf{r}(t) = t^2\mathbf{i} + (t-3)\mathbf{j} + t\mathbf{k}$

(a) $\mathbf{s}(t) = \mathbf{r}(t) + 3\mathbf{k} = t^2\mathbf{i} + (t-3)\mathbf{j} + (t+3)\mathbf{k}$

(b) $\mathbf{s}(t) = \mathbf{r}(t) - 2\mathbf{i} = (t^2-2)\mathbf{i} + (t-3)\mathbf{j} + t\mathbf{k}$

(c) $\mathbf{s}(t) = \mathbf{r}(t) + 5\mathbf{j} = t^2\mathbf{i} + (t+2)\mathbf{j} + t\mathbf{k}$

84. A vector-valued function $\mathbf{r}$ is continuous at $t = a$ if the limit of $\mathbf{r}(t)$ exists as $t \to a$ and

$$\lim_{t\to a} \mathbf{r}(t) = \mathbf{r}(a).$$

The function $\mathbf{r}(t) = \begin{cases} \mathbf{i} + \mathbf{j} & t \ge 0 \\ -\mathbf{i} + \mathbf{j} & t < 0 \end{cases}$ is not continuous at $t = 0$.

85. Let $\mathbf{r}(t) = x_1(t) + y_1(t)\mathbf{j} + z_1(t)\mathbf{k}$ and $\mathbf{u}(t) = x_2(t)\mathbf{i} + y_2(t)\mathbf{j} + z_2(t)\mathbf{k}$. Then:

$$\begin{aligned}\lim_{t\to c}[\mathbf{r}(t)\times\mathbf{u}(t)] &= \lim_{t\to c}\{[y_1(t)z_2(t) - y_2(t)z_1(t)]\mathbf{i} - [x_1(t)z_2(t) - x_2(t)z_1(t)]\mathbf{j} + [x_1(t)y_2(t) - x_2(t)y_1(t)]\mathbf{k}\}\\ &= \left[\lim_{t\to c} y_1(t)\lim_{t\to c} z_2(t) - \lim_{t\to c} y_2(t)\lim_{t\to c} z_1(t)\right]\mathbf{i} - \left[\lim_{t\to c} x_1(t)\lim_{t\to c} z_2(t) - \lim_{t\to c} x_2(t)\lim_{t\to c} z_1(t)\right]\mathbf{j}\\ &\qquad + \left[\lim_{t\to c} x_1(t)\lim_{t\to c} y_2(t) - \lim_{t\to c} x_2(t)\lim_{t\to c} y_1(t)\right]\mathbf{k}\\ &= \left[\lim_{t\to c} x_1(t)\mathbf{i} + \lim_{t\to c} y_1(t)\mathbf{j} + \lim_{t\to c} z_1(t)\mathbf{k}\right]\times\left[\lim_{t\to c} x_2(t)\mathbf{i} + \lim_{t\to c} y_2(t)\mathbf{j} + \lim_{t\to c} z_2(t)\mathbf{k}\right]\\ &= \lim_{t\to c}\mathbf{r}(t)\times\lim_{t\to c}\mathbf{u}(t)\end{aligned}$$

86. Let $\mathbf{r}(t) = x_1(t)\mathbf{i} + y_1(t)\mathbf{j} + z_1(t)\mathbf{k}$ and $\mathbf{u}(t) = x_2(t)\mathbf{i} + y_2(t)\mathbf{j} + z_2(t)\mathbf{k}$. Then:

$$\begin{aligned}\lim_{t\to c}[\mathbf{r}(t)\cdot\mathbf{u}(t)] &= \lim_{t\to c}[x_1(t)x_2(t) + y_1(t)y_2(t) + z_1(t)z_2(t)]\\ &= \lim_{t\to c} x_1(t)\lim_{t\to c} x_2(t) + \lim_{t\to c} y_1(t)\lim_{t\to c} y_2(t) + \lim_{t\to c} z_1(t)\lim_{t\to c} z_2(t)\\ &= \left[\lim_{t\to c} x_1(t)\mathbf{i} + \lim_{t\to c} y_1(t)\mathbf{j} + \lim_{t\to c} z_1(t)\mathbf{k}\right]\cdot\left[\lim_{t\to c} x_2(t)\mathbf{i} + \lim_{t\to c} y_2(t)\mathbf{j} + \lim_{t\to c} z_2(t)\mathbf{k}\right]\\ &= \lim_{t\to c}\mathbf{r}(t)\cdot\lim_{t\to c}\mathbf{u}(t)\end{aligned}$$

87. Let $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$. Since $\mathbf{r}$ is continuous at $t = c$, then $\lim_{t\to c}\mathbf{r}(t) = \mathbf{r}(c)$.

$$\mathbf{r}(c) = x(c)\mathbf{i} + y(c)\mathbf{j} + z(c)\mathbf{k} \Rightarrow x(c),\ y(c),\ z(c)$$

are defined at c.

$$\|\mathbf{r}\| = \sqrt{(x(t))^2 + (y(t))^2 + (z(t))^2}$$

$$\lim_{t\to c}\|\mathbf{r}\| = \sqrt{(x(c))^2 + (y(c))^2 + (z(c))^2} = \|\mathbf{r}(c)\|$$

Therefore, $\|\mathbf{r}\|$ is continuous at c.

88. Let

$$f(t) = \begin{cases} 1, & \text{if } t \ge 0 \\ -1, & \text{if } t < 0 \end{cases}$$

and $\mathbf{r}(t) = f(t)\mathbf{i}$. Then $\mathbf{r}$ is not continuous at $c = 0$, whereas, $\|\mathbf{r}\| = 1$ is continuous for all t.

89. True

90. False. The graph of $x = y = z = t^3$ represents a line.

91. False. Although $\mathbf{r}(4) = \langle 4, 16 \rangle = \mathbf{u}(2)$, they do not collide. Their paths cross this point at different times.

92. True. $y^2 + z^2 = t^2 \sin^2 t + t^2 \cos^2 t = t^2 = x$

Section 12.2 Differentiation and Integration of Vector-Valued Functions

1. $\mathbf{r}(t) = t^2\mathbf{i} + t\mathbf{j},\ t_0 = 2$

$x(t) = t^2,\ y(t) = t$

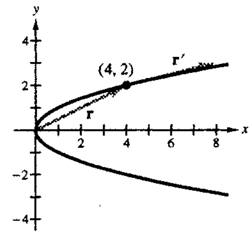
$x = y^2$

$\mathbf{r}(2) = 4\mathbf{i} + 2\mathbf{j}$

$\mathbf{r}'(t) = 2t\mathbf{i} + \mathbf{j}$

$\mathbf{r}'(2) = 4\mathbf{i} + \mathbf{j}$

$\mathbf{r}'(t_0)$ is tangent to the curve.



2. $\mathbf{r}(t) = t\mathbf{i} + t^3\mathbf{j},\ t_0 = 1$

$x(t) = t,\ y(t) = t^3$

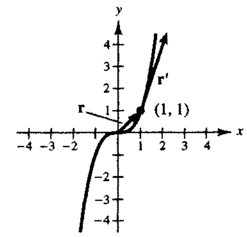
$y = x^3$

$\mathbf{r}(1) = \mathbf{i} + \mathbf{j}$

$\mathbf{r}'(t) = \mathbf{i} + 3t^2\mathbf{j}$

$\mathbf{r}'(1) = \mathbf{i} + 3\mathbf{j}$

$\mathbf{r}'(t_0)$ is tangent to the curve.



3. $\mathbf{r}(t) = t^2\mathbf{i} + \frac{1}{t}\mathbf{j},\ t_0 = 2$

$x(t) = t^2,\ y(t) = \frac{1}{t}$

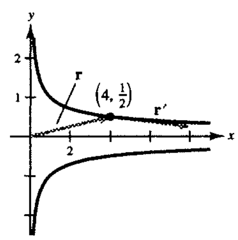
$x = \frac{1}{y^2}$

$\mathbf{r}(2) = 4\mathbf{i} + \frac{1}{2}\mathbf{j}$

$\mathbf{r}'(t) = 2t\mathbf{i} - \frac{1}{t^2}\mathbf{j}$

$\mathbf{r}'(2) = 4\mathbf{i} - \frac{1}{4}\mathbf{j}$

$\mathbf{r}'(t_0)$ is tangent to the curve.



4. (a) $\mathbf{r}(t) = (1 + t)\mathbf{i} + t^3\mathbf{j},\ t_0 = 1$

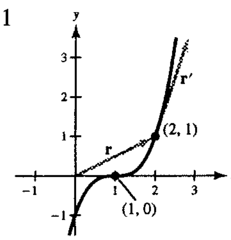
$x = 1 + t$
$y = t^3 = (x - 1)^3$

(b) $\mathbf{r}(1) = 2\mathbf{i} + \mathbf{j}$

$\mathbf{r}'(t) = \mathbf{i} + 3t^2\mathbf{j}$

$\mathbf{r}'(1) = \mathbf{i} + 3\mathbf{j}$

$\mathbf{r}'(1)$ is tangent to the curve.



5. $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j},\ t_0 = \frac{\pi}{2}$

$x(t) = \cos t,\ y(t) = \sin t$

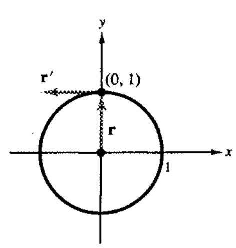
$x^2 + y^2 = 1$

$\mathbf{r}\left(\frac{\pi}{2}\right) = \mathbf{j}$

$\mathbf{r}'(t) = -\sin t\mathbf{i} + \cos t\mathbf{j}$

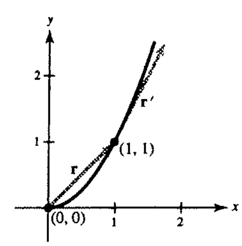
$\mathbf{r}'\left(\frac{\pi}{2}\right) = -\mathbf{i}$

$\mathbf{r}'(t_0)$ is tangent to the curve.



6. (a) $\mathbf{r}(t) = e^t\mathbf{i} + e^{2t}\mathbf{j},\ t_0 = 0$

$x = e^t, y = e^{2t} = x^2 \implies y = x^2, x > 0$



(b) $\mathbf{r}(0) = \mathbf{i} + \mathbf{j}$

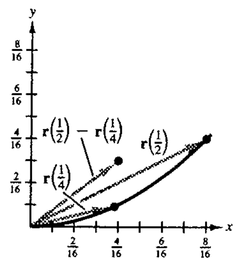
$\mathbf{r}'(t) = e^t\mathbf{i} + 2e^{2t}\mathbf{j}$

$\mathbf{r}'(0) = \mathbf{i} + 2\mathbf{j}$

$\mathbf{r}'(0)$ is tangent to the curve.

7. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}$

(a)



(b)

$$\mathbf{r}\left(\frac{1}{4}\right) = \frac{1}{4}\mathbf{i} + \frac{1}{16}\mathbf{j}$$

$$\mathbf{r}\left(\frac{1}{2}\right) = \frac{1}{2}\mathbf{i} + \frac{1}{4}\mathbf{j}$$

$$\mathbf{r}\left(\frac{1}{2}\right) - \mathbf{r}\left(\frac{1}{4}\right) = \frac{1}{4}\mathbf{i} + \frac{3}{16}\mathbf{j}$$

(c)

$$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j}$$

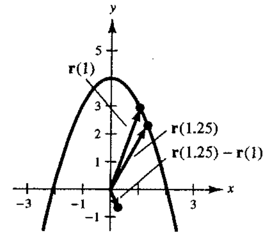
$$\mathbf{r}'\left(\frac{1}{4}\right) = \mathbf{i} + \frac{1}{2}\mathbf{j}$$

$$\frac{\mathbf{r}(1/2) - \mathbf{r}(1/4)}{(1/2) - (1/4)} = \frac{(1/4)\mathbf{i} + (3/16)\mathbf{j}}{1/4} = \mathbf{i} + \frac{3}{4}\mathbf{j}$$

This vector approximates $\mathbf{r}'(\frac{1}{4})$.

8. $\mathbf{r}(t) = t\mathbf{i} + (4 - t^2)\mathbf{j}$

(a)



(b)

$$\mathbf{r}(1) = \mathbf{i} + 3\mathbf{j}$$

$$\mathbf{r}(1.25) = 1.25\mathbf{i} + 2.4375\mathbf{j}$$

$$\mathbf{r}(1.25) - \mathbf{r}(1) = 0.25\mathbf{i} - 0.5625\mathbf{j}$$

(c)

$$\mathbf{r}'(t) = \mathbf{i} - 2t\mathbf{j}$$

$$\mathbf{r}'(1) = \mathbf{i} - 2\mathbf{j}$$

$$\frac{\mathbf{r}(1.25) - \mathbf{r}(1)}{1.25 - 1} = \frac{0.25\mathbf{i} - 0.5625\mathbf{j}}{0.25} = \mathbf{i} - 2.25\mathbf{j}$$

This vector approximates $\mathbf{r}'(1)$.

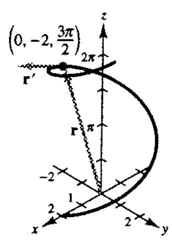
9. (a) and (b) $\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j} + t\mathbf{k},\ t_0 = \dfrac{3\pi}{2}$

$x^2 + y^2 = 4,\ z = t$

$$\mathbf{r}'(t) = -2\sin t\mathbf{i} + 2\cos t\mathbf{j} + \mathbf{k}$$

$$\mathbf{r}\left(\frac{3\pi}{2}\right) = -2\mathbf{j} + \frac{3\pi}{2}\mathbf{k}$$

$$\mathbf{r}'\left(\frac{3\pi}{2}\right) = 2\mathbf{i} + \mathbf{k}$$



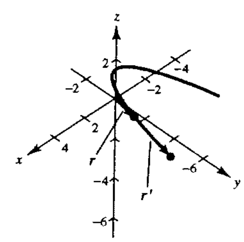
10. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \dfrac{3}{2}\mathbf{k},\ t_0 = 2$

$y = x^2,\ z = \dfrac{3}{2}$

$$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j}$$

$$\mathbf{r}(2) = 2\mathbf{i} + 4\mathbf{j} + \frac{3}{2}\mathbf{k}$$

$$\mathbf{r}'(2) = \mathbf{i} + 4\mathbf{j}$$



11. $\mathbf{r}(t) = 6t\mathbf{i} - 7t^2\mathbf{j} + t^3\mathbf{k}$

$\mathbf{r}'(t) = 6\mathbf{i} - 14t\mathbf{j} + 3t^2\mathbf{k}$

12. $\mathbf{r}(t) = \dfrac{1}{t}\mathbf{i} + 16t\mathbf{j} + \dfrac{t^2}{2}\mathbf{k}$

$$\mathbf{r}'(t) = -\frac{1}{t^2}\mathbf{i} + 16\mathbf{j} + t\mathbf{k}$$

13. $\mathbf{r}(t) = a\cos^3 t\mathbf{i} + a\sin^3 t\mathbf{j} + \mathbf{k}$

$\mathbf{r}'(t) = -3a\cos^2 t\sin t\mathbf{i} + 3a\sin^2 t\cos t\mathbf{j}$

14. $\mathbf{r}(t) = 4\sqrt{t}\mathbf{i} + t^2\sqrt{t}\mathbf{j} + \ln t^2\mathbf{k}$

$$\mathbf{r}'(t) = \frac{2}{\sqrt{t}}\mathbf{i} + \left(2t\sqrt{t} + \frac{t^2}{2\sqrt{t}}\right)\mathbf{j} + \frac{2}{t}\mathbf{k}$$

$$= \frac{2}{\sqrt{t}}\mathbf{i} + \frac{5t^{3/2}}{2}\mathbf{j} + \frac{2}{t}\mathbf{k}$$

15. $\mathbf{r}(t) = e^{-t}\mathbf{i} + 4\mathbf{j}$

$\mathbf{r}'(t) = -e^{-t}\mathbf{i}$

16. $\mathbf{r}(t) = \langle \sin t - t\cos t, \cos t + t\sin t, t^2\rangle$

$\mathbf{r}'(t) = \langle t\sin t, t\cos t, 2t\rangle$

17. $\mathbf{r}(t) = \langle t\sin t, t\cos t, t\rangle$

$\mathbf{r}'(t) = \langle \sin t + t\cos t, \cos t - t\sin t, 1\rangle$

18. $\mathbf{r}(t) = \langle \arcsin t, \arccos t, 0\rangle$

$\mathbf{r}'(t) = \left\langle \frac{1}{\sqrt{1-t^2}}, -\frac{1}{\sqrt{1-t^2}}, 0\right\rangle$

19. $\mathbf{r}(t) = t^3\mathbf{i} + \frac{1}{2}t^2\mathbf{j}$

(a) $\mathbf{r}'(t) = 3t^2\mathbf{i} + t\mathbf{j}$

$\mathbf{r}''(t) = 6t\mathbf{i} + \mathbf{j}$

(b) $\mathbf{r}'(t)\cdot\mathbf{r}''(t) = 3t^2(6t) + t = 18t^3 + t$

20. $\mathbf{r}(t) = (t^2 + t)\mathbf{i} + (t^2 - t)\mathbf{j}$

(a) $\mathbf{r}'(t) = (2t+1)\mathbf{i} + (2t-1)\mathbf{j}$

$\mathbf{r}''(t) = 2\mathbf{i} + 2\mathbf{j}$

(b) $\mathbf{r}'(t)\cdot\mathbf{r}''(t) = (2t+1)(2) + (2t-1)(2) = 8t$

21. $\mathbf{r}(t) = 4\cos t\mathbf{i} + 4\sin t\mathbf{j}$

(a) $\mathbf{r}'(t) = -4\sin t\mathbf{i} + 4\cos t\mathbf{j}$

$\mathbf{r}''(t) = -4\cos t\mathbf{i} - 4\sin t\mathbf{j}$

(b) $\mathbf{r}'(t)\cdot\mathbf{r}''(t) = (-4\sin t)(-4\cos t) + 4\cos t(-4\sin t)$

$= 0$

22. $\mathbf{r}(t) = 8\cos t\mathbf{i} + 3\sin t\mathbf{j}$

(a) $\mathbf{r}'(t) = -8\sin t\mathbf{i} + 3\cos t\mathbf{j}$

$\mathbf{r}''(t) = -8\cos t\mathbf{i} - 3\sin t\mathbf{j}$

(b) $\mathbf{r}'(t)\cdot\mathbf{r}''(t) = (-8\sin t)(-8\cos t) + 3\cos t(-3\sin t)$

$= 55\sin t\cos \mathrm{t}$

23. $\mathbf{r}(t) = \frac{1}{2}t^2\mathbf{i} - t\mathbf{j} + \frac{1}{6}t^3\mathbf{k}$

(a) $\mathbf{r}'(t) = t\mathbf{i} - \mathbf{j} + \frac{1}{2}t^2\mathbf{k}$

$\mathbf{r}''(t) = \mathbf{i} + t\mathbf{k}$

(b) $\mathbf{r}'(t)\cdot\mathbf{r}''(t) = t(1) - 1(0) + \frac{1}{2}t^2(t) = t + \frac{t^3}{2}$

24. $\mathbf{r}(t) = t\mathbf{i} + (2t+3)\mathbf{j} + (3\mathrm{t} - 5)\mathbf{k}$

(a) $\mathbf{r}'(t) = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$

$\mathbf{r}''(t) = 0$

(b) $\mathbf{r}'(t)\cdot\mathbf{r}''(t) = 0$

25. $\mathbf{r}(t) = \langle \cos t + t\sin t, \sin t - t\cos t, t\rangle$

(a) $\mathbf{r}'(t) = \langle -\sin t + \sin t + t\cos t, \cos t - \cos t + t\sin t, 1\rangle$

$= \langle t\cos t, t\sin t, 1\rangle$

$\mathbf{r}''(t) = \langle \cos t - t\sin t, \sin t + t\cos t, 0\rangle$

(b) $\mathbf{r}'(t)\cdot\mathbf{r}''(t) = (t\cos t)(\cos t - t\sin t) + (t\sin t)(\sin t + t\cos t) = t$

26. $\mathbf{r}(t) = \langle e^{-t}, t^2, \tan(t)\rangle$

(a) $\mathbf{r}'(t) = \langle -e^{-t}, 2t, \sec^2 t\rangle$

$\mathbf{r}''(t) = \langle e^{-t}, 2, 2\sec^2 t\tan t\rangle$

(b) $\mathbf{r}'(t)\cdot\mathbf{r}''(t) = -e^{-2t} + 4t + 2\sec^4 t\tan t$

27. $\mathbf{r}(t) = \cos(\pi t)\mathbf{i} + \sin(\pi t)\mathbf{j} + t^2\mathbf{k},\ t_0 = -\frac{1}{4}$

$\mathbf{r}'(t) = -\pi\sin(\pi t)\mathbf{i} + \pi\cos(\pi t)\mathbf{j} + 2t\mathbf{k}$

$\mathbf{r}'\left(-\frac{1}{4}\right) = \frac{\sqrt{2}\pi}{2}\mathbf{i} + \frac{\sqrt{2}\pi}{2}\mathbf{j} - \frac{1}{2}\mathbf{k}$

$\left\|\mathbf{r}'\left(\frac{1}{4}\right)\right\| = \sqrt{\left(\frac{\sqrt{2}\pi}{2}\right)^2 + \left(\frac{\sqrt{2}\pi}{2}\right)^2 + \left(-\frac{1}{2}\right)^2} = \sqrt{\pi^2 + \frac{1}{4}} = \frac{\sqrt{4\pi^2+1}}{2}$

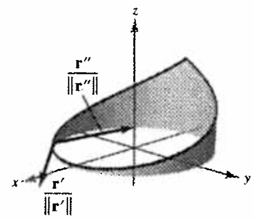
$\frac{\mathbf{r}'(-1/4)}{\|\mathbf{r}'(-1/4)\|} = \frac{1}{\sqrt{4\pi^2+1}}\left(\sqrt{2}\pi\mathbf{i} + \sqrt{2}\pi\mathbf{j} - \mathbf{k}\right)$

$\mathbf{r}''(t) = -\pi^2\cos(\pi t)\mathbf{i} - \pi^2\sin(\pi t)\mathbf{j} + 2\mathbf{k}$

$\mathbf{r}''\left(-\frac{1}{4}\right) = -\frac{\sqrt{2}\pi^2}{2}\mathbf{i} + \frac{\sqrt{2}\pi^2}{2}\mathbf{j} + 2\mathbf{k}$

$\left\|\mathbf{r}''\left(-\frac{1}{4}\right)\right\| = \sqrt{\left(-\frac{\sqrt{2}\pi^2}{2}\right)^2 + \left(\frac{\sqrt{2}\pi^2}{2}\right)^2 + (2)^2} = \sqrt{\pi^4+4}$

$\frac{\mathbf{r}''(-1/4)}{\|\mathbf{r}'(-1/4)\|} = \frac{1}{2\sqrt{\pi^4+4}}\left(-\sqrt{2}\pi^2\mathbf{i} + \sqrt{2}\pi^2\mathbf{j} + 4\mathbf{k}\right)$



28. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + e^{0.75t}\mathbf{k},\ t_0 = \frac{1}{4}$

$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j} + 0.75e^{0.75t}\mathbf{k}$

$\mathbf{r}'\left(\frac{1}{4}\right) = \mathbf{i} + \frac{1}{2}\mathbf{j} + 0.75e^{0.1875}\mathbf{k} = \mathbf{i} + \frac{1}{2}\mathbf{j} + \frac{3}{4}e^{3/16}\mathbf{k}$

$\left\|\mathbf{r}'\left(\frac{1}{4}\right)\right\| = \sqrt{1^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{3}{4}e^{3/16}\right)^2} = \sqrt{\frac{5}{4} + \frac{9}{16}e^{3/8}} = \frac{\sqrt{20+9e^{3/8}}}{4}$

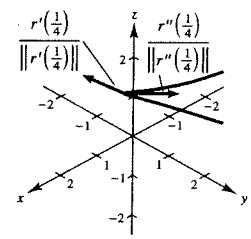
$\frac{\mathbf{r}'(1/4)}{\|\mathbf{r}'(1/4)\|} = \frac{1}{\sqrt{20+9e^{3/8}}}(4\mathbf{i} + 2\mathbf{j} + 3e^{3/16}\mathbf{k})$

$\mathbf{r}''(t) = 2\mathbf{i} + \frac{9}{16}e^{0.75t}\mathbf{k}$

$\mathbf{r}''\left(\frac{1}{4}\right) = 2\mathbf{i} + \frac{9}{16}e^{3/16}\mathbf{k}$

$\left\|\mathbf{r}''\left(\frac{1}{4}\right)\right\| = \sqrt{2^2 + \left(\frac{9}{16}e^{3/16}\right)^2} = \sqrt{4 + \frac{81}{256}e^{3/8}} = \frac{\sqrt{1024+81e^{3/8}}}{16}$

$\frac{\mathbf{r}''(1/4)}{\|\mathbf{r}'(1/4)\|} = \frac{1}{\sqrt{1024+81e^{3/8}}}(32\mathbf{j} + 9e^{3/16}\mathbf{k})$



29. $\mathbf{r}(t) = t^2\mathbf{i} + t^3\mathbf{j}$

$\mathbf{r}'(t) = 2t\mathbf{i} + 3t^2\mathbf{j}$

$\mathbf{r}'(0) = \mathbf{0}$

Smooth on $(-\infty, 0),\ (0, \infty)$

30. $\mathbf{r}(t) = \frac{1}{t-1}\mathbf{i} + 3t\mathbf{j}$

$\mathbf{r}'(t) = -\frac{1}{(t-1)^2}\mathbf{i} + 3\mathbf{j}$

Not continuous when $t = 1$

Smooth on $(-\infty, 1),\ (1, \infty)$

31. $\mathbf{r}(\theta) = 2\cos^3\theta\mathbf{i} + 3\sin^3\theta\mathbf{j}$

$\mathbf{r}'(\theta) = -6\cos^2\theta\sin\theta\mathbf{i} + 9\sin^2\theta\cos\theta\mathbf{j}$

$\mathbf{r}'\left(\frac{n\pi}{2}\right) = \mathbf{0}$

Smooth on $\left(\frac{n\pi}{2}, \frac{(n+1)\pi}{2}\right)$, n any integer.

32. $\mathbf{r}(\theta) = (\theta + \sin\theta)\mathbf{i} + (1 - \cos\theta)\mathbf{j}$

$\mathbf{r}'(\theta) = (1 + \cos\theta)\mathbf{i} + \sin\theta\mathbf{j}$

$\mathbf{r}'((2n-1)\pi) = \mathbf{0}$, n any integer

Smooth on $((2n-1)\pi, (2n+1)\pi)$

33. $\mathbf{r}(\theta) = (\theta - 2\sin\theta)\mathbf{i} + (1 - 2\cos\theta)\mathbf{j}$

$\mathbf{r}'(\theta) = (1 - 2\cos\theta)\mathbf{i} + (2\sin\theta)\mathbf{j}$

$\mathbf{r}'(\theta) \neq \mathbf{0}$ for any value of θ

Smooth on $(-\infty, \infty)$

34. $\mathbf{r}(t) = \frac{2t}{8+t^3}\mathbf{i} + \frac{2t^2}{8+t^3}\mathbf{j}$

$\mathbf{r}'(t) = \frac{16 - 4t^3}{(t^3+8)^2}\mathbf{i} + \frac{32t - 2t^4}{(t^3+8)^2}\mathbf{j}$

$\mathbf{r}'(t) \neq \mathbf{0}$ for any value of t.

$\mathbf{r}$ is not continuous when $t = -2$.

Smooth on $(-\infty, -2), (-2, \infty)$

35. $\mathbf{r}(t) = (t-1)\mathbf{i} + \frac{1}{t}\mathbf{j} - t^2\mathbf{k}$

$\mathbf{r}'(t) = \mathbf{i} - \frac{1}{t^2}\mathbf{j} - 2t\mathbf{k} \neq \mathbf{0}$

$\mathbf{r}$ is smooth for all $t \neq 0$: $(-\infty, 0), (0, \infty)$

36. $\mathbf{r}(t) = e^t\mathbf{i} - e^{-t}\mathbf{j} + 3t\mathbf{k}$

$\mathbf{r}'(t) = e^t\mathbf{i} + e^{-t}\mathbf{j} + 3\mathbf{k} \neq \mathbf{0}$

$\mathbf{r}$ is smooth for all t: $(-\infty, \infty)$

37. $\mathbf{r}(t) = t\mathbf{i} - 3t\mathbf{j} + \tan t\mathbf{k}$

$\mathbf{r}'(t) = \mathbf{i} - 3\mathbf{j} + \sec^2 t\mathbf{k} \neq \mathbf{0}$

$\mathbf{r}$ is smooth for all $t \neq \frac{\pi}{2} + n\pi = \frac{2n+1}{2}\pi$.

Smooth on intervals of form $\left(-\frac{\pi}{2} + n\pi, \frac{\pi}{2} + n\pi\right)$

38. $\mathbf{r}(t) = \sqrt{t}\mathbf{i} + (t^2 - 1)\mathbf{j} + \frac{1}{4}t\mathbf{k}$

$\mathbf{r}'(t) = \frac{1}{2\sqrt{t}}\mathbf{i} + 2t\mathbf{j} + \frac{1}{4}\mathbf{k} \neq \mathbf{0}$

$\mathbf{r}$ is smooth for all $t > 0$: $(0, \infty)$

39. $\mathbf{r}(t) = t\mathbf{i} + 3t\mathbf{j} + t^2\mathbf{k}$, $\mathbf{u}(t) = 4t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$

(a) $\mathbf{r}'(t) = \mathbf{i} + 3\mathbf{j} + 2t\mathbf{k}$

(b) $\mathbf{r}''(t) = 2\mathbf{k}$

(c) $\mathbf{r}(t) \cdot \mathbf{u}(t) = 4t^2 + 3t^3 + t^5$

$D_t[\mathbf{r}(t) \cdot \mathbf{u}(t)] = 8t + 9t^2 + 5t^4$

(d) $3\mathbf{r}(t) - \mathbf{u}(t) = -t\mathbf{i} + (9t - t^2)\mathbf{j} + (3t^2 - t^3)\mathbf{k}$

$D_t[3\mathbf{r}(t) - \mathbf{u}(t)] = -\mathbf{i} + (9 - 2t)\mathbf{j} + (6t - 3t^2)\mathbf{k}$

(e) $\mathbf{r}(t) \times \mathbf{u}(t) = 2t^4\mathbf{i} - (t^4 - 4t^3)\mathbf{j} + (t^3 - 12t^2)\mathbf{k}$

$D_t[\mathbf{r}(t) \times \mathbf{u}(t)] = 8t^3\mathbf{i} + (12t^2 - 4t^3)\mathbf{j} + (3t^2 - 24t)\mathbf{k}$

(f) $\|\mathbf{r}(t)\| = \sqrt{10t^2 + t^4} = t\sqrt{10 + t^2}$

$D_t[\|\mathbf{r}(t)\|] = \frac{10 + 2t^2}{\sqrt{10 + t^2}}$

40. $\mathbf{r}(t) = t\mathbf{i} + 2\sin t\mathbf{j} + 2\cos t\mathbf{k}$

$\mathbf{u}(t) = \frac{1}{t}\mathbf{i} + 2\sin t\mathbf{j} + 2\cos t\mathbf{k}$

(a) $\mathbf{r}'(t) = \mathbf{i} + 2\cos t\mathbf{j} - 2\sin t\mathbf{k}$

(b) $\mathbf{r}''(t) = -2\sin t\mathbf{j} - 2\cos t\mathbf{k}$

(c) $\mathbf{r}(t) \cdot \mathbf{u}(t) = 1 + 4\sin^2 t + 4\cos^2 t = 5$

$D_t[\mathbf{r}(t) \cdot \mathbf{u}(t)] = 0, t \neq 0$

(d) $3\mathbf{r}(t) - \mathbf{u}(t) = \left(3t - \frac{1}{t}\right)\mathbf{i} + 4\sin t\mathbf{j} + 4\cos t\mathbf{k}$

$D_t[3\mathbf{r}(t) - \mathbf{u}(t)] = \left(3 - \frac{1}{t^2}\right)\mathbf{i} + 4\cos t\mathbf{j} - 4\sin t\mathbf{k}$

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40. —CONTINUED—

(e) $\mathbf{r}(t) \times \mathbf{u}(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t & 2\sin t & 2\cos t \\ 1/t & 2\sin t & 2\cos t \end{vmatrix}$

$= 2\cos t\left(\frac{1}{t} - t\right)\mathbf{j} + 2\sin t\left(t - \frac{1}{t}\right)\mathbf{k}$

$D_t[\mathbf{r}(t) - \mathbf{u}(t)] = \left[-2\sin t\left(\frac{1}{t} - t\right) + 2\cos t\left(-\frac{1}{t^2} - 1\right)\right]\mathbf{j}$

$+ \left[2\cos t\left(t - \frac{1}{t}\right) + 2\sin t\left(1 + \frac{1}{t^2}\right)\right]\mathbf{k}$

(f) $\|\mathbf{r}(t)\| = \sqrt{t^2 + 4}$

$D_t(\|\mathbf{r}(t)\|) = \frac{1}{2}(t^2 + 4)^{-1/2}(2t) = \frac{t}{\sqrt{t^2 + 4}}$

41. $\mathbf{r}(t) = t\mathbf{i} + 2t^2\mathbf{j} + t^3\mathbf{k},\ \mathbf{u}(t) = t^4\mathbf{k}$

(a) $\mathbf{r}(t) \cdot \mathbf{u}(t) = t^7$

$D_t[\mathbf{r}(t) \cdot \mathbf{u}(t)] = 7t^6$

Alternate Solution:

$D_t[\mathbf{r}(t) \cdot \mathbf{u}(t)] = \mathbf{r}(t) \cdot \mathbf{u}'(t) + \mathbf{r}'(t) \cdot \mathbf{u}(t)$

$= (t\mathbf{i} + 2t^2\mathbf{j} + t^3\mathbf{k}) \cdot (4t^3\mathbf{k}) + (\mathbf{i} + 4t\mathbf{j} + 3t^2\mathbf{k}) \cdot (t^4\mathbf{k})$

$= 4t^6 + 3t^6 = 7t^6$

(b) $\mathbf{r}(t) \times \mathbf{u}(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t & 2t^2 & t^3 \\ 0 & 0 & t^4 \end{vmatrix} = 2t^6\mathbf{i} - t^5\mathbf{j}$

$D_t[\mathbf{r}(t) \times \mathbf{u}(t)] = 12t^5\mathbf{i} - 5t^4\mathbf{j}$

Alternate Solution:

$D_t[\mathbf{r}(t) \times \mathbf{u}(t)] = \mathbf{r}(t) \times \mathbf{u}'(t) + \mathbf{r}'(t) \times \mathbf{u}(t)$

$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t & 2t^2 & t^3 \\ 0 & 0 & 4t^3 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 4t & 3t^2 \\ 0 & 0 & t^4 \end{vmatrix} = 12t^5\mathbf{i} - 5t^4\mathbf{j}$

42. $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k},\ \mathbf{u}(t) = \mathbf{j} + t\mathbf{k}$

(a) $\mathbf{r}(t) \cdot \mathbf{u}(t) = \sin t + t^2$

$D_t[\mathbf{r}(t) \cdot \mathbf{u}(t)] = \cos t + 2t$

Alternate Solution:

$D_t[\mathbf{r}(t) \cdot \mathbf{u}(t)] = \mathbf{r}(t) \cdot \mathbf{u}'(t) + \mathbf{r}'(t) \cdot \mathbf{u}(t)$

$= (\cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}) \cdot \mathbf{k} + (-\sin t\mathbf{i} + \cos t\mathbf{j} + \mathbf{k}) \cdot (\mathbf{j} + t\mathbf{k})$

$= t + \cos t + t = 2t + \cos t$

(b) $\mathbf{r}(t) \times \mathbf{u}(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos t & \sin t & t \\ 0 & 1 & t \end{vmatrix} = (t\sin t - t)\mathbf{i} - (t\cos t)\mathbf{j} + \cos t\mathbf{k}$

$D_t[\mathbf{r}(t) \times \mathbf{u}(t)] = (t\cos t + \sin t - 1)\mathbf{i} - (\cos t - t\sin t)\mathbf{j} - \sin t\mathbf{k}$

Alternate Solution:

$D_t[\mathbf{r}(t) \times \mathbf{u}(t)] = \mathbf{r}(t) \times \mathbf{u}'(t) + \mathbf{r}'(t) \times \mathbf{u}(t)$

$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos t & \sin t & t \\ 0 & 0 & 1 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin t & \cos t & 1 \\ 0 & 1 & t \end{vmatrix} = (\sin t + t\cos t - 1)\mathbf{i} + (t\sin t - \cos t)\mathbf{j} - \sin t\mathbf{k}$

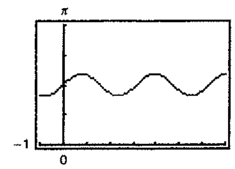
43. $\mathbf{r}(t) = 3\sin t\mathbf{i} + 4\cos t\mathbf{j}$

$\mathbf{r}'(t) = 3\cos t\mathbf{i} - 4\sin t\mathbf{j}$

$\mathbf{r}(t)\cdot\mathbf{r}'(t) = 9\sin t\cos t - 16\cos t\sin t = -7\sin t\cos t$

$$\cos\theta = \frac{\mathbf{r}(t)\cdot\mathbf{r}'(t)}{\|\mathbf{r}(t)\|\,\|\mathbf{r}'(t)\|} = \frac{-7\sin t\cos t}{\sqrt{9\sin^2 t + 16\cos^2 t}\sqrt{9\cos^2 t + 16\sin^2 t}}$$

$$\theta = \arccos\left[\frac{-7\sin t\cos t}{\sqrt{(9\sin^2 t + 16\cos^2 t)(9\cos^2 t + 16\sin^2 t)}}\right]$$



$\theta = 1.855$ maximum at $t = 3.927 = \left(\frac{5\pi}{4}\right)$ and $t = 0.785 = \left(\frac{\pi}{4}\right)$.

$\theta = 1.287$ minimum at $t = 2.356 = \left(\frac{3\pi}{4}\right)$ and $t = 5.498 = \left(\frac{7\pi}{4}\right)$.

$\theta = \frac{\pi}{2} = (1.571)$ for $t = \frac{n\pi}{2}, n = 0, 1, 2, 3, \ldots$

44. $\mathbf{r}(t) = t^2\mathbf{i} + t\mathbf{j}$

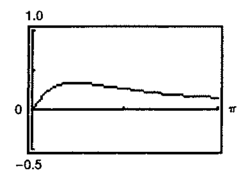
$\mathbf{r}'(t) = 2t\mathbf{i} + \mathbf{j}$

$\mathbf{r}(t)\cdot\mathbf{r}'(t) = 2t^3 + t$

$\|\mathbf{r}(t)\| = \sqrt{t^4 + t^2},\ \|\mathbf{r}'(t)\| = \sqrt{4t^2 + 1}$

$$\cos\theta = \frac{2t^3 + t}{\sqrt{t^4 + t^2}\sqrt{4t^2 + 1}}$$

$$\theta = \arccos\frac{2t^3 + t}{\sqrt{t^4 + t^2}\sqrt{4t^2 + 1}}$$



$\theta = 0.340$ ($\approx 19.47°$)maximum at $t = 0.707 = \left(\frac{\sqrt{2}}{2}\right)$.

$\theta \neq \frac{\pi}{2}$ for any t.

45. $$\mathbf{r}'(t) = \lim_{\Delta t\to 0}\frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t} = \lim_{\Delta t\to 0}\frac{[3(t+\Delta t) + 2]\mathbf{i} + [1 - (t+\Delta t)^2]\mathbf{j} - (3t+2)\mathbf{i} - (1 - t^2)\mathbf{j}}{\Delta t}$$

$$= \lim_{\Delta t\to 0}\frac{(3\Delta t)\mathbf{i} - (2t(\Delta t) + (\Delta t)^2)\mathbf{j}}{\Delta t} = \lim_{\Delta t\to 0} 3\mathbf{i} - (2t + \Delta t)\mathbf{j} = 3\mathbf{i} - 2t\mathbf{j}$$

46. $$\mathbf{r}'(t) = \lim_{\Delta t\to 0}\frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t}$$

$$= \lim_{\Delta t\to 0}\frac{\left[\sqrt{t+\Delta t}\,\mathbf{i} + \frac{3}{t+\Delta t}\mathbf{j} - 2(t+\Delta t)\mathbf{k}\right] - \left[\sqrt{t}\,\mathbf{i} + \frac{3}{t}\mathbf{j} - 2t\mathbf{k}\right]}{\Delta t}$$

$$= \lim_{\Delta t\to 0}\left[\frac{\sqrt{t+\Delta t} - \sqrt{t}}{\Delta t}\mathbf{i} + \frac{\frac{3}{t+\Delta t} - \frac{3}{t}}{\Delta t}\mathbf{j} - 2\mathbf{k}\right]$$

$$= \lim_{\Delta t\to 0}\left[\frac{\Delta t}{\Delta t\left(\sqrt{t+\Delta t} + \sqrt{t}\right)}\mathbf{i} + \frac{-3\Delta t}{(t+\Delta t)t(\Delta t)}\mathbf{j} - 2\mathbf{k}\right]$$

$$= \lim_{\Delta t\to 0}\left[\frac{1}{\sqrt{t+\Delta t} + \sqrt{t}}\mathbf{i} - \frac{3}{(t+\Delta t)t}\mathbf{j} - 2\mathbf{k}\right]$$

$$= \frac{1}{2\sqrt{t}}\mathbf{i} - \frac{3}{t^2}\mathbf{j} - 2\mathbf{k}$$

47. $\mathbf{r}'(t) = \lim_{\Delta t \to 0} \dfrac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t}$

$= \lim_{\Delta t \to 0} \dfrac{\langle (t + \Delta t)^2, 0, 2(t + \Delta t)\rangle - \langle t^2, 0, 2t\rangle}{\Delta t}$

$= \lim_{\Delta t \to 0} \dfrac{\langle 2t\Delta t + (\Delta t)^2, 0, 2\Delta t\rangle}{\Delta t}$

$= \lim_{\Delta t \to 0} \langle 2t + \Delta t, 0, 2\rangle$

$= \langle 2t, 0, 2\rangle$

48. $\mathbf{r}'(t) = \lim_{\Delta t \to 0} \dfrac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t}$

$= \lim_{\Delta t \to 0} \dfrac{\langle 0, \sin(t + \Delta t), 4(t + \Delta t)\rangle - \langle 0, \sin t, 4t\rangle}{\Delta t}$

$= \lim_{\Delta t \to 0} \dfrac{\langle 0, \sin t \cdot \cos(\Delta t) + \sin(\Delta t)\cos t - \sin t, 4\Delta t\rangle}{\Delta t}$

$= \lim_{\Delta t \to 0} \left\langle 0, \dfrac{\sin t(\cos(\Delta t) - 1)}{\Delta t} + \cos t\left(\dfrac{\sin(\Delta t)}{\Delta t}\right), 4\right\rangle$

$= \langle 0, 0 + \cos t, 4\rangle$

$= \langle 0, \cos t, 4\rangle$

49. $\displaystyle\int (2t\mathbf{i} + \mathbf{j} + \mathbf{k})\,dt = t^2\mathbf{i} + t\mathbf{j} + t\mathbf{k} + \mathbf{C}$

50. $\displaystyle\int \left(4t^3\mathbf{i} + 6t\mathbf{j} - 4\sqrt{t}\mathbf{k}\right) dt = t^4\mathbf{i} + 3t^2\mathbf{j} - \frac{8}{3}t^{3/2}\mathbf{k} + \mathbf{C}$

51. $\displaystyle\int \left(\frac{1}{t}\mathbf{i} + \mathbf{j} - t^{3/2}\mathbf{k}\right) dt = \ln t\mathbf{i} + t\mathbf{j} - \frac{2}{5}t^{5/2}\mathbf{k} + \mathbf{C}$

52. $\displaystyle\int \left[\ln t\mathbf{i} + \frac{1}{t}\mathbf{j} + \mathbf{k}\right] dt = (t\ln t - t)\mathbf{i} + \ln t\mathbf{j} + t\mathbf{k} + \mathbf{C}$

(Integration by parts)

53. $\displaystyle\int \left[(2t - 1)\mathbf{i} + 4t^3\mathbf{j} + 3\sqrt{t}\mathbf{k}\right] dt = (t^2 - t)\mathbf{i} + t^4\mathbf{j} + 2t^{3/2}\mathbf{k} + \mathbf{C}$

54. $\displaystyle\int [e^t\mathbf{i} + \sin t\mathbf{j} + \cos t\mathbf{k}]\,dt = e^t\mathbf{i} - \cos t\mathbf{j} + \sin t\mathbf{k} + \mathbf{C}$

55. $\displaystyle\int \left[\sec^2 t\mathbf{i} + \frac{1}{1 + t^2}\mathbf{j}\right] dt = \tan t\mathbf{i} + \arctan t\mathbf{j} + \mathbf{C}$

56. $\displaystyle\int [e^{-t}\sin t\mathbf{i} + e^{-t}\cos t\mathbf{j}]\,dt = \frac{e^{-t}}{2}(-\sin t - \cos t)\mathbf{i} + \frac{e^{-t}}{2}(-\cos t + \sin t)\mathbf{j} + \mathbf{C}$

57. $\displaystyle\int_0^1 (8t\mathbf{i} + t\mathbf{j} - \mathbf{k})\,dt = \Big[4t^2\mathbf{i}\Big]_0^1 + \left[\frac{t^2}{2}\mathbf{j}\right]_0^1 - \Big[t\mathbf{k}\Big]_0^1 = 4\mathbf{i} + \frac{1}{2}\mathbf{j} - \mathbf{k}$

58. $\displaystyle\int_{-1}^1 \left(t\mathbf{i} + t^3\mathbf{j} + \sqrt[3]{t}\mathbf{k}\right) dt = \left[\frac{t^2}{2}\mathbf{i}\right]_{-1}^1 + \left[\frac{t^4}{4}\mathbf{j}\right]_{-1}^1 + \left[\frac{3}{4}t^{4/3}\mathbf{k}\right]_{-1}^1 = \mathbf{0}$

59. $\displaystyle\int_0^{\pi/2} [(a\cos t)\mathbf{i} + (a\sin t)\mathbf{j} + \mathbf{k}]\,dt = \Big[a\sin t\mathbf{i}\Big]_0^{\pi/2} - \Big[a\cos t\mathbf{j}\Big]_0^{\pi/2} + \Big[t\mathbf{k}\Big]_0^{\pi/2} = a\mathbf{i} + a\mathbf{j} + \frac{\pi}{2}\mathbf{k}$

60. $\displaystyle\int_0^{\pi/4} [(\sec t\tan t)\mathbf{i} + (\tan t)\mathbf{j} + (2\sin t\cos t)\mathbf{k}]\,dt = \Big[\sec t\mathbf{i} + \ln|\sec t|\mathbf{j} + \sin^2 t\mathbf{k}\Big]_0^{\pi/4}$

$\displaystyle= \left(\sqrt{2} - 1\right)\mathbf{i} + \ln\sqrt{2}\mathbf{j} + \frac{1}{2}\mathbf{k}$

61. $\displaystyle\int_0^2 (t\mathbf{i} + e^t\mathbf{j} - te^t\mathbf{k})\,dt = \left[\frac{t^2}{2}\mathbf{i}\right]_0^2 + \Big[e^t\mathbf{j}\Big]_0^2 - \Big[(t - 1)e^t\mathbf{k}\Big]_0^2$

$= 2\mathbf{i} + (e^2 - 1)\mathbf{j} - (e^2 + 1)\mathbf{k}$

62. $\|t\mathbf{i} + t^2\mathbf{j}\| = \sqrt{t^2 + t^4} = t\sqrt{1 + t^2}$ for $t \geq 0$

$\displaystyle\int_0^3 \|t\mathbf{i} + t^2\mathbf{j}\|\,dt = \int_0^3 t\sqrt{1 + t^2}\,dt$

$\displaystyle= \left[\frac{1}{3}(1 + t^2)^{3/2}\right]_0^3$

$\displaystyle= \frac{1}{3}(10^{3/2} - 1)$

63. $\mathbf{r}(t) = \int (4e^{2t}\mathbf{i} + 3e^t\mathbf{j})\,dt = 2e^{2t}\mathbf{i} + 3e^t\mathbf{j} + \mathbf{C}$

$\mathbf{r}(0) = 2\mathbf{i} + 3\mathbf{j} + \mathbf{C} = 2\mathbf{i} \Rightarrow \mathbf{C} = -3\mathbf{j}$

$\mathbf{r}(t) = 2e^{2t}\mathbf{i} + 3(e^t - 1)\mathbf{j}$

64. $\mathbf{r}(t) = \int \left(3t^2\mathbf{j} + 6\sqrt{t}\mathbf{k}\right)dt = t^3\mathbf{j} + 4t^{3/2}\mathbf{k} + \mathbf{C}$

$\mathbf{r}(0) = C = \mathbf{i} + 2\mathbf{j}$

$\mathbf{r}(t) = \mathbf{i} + (2 + t^3)\mathbf{j} + 4t^{3/2}\mathbf{k}$

65. $\mathbf{r}'(t) = \int -32\mathbf{j}\,dt = -32t\mathbf{j} + \mathbf{C}_1$

$\mathbf{r}'(0) = \mathbf{C}_1 = 600\sqrt{3}\mathbf{i} + 600\mathbf{j}$

$\mathbf{r}'(t) = 600\sqrt{3}\mathbf{i} + (600 - 32t)\mathbf{j}$

$\mathbf{r}(t) = \int \left[600\sqrt{3}\mathbf{i} + (600 - 32t)\mathbf{j}\right]dt$

$= 600\sqrt{3}t\mathbf{i} + (600t - 16t^2)\mathbf{j} + \mathbf{C}$

$\mathbf{r}(0) = \mathbf{C} = \mathbf{0}$

$\mathbf{r}(t) = 600\sqrt{3}t\mathbf{i} + (600t - 16t^2)\mathbf{j}$

66. $\mathbf{r}''(t) = -4\cos t\mathbf{j} - 3\sin t\mathbf{k}$

$\mathbf{r}'(t) = -4\sin t\mathbf{j} + 3\cos t\mathbf{k} + \mathbf{C}_1$

$\mathbf{r}'(0) = 3\mathbf{k} = 3\mathbf{k} + \mathbf{C}_1 \Rightarrow \mathbf{C}_1 = \mathbf{0}$

$\mathbf{r}(t) = 4\cos t\mathbf{j} + 3\sin t\mathbf{k} + \mathbf{C}_2$

$\mathbf{r}(0) = 4j + \mathbf{C}_2 = 4\mathbf{j} \Rightarrow \mathbf{C}_2 = 0$

$\mathbf{r}(t) = 4\cos t\mathbf{j} + 3\sin t\mathbf{k}$

67. $\mathbf{r}(t) = \int (te^{-t^2}\mathbf{i} - e^{-t}\mathbf{j} + \mathbf{k})\,dt = -\frac{1}{2}e^{-t^2}\mathbf{i} + e^{-t}\mathbf{j} + t\mathbf{k} + \mathbf{C}$

$\mathbf{r}(0) = -\frac{1}{2}\mathbf{i} + \mathbf{j} + \mathbf{C} = \frac{1}{2}\mathbf{i} - \mathbf{j} + \mathbf{k} \Rightarrow \mathbf{C} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$

$\mathbf{r}(t) = \left(1 - \frac{1}{2}e^{-t^2}\right)\mathbf{i} + (e^{-t} - 2)\mathbf{j} + (t + 1)\mathbf{k} = \left(\frac{2 - e^{-t^2}}{2}\right)\mathbf{i} + (e^{-t} - 2)\mathbf{j} + (t + 1)\mathbf{k}$

68. $\mathbf{r}(t) = \int \left[\frac{1}{1 + t^2}\mathbf{i} + \frac{1}{t^2}\mathbf{j} + \frac{1}{t}\mathbf{k}\right]dt = \arctan t\,\mathbf{i} - \frac{1}{t}\mathbf{j} + \ln t\,\mathbf{k} + \mathbf{C}$

$\mathbf{r}(1) = \frac{\pi}{4}\mathbf{i} - \mathbf{j} + \mathbf{C} = 2\mathbf{i} \Rightarrow \mathbf{C} = \left(2 - \frac{\pi}{4}\right)\mathbf{i} + \mathbf{j}$

$\mathbf{r}(t) = \left[2 - \frac{\pi}{4} + \arctan t\right]\mathbf{i} + \left(1 - \frac{1}{t}\right)\mathbf{j} + \ln t\,\mathbf{k}$

69. See "Definition of the Derivative of a Vector-Valued Function" and Figure 12.8 on page 840.

70. To find the integral of a vector-valued function, you integrate each component function separately. The constant of integration **C** is a constant vector.

71. At $t = t_0$, the graph of $\mathbf{u}(t)$ is increasing in the x, y, and z directions simultaneously.

72. The graph of $\mathbf{u}(t)$ does not change position relative to the xy-plane.

73. Let $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$. Then $c\mathbf{r}(t) = cx(t)\mathbf{i} + cy(t)\mathbf{j} + cz(t)\mathbf{k}$ and

$$D_t[c\mathbf{r}(t)] = cx'(t)\mathbf{i} + cy'(t)\mathbf{j} + cz'(t)\mathbf{k}$$
$$= c[x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}] = c\mathbf{r}'(t).$$

74. Let $\mathbf{r}(t) = x_1(t)\mathbf{i} + y_1(t)\mathbf{j} + z_1(t)\mathbf{k}$ and $\mathbf{u}(t) = x_2(t)\mathbf{i} + y_2(t)\mathbf{j} + z_2(t)\mathbf{k}$.

$$\mathbf{r}(t) \pm \mathbf{u}(t) = [x_1(t) \pm x_2(t)]\mathbf{i} + [y_1(t) \pm y_2(t)]\mathbf{j} + [z_1(t) \pm z_2(t)]\mathbf{k}$$
$$D_t[\mathbf{r}(t) \pm \mathbf{u}(t)] = [x_1'(t) \pm x_2'(t)]\mathbf{i} + [y_1'(t) \pm y_2'(t)]\mathbf{j} + [z_1'(t) \pm z_2'(t)]\mathbf{k}$$
$$= [x_1'(t)\mathbf{i} + y_1'(t)\mathbf{j} + z_1'(t)\mathbf{k}] \pm [x_2'(t)\mathbf{i} + y_2'(t)\mathbf{j} + z_2'(t)\mathbf{k}]$$
$$= \mathbf{r}'(t) \pm \mathbf{u}'(t)$$

75. Let $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, then $f(t)\mathbf{r}(t) = f(t)x(t)\mathbf{i} + f(t)y(t)\mathbf{j} + f(t)z(t)\mathbf{k}$.

$$\begin{aligned} D_t[f(t)\mathbf{r}(t)] &= [f(t)x'(t) + f'(t)x(t)]\mathbf{i} + [f(t)y'(t) + f'(t)y(t)]\mathbf{j} + [f(t)z'(t) + f'(t)z(t)]\mathbf{k} \\ &= f(t)[x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}] + f'(t)[x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}] \\ &= f(t)\mathbf{r}'(t) + f'(t)\mathbf{r}(t) \end{aligned}$$

76. Let $\mathbf{r}(t) = x_1(t)\mathbf{i} + y_1(t)\mathbf{j} + z_1(t)\mathbf{k}$ and $\mathbf{u}(t) = x_2(t)\mathbf{i} + y_2(t)\mathbf{j} + z_2(t)\mathbf{k}$.

$$\mathbf{r}(t) \times \mathbf{u}(t) = [y_1(t)z_2(t) - z_1(t)y_2(t)]\mathbf{i} - [x_1(t)z_2(t) - z_1(t)x_2(t)]\mathbf{j} + [x_1(t)y_2(t) - y_1(t)x_2(t)]\mathbf{k}$$

$$\begin{aligned} D_t[\mathbf{r}(t) \times \mathbf{u}(t)] &= [y_1(t)z_2'(t) + y_1'(t)z_2(t) - z_1(t)y_2'(t) - z_1'(t)y_2(t)]\mathbf{i} - [x_1(t)z_2'(t) + x_1'(t)z_2(t) - z_1(t)x_2'(t) - z_1'(t)x_2(t)]\mathbf{j} + \\ &\quad [x_1(t)y_2'(t) + x_1'(t)y_2(t) - y_1(t)x_2'(t) - y_1'(t)x_2(t)]\mathbf{k} \\ &= \{[y_1(t)z_2'(t) - z_1(t)y_2'(t)]\mathbf{i} - [x_1(t)z_2'(t) - z_1(t)x_2'(t)]\mathbf{j} + [x_1(t)y_2'(t) - y_1(t)x_2'(t)]\mathbf{k}\} + \\ &\quad \{[y_1'(t)z_2(t) - z_1'(t)y_2(t)]\mathbf{i} - [x_1'(t)z_2(t) - z_1'(t)x_2(t)]\mathbf{j} + [x_1'(t)y_2(t) - y_1'(t)x_2(t)]\mathbf{k}\} \\ &= \mathbf{r}(t) \times \mathbf{u}'(t) + \mathbf{r}'(t) \times \mathbf{u}(t) \end{aligned}$$

77. Let $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$. Then $\mathbf{r}(f(t)) = x(f(t))\mathbf{i} + y(f(t))\mathbf{j} + z(f(t))\mathbf{k}$ and

$$\begin{aligned} D_t[\mathbf{r}(f(t))] &= x'(f(t))f'(t)\mathbf{i} + y'(f(t))f'(t)\mathbf{j} + z'(f(t))f'(t)\mathbf{k} \quad \text{(Chain Rule)} \\ &= f'(t)[x'(f(t))\mathbf{i} + y'(f(t))\mathbf{j} + z'(f(t))\mathbf{k}] = f'(t)\mathbf{r}'(f(t)). \end{aligned}$$

78. Let $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$. Then $\mathbf{r}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}$.

$$\mathbf{r}(t) \times \mathbf{r}'(t) = [y(t)z'(t) - z(t)y'(t)]\mathbf{i} - [x(t)z'(t) - z(t)x'(t)]\mathbf{j} + [x(t)y'(t) - y(t)x'(t)]\mathbf{k}$$

$$\begin{aligned} D_t[\mathbf{r}(t) \times \mathbf{r}'(t)] &= [y(t)z''(t) + y'(t)z'(t) - z(t)y''(t) - z'(t)y'(t)]\mathbf{i} - [x(t)z''(t) + x'(t)z'(t) - z(t)x''(t) - z'(t)x'(t)]\mathbf{j} + \\ &\quad [x(t)y''(t) + x'(t)y'(t) - y(t)x''(t) - y'(t)x'(t)]\mathbf{k} \\ &= [y(t)z''(t) - z(t)y''(t)]\mathbf{i} - [x(t)z''(t) - z(t)x''(t)]\mathbf{j} + [x(t)y''(t) - y(t)x''(t)]\mathbf{k} = \mathbf{r}(t) \times \mathbf{r}''(t) \end{aligned}$$

79. Let $\mathbf{r}(t) = x_1(t)\mathbf{i} + y_1(t)\mathbf{j} + z_1(t)\mathbf{k}$, $\mathbf{u}(t) = x_2(t)\mathbf{i} + y_2(t)\mathbf{j} + z_2(t)\mathbf{k}$, and $\mathbf{v}(t) = x_3(t)\mathbf{i} + y_3(t)\mathbf{j} + z_3(t)\mathbf{k}$. Then:

$$\mathbf{r}(t) \cdot [\mathbf{u}(t) \times \mathbf{v}(t)] = x_1(t)[y_2(t)z_3(t) - z_2(t)y_3(t)] - y_1(t)[x_2(t)z_3(t) - z_2(t)x_3(t)] + z_1(t)[x_2(t)y_3(t) - y_2(t)x_3(t)]$$

$$\begin{aligned} D_t[\mathbf{r}(t) \cdot (\mathbf{u}(t) \times \mathbf{v}(t))] &= x_1(t)y_2(t)z_3'(t) + x_1(t)y_2'(t)z_3(t) + x_1'(t)y_2(t)z_3(t) - x_1(t)y_3(t)z_2'(t) - \\ &\quad x_1(t)y_3'(t)z_2(t) - x_1'(t)y_3(t)z_2(t) - y_1(t)x_2(t)z_3'(t) - y_1(t)x_2'(t)z_3(t) - y_1'(t)x_2(t)z_3(t) + \\ &\quad y_1(t)z_2(t)x_3'(t) + y_1(t)z_2'(t)x_3(t) + y_1'(t)z_2(t)x_3(t) + z_1(t)x_2(t)y_3'(t) + z_1(t)x_2'(t)y_3(t) + \\ &\quad z_1'(t)x_2(t)y_3(t) - z_1(t)y_2(t)x_3'(t) - z_1(t)y_2'(t)x_3(t) - z_1'(t)y_2(t)x_3(t) \\ &= \{x_1'(t)[y_2(t)z_3(t) - y_3(t)z_2(t)] + y_1'(t)[-x_2(t)z_3(t) + z_2(t)x_3(t)] + z_1'(t)[x_2(t)y_3(t) - y_2(t)x_3(t)]\} + \\ &\quad \{x_1(t)[y_2'(t)z_3(t) - y_3(t)z_2'(t)] + y_1(t)[-x_2'(t)z_3(t) + z_2'(t)x_3(t)] + z_1(t)[x_2'(t)y_3(t) - y_2'(t)x_3(t)]\} + \\ &\quad \{x_1(t)[y_2(t)z_3'(t) - y_3'(t)z_2(t)] + y_1(t)[-x_2(t)z_3'(t) + z_2(t)x_3'(t)] + z_1(t)[x_2(t)y_3'(t) - y_2(t)x_3'(t)]\} \\ &= \mathbf{r}'(t) \cdot [\mathbf{u}(t) \times \mathbf{v}(t)] + \mathbf{r}(t) \cdot [\mathbf{u}'(t) \times \mathbf{v}(t)] + \mathbf{r}(t) \cdot [\mathbf{u}(t) \times \mathbf{v}'(t)] \end{aligned}$$

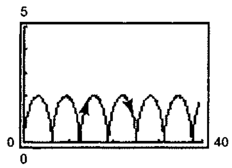
80. Let $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$. If $\mathbf{r}(t) \cdot \mathbf{r}(t)$ is constant, then:

$$\begin{aligned} x^2(t) + y^2(t) + z^2(t) &= C \\ D_t[x^2(t) + y^2(t) + z^2(t)] &= D_t[C] \\ 2x(t)x'(t) + 2y(t)y'(t) + 2z(t)z'(t) &= 0 \\ 2[x(t)x'(t) + y(t)y'(t) + z(t)z'(t)] &= 0 \\ 2[\mathbf{r}(t) \cdot \mathbf{r}'(t)] &= 0 \end{aligned}$$

Therefore, $\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$.

81. $\mathbf{r}(t) = (t - \sin t)\mathbf{i} + (1 - \cos t)\mathbf{j}$

(a)



The curve is a cycloid.

(b) $\mathbf{r}'(t) = (1 - \cos t)\mathbf{i} + \sin t\mathbf{j}$

$\mathbf{r}''(t) = \sin t\mathbf{i} + \cos t\mathbf{j}$

$\|\mathbf{r}'(t)\| = \sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t} = \sqrt{2 - 2\cos t}$

Minimum of $\|\mathbf{r}'(t)\|$ is 0, $(t = 0)$

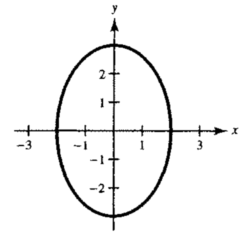
Maximum of $\|\mathbf{r}'(t)\|$ is 2, $(t = \pi)$

$\|\mathbf{r}''(t)\| = \sqrt{\sin^2 t + \cos^2 t} = 1$

Minimum and maximum of $\|\mathbf{r}'(t)\|$ is 1.

82. $\mathbf{r}(t) = 2\cos t\mathbf{i} + 3\sin t\mathbf{j}$

(a) Ellipse



(b) $\mathbf{r}'(t) = -2\sin t\mathbf{i} + 3\cos t\mathbf{j}$

$\mathbf{r}''(t) = -2\cos t\mathbf{i} - 3\sin t\mathbf{j}$

$\|\mathbf{r}'(t)\| = \sqrt{4\sin^2 t + 9\cos^2 t}$

Minimum of $\|\mathbf{r}'(t)\|$ is 2, $(t = \pi/2)$

Maximum of $\|\mathbf{r}'(t)\|$ is 3, $(t = 0)$

$\|\mathbf{r}''(t)\| = \sqrt{4\cos^2 t + 9\sin^2 t}$

Minimum of $\|\mathbf{r}''(t)\|$ is 2, $(t = 0)$

Maximum of $\|\mathbf{r}''(t)\|$ is 3, $(t = \pi/2)$

83. True

84. False. The definite integral is a vector, not a real number.

85. False. Let $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + \mathbf{k}$.

$\|\mathbf{r}(t)\| = \sqrt{2}$

$\frac{d}{dt}[\|\mathbf{r}(t)\|] = 0$

$\mathbf{r}'(t) = -\sin t\mathbf{i} + \cos t\mathbf{j}$

$\|\mathbf{r}'(t)\| = 1$

86. False

$D_t[\mathbf{r}(t) \cdot \mathbf{u}(t)] = \mathbf{r}(t) \cdot \mathbf{u}'(t) + \mathbf{r}'(t) \cdot \mathbf{u}(t)$

(See Theorem 11.2, part 4)

87. $\mathbf{r}(t) = e^t \sin t\mathbf{i} + e^t \cos t\mathbf{j}$

$\mathbf{r}'(t) = (e^t\cos t + e^t\sin t)\mathbf{i} + (e^t\cos t - e^t\sin t)\mathbf{j}$

$\mathbf{r}''(t) = (-e^t\sin t + e^t\cos t + e^t\sin t + e^t\cos t)\mathbf{i} + (e^t\cos t - e^t\sin t - e^t\sin t - e^t\cos t)\mathbf{j}$

$= 2e^t\cos t\mathbf{i} - 2e^t\sin t\mathbf{j}$

$\mathbf{r}(t) \cdot \mathbf{r}''(t) = 2e^{2t}\sin t\cos t - 2e^{2t}\sin t\cos t = 0$

Hence, $\mathbf{r}(t)$ is always perpendicular to $\mathbf{r}''(t)$.

Section 12.3 Velocity and Acceleration

1. $\mathbf{r}(t) = 3t\mathbf{i} + (t - 1)\mathbf{j}$

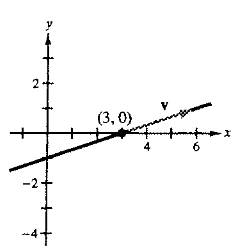
$\mathbf{v}(t) = \mathbf{r}'(t) = 3\mathbf{i} + \mathbf{j}$

$\mathbf{a}(t) = \mathbf{r}''(t) = \mathbf{0}$

$x = 3t,\ y = t - 1,\ y = \frac{x}{3} - 1$

At $(3, 0)$, $t = 1$.

$\mathbf{v}(1) = 3\mathbf{i} + \mathbf{j}, \mathbf{a}(1) = \mathbf{0}$

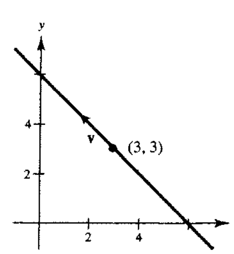


2. $\mathbf{r}(t) = (6 - t)\mathbf{i} + t\mathbf{j}$

$\mathbf{v}(t) = \mathbf{r}'(t) = -\mathbf{i} + \mathbf{j}$

$\mathbf{a}(t) = \mathbf{r}''(t) = \mathbf{0}$

$x = 6 - t,\ y = t,\ y = 6 - x$



3. $\mathbf{r}(t) = t^2\mathbf{i} + t\mathbf{j}$

$\mathbf{v}(t) = \mathbf{r}'(t) = 2t\mathbf{i} + \mathbf{j}$

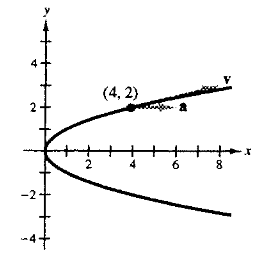
$\mathbf{a}(t) = \mathbf{r}''(t) = 2\mathbf{i}$

$x = t^2,\ y = t,\ x = y^2$

At $(4, 2)$, $t = 2$.

$\mathbf{v}(2) = 4\mathbf{i} + \mathbf{j}$

$\mathbf{a}(2) = 2\mathbf{i}$



4. $\mathbf{r}(t) = t^2\mathbf{i} + t^3\mathbf{j}$

$\mathbf{v}(t) = \mathbf{r}'(t) = 2t\mathbf{i} + 3t^2\mathbf{j}$

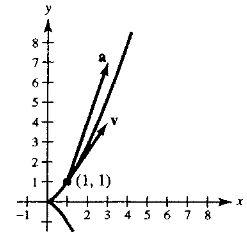
$\mathbf{a}(t) = \mathbf{r}''(t) = 2\mathbf{i} + 6t\mathbf{j}$

$x = t^2, y = t^3,\ x = y^{2/3}$

At $(1, 1)$, $t = 1$.

$\mathbf{v}(1) = 2\mathbf{i} + 3\mathbf{j}$

$\mathbf{a}(1) = 2\mathbf{i} + 6\mathbf{j}$



5. $\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j}$

$\mathbf{v}(t) = \mathbf{r}'(t) = -2\sin t\mathbf{i} + 2\cos t\mathbf{j}$

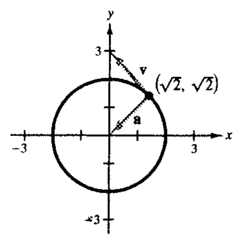
$\mathbf{a}(t) = \mathbf{r}''(t) = -2\cos t\mathbf{i} - 2\sin t\mathbf{j}$

$x = 2\cos t,\ y = 2\sin t,\ x^2 + y^2 = 4$

At $(\sqrt{2}, \sqrt{2})$, $t = \frac{\pi}{4}$.

$\mathbf{v}\left(\frac{\pi}{4}\right) = -\sqrt{2}\mathbf{i} + \sqrt{2}\mathbf{j}$

$\mathbf{a}\left(\frac{\pi}{4}\right) = -\sqrt{2}\mathbf{i} - \sqrt{2}\mathbf{j}$



6. $\mathbf{r}(t) = 3\cos t\mathbf{i} + 2\sin t\mathbf{j}$

$\mathbf{v}(t) = -3\sin t\mathbf{i} + 2\cos t\mathbf{j}$

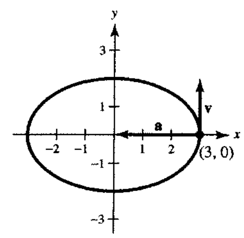
$\mathbf{a}(t) = -3\cos t\mathbf{i} - 2\sin t\mathbf{j}$

$x = 3\cos t, y = 2\sin t, \frac{x^2}{9} + \frac{y^2}{4} = 1$ Ellipse

At $(3, 0)$, $t = 0$.

$\mathbf{v}(0) = 2\mathbf{j}$

$\mathbf{a}(0) = -3\mathbf{i}$



7. $\mathbf{r}(t) = \langle t - \sin t, 1 - \cos t\rangle$

$\mathbf{v}(t) = \mathbf{r}'(t) = \langle 1 - \cos t, \sin t\rangle$

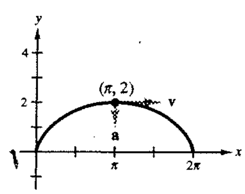
$\mathbf{a}(t) = \mathbf{r}''(t) = \langle \sin t, \cos t\rangle$

$x = t - \sin t,\ y = 1 - \cos t$ (cycloid)

At $(\pi, 2)$, $t = \pi$.

$\mathbf{v}(\pi) = \langle 2, 0\rangle = 2\mathbf{i}$

$\mathbf{a}(\pi) = \langle 0, -1\rangle = -\mathbf{j}$



8. $\mathbf{r}(t) = \langle e^{-t}, e^t\rangle$

$\mathbf{v}(t) = \mathbf{r}'(t) = \langle -e^{-t}, e^t\rangle$

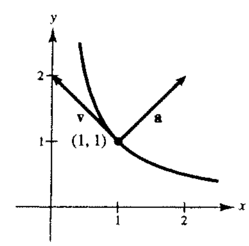
$\mathbf{a}(t) = \mathbf{r}''(t) = \langle e^{-t}, e^t\rangle$

$x = e^{-t} = \frac{1}{e^t},\ y = e^t,\ y = \frac{1}{x}$

At $(1, 1)$, $t = 0$.

$\mathbf{v}(0) = \langle -1, 1\rangle = -\mathbf{i} + \mathbf{j}$

$\mathbf{a}(0) = \langle 1, 1\rangle = \mathbf{i} + \mathbf{j}$



9. $\mathbf{r}(t) = t\mathbf{i} + (2t - 5)\mathbf{j} + 3t\mathbf{k}$

$\mathbf{v}(t) = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$

$s(t) = \|\mathbf{v}(t)\| = \sqrt{1 + 4 + 9} = \sqrt{14}$

$\mathbf{a}(t) = \mathbf{0}$

10. $\mathbf{r}(t) = 4t\mathbf{i} + 4t\mathbf{j} + 2t\mathbf{k}$

$\mathbf{v}(t) = 4\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$

$s(t) = \|\mathbf{v}(t)\| = \sqrt{16 + 16 + 4} = 6$

$\mathbf{a}(t) = \mathbf{0}$

11. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{t^2}{2}\mathbf{k}$

$\mathbf{v}(t) = \mathbf{i} + 2t\mathbf{j} + t\mathbf{k}$

$s(t) = \sqrt{1 + 4t^2 + t^2} = \sqrt{1 + 5t^2}$

$\mathbf{a}(t) = 2\mathbf{j} + \mathbf{k}$

12. $\mathbf{r}(t) = 3t\mathbf{i} + t\mathbf{j} + \frac{1}{4}t^2\mathbf{k}$

$\mathbf{v}(t) = 3\mathbf{i} + \mathbf{j} + \frac{1}{2}t\mathbf{k}$

$s(t) = \sqrt{9 + 1 + \frac{1}{4}t^2} = \sqrt{10 + \frac{1}{4}t^2}$

$\mathbf{a}(t) = \frac{1}{2}\mathbf{k}$

13. $\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + \sqrt{9 - t^2}\mathbf{k}$

$\mathbf{v}(t) = \mathbf{i} + \mathbf{j} - \frac{t}{\sqrt{9 - t^2}}\mathbf{k}$

$s(t) = \sqrt{1 + 1 + \frac{t^2}{9 - t^2}} = \sqrt{\frac{18 - t^2}{9 - t^2}}$

$\mathbf{a}(t) = -\frac{9}{(9 - t^2)^{3/2}}\mathbf{k}$

14. $\mathbf{r}(t) = t^2\mathbf{i} + t\mathbf{j} + 2t^{3/2}\mathbf{k}$

$\mathbf{v}(t) = 2t\mathbf{i} + \mathbf{j} + 3\sqrt{t}\mathbf{k}$

$s(t) = \sqrt{4t^2 + 1 + 9t} = \sqrt{4t^2 + 9t + 1}$

$\mathbf{a}(t) = 2\mathbf{i} + \dfrac{3}{2\sqrt{t}}\mathbf{k}$

15. $\mathbf{r}(t) = \langle 4t, 3\cos t, 3\sin t\rangle$

$\mathbf{v}(t) = \langle 4, -3\sin t, 3\cos t\rangle = 4\mathbf{i} - 3\sin t\mathbf{j} + 3\cos t\mathbf{k}$

$s(t) = \sqrt{16 + 9\sin^2 t + 9\cos^2 t} = 5$

$\mathbf{a}(t) = \langle 0, -3\cos t, -3\sin t\rangle = -3\cos t\mathbf{j} - 3\sin t\mathbf{k}$

16. $\mathbf{r}(t) = \langle e^t\cos t, e^t\sin t, e^t\rangle$

$\mathbf{v}(t) = (e^t\cos t - e^t\sin t)\mathbf{i} + (e^t\sin t + e^t\cos t)\mathbf{j} + e^t\mathbf{k}$

$s(t) = \sqrt{e^{2t}(\cos t - \sin t)^2 + e^{2t}(\cos t + \sin t)^2 + e^{2t}} = e^t\sqrt{3}$

$\mathbf{a}(t) = -2e^t\sin t\mathbf{i} + 2e^t\cos t\mathbf{j} + e^t\mathbf{k}$

17. (a) $\mathbf{r}(t) = \left\langle t, -t^2, \dfrac{t^3}{4}\right\rangle, t_0 = 1$

$\mathbf{r}'(t) = \left\langle 1, -2t, \dfrac{3t^2}{4}\right\rangle$

$\mathbf{r}'(1) = \left\langle 1, -2, \dfrac{3}{4}\right\rangle$

$x = 1 + t,\ y = -1 - 2t,\ z = \dfrac{1}{4} + \dfrac{3}{4}t$

(b) $\mathbf{r}(1 + 0.1) \approx \left\langle 1 + 0.1, -1 - 2(0.1), \dfrac{1}{4} + \dfrac{3}{4}(0.1)\right\rangle$

$= \langle 1.100, -1.200, 0.325\rangle$

18. (a) $\mathbf{r}(t) = \langle t, \sqrt{25 - t^2}, \sqrt{25 - t^2}\rangle, t_0 = 3$

$\mathbf{r}'(t) = \left\langle 1, \dfrac{-t}{\sqrt{25 - t^2}}, \dfrac{-t}{\sqrt{25 - t^2}}\right\rangle$

$\mathbf{r}'(3) = \left\langle 1, -\dfrac{3}{4}, -\dfrac{3}{4}\right\rangle$

$x = 3 + t,\ y = z = 4 - \dfrac{3}{4}t$

(b) $\mathbf{r}(3 + 0.1) \approx \left\langle 3 + 0.1, 4 - \dfrac{3}{4}(0.1), 4 - \dfrac{3}{4}(0.1)\right\rangle$

$= \langle 3.100, 3.925, 3.925\rangle$

19. $\mathbf{a}(t) = \mathbf{i} + \mathbf{j} + \mathbf{k}, \mathbf{v}(0) = \mathbf{0}, \mathbf{r}(0) = \mathbf{0}$

$\mathbf{v}(t) = \displaystyle\int(\mathbf{i} + \mathbf{j} + \mathbf{k})\,dt = t\mathbf{i} + t\mathbf{j} + t\mathbf{k} + \mathbf{C}$

$\mathbf{v}(0) = \mathbf{C} = \mathbf{0}, \mathbf{v}(t) = t\mathbf{i} + t\mathbf{j} + t\mathbf{k}, \mathbf{v}(t) = t(\mathbf{i} + \mathbf{j} + \mathbf{k})$

$\mathbf{r}(t) = \displaystyle\int(t\mathbf{i} + t\mathbf{j} + t\mathbf{k})\,dt = \dfrac{t^2}{2}(\mathbf{i} + \mathbf{j} + \mathbf{k}) + \mathbf{C}$

$\mathbf{r}(0) = \mathbf{C} = \mathbf{0}, \mathbf{r}(t) = \dfrac{t^2}{2}(\mathbf{i} + \mathbf{j} + \mathbf{k}),$

$\mathbf{r}(2) = 2(\mathbf{i} + \mathbf{j} + \mathbf{k}) = 2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$

20. $\mathbf{a}(t) = 2\mathbf{i} + 3\mathbf{k}$

$\mathbf{v}(t) = \displaystyle\int(2\mathbf{i} + 3\mathbf{k})\,dt = 2t\mathbf{i} + 3t\mathbf{k} + \mathbf{C}$

$\mathbf{v}(0) = \mathbf{C} = 4\mathbf{j} \implies \mathbf{v}(t) = 2t\mathbf{i} + 4\mathbf{j} + 3t\mathbf{k}$

$\mathbf{r}(t) = \displaystyle\int(2t\mathbf{i} + 4\mathbf{j} + 3t\mathbf{k})\,dt = t^2\mathbf{i} + 4t\mathbf{j} + \dfrac{3}{2}t^2\mathbf{k} + \mathbf{C}$

$\mathbf{r}(0) = \mathbf{C} = 0 \implies \mathbf{r}(t) = t^2\mathbf{i} + 4t\mathbf{j} + \dfrac{3}{2}t^2\mathbf{k}$

$\mathbf{r}(2) = 4\mathbf{i} + 8\mathbf{j} + 6\mathbf{k}$

21. $\mathbf{a}(t) = t\mathbf{j} + t\mathbf{k}, \mathbf{v}(1) = 5\mathbf{j}, \mathbf{r}(1) = \mathbf{0}$

$\mathbf{v}(t) = \displaystyle\int(t\mathbf{j} + t\mathbf{k})\,dt = \dfrac{t^2}{2}\mathbf{j} + \dfrac{t^2}{2}\mathbf{k} + \mathbf{C}$

$\mathbf{v}(1) = \dfrac{1}{2}\mathbf{j} + \dfrac{1}{2}\mathbf{k} + \mathbf{C} = 5\mathbf{j} \implies \mathbf{C} = \dfrac{9}{2}\mathbf{j} - \dfrac{1}{2}\mathbf{k}$

$\mathbf{v}(t) = \left(\dfrac{t^2}{2} + \dfrac{9}{2}\right)\mathbf{j} + \left(\dfrac{t^2}{2} - \dfrac{1}{2}\right)\mathbf{k}$

$\mathbf{r}(t) = \displaystyle\int\left[\left(\dfrac{t^2}{2} + \dfrac{9}{2}\right)\mathbf{j} + \left(\dfrac{t^2}{2} - \dfrac{1}{2}\right)\mathbf{k}\right]dt = \left(\dfrac{t^3}{6} + \dfrac{9}{2}t\right)\mathbf{j} + \left(\dfrac{t^3}{6} - \dfrac{1}{2}t\right)\mathbf{k} + \mathbf{C}$

$\mathbf{r}(1) = \dfrac{14}{3}\mathbf{j} - \dfrac{1}{3}\mathbf{k} + \mathbf{C} = \mathbf{0} \implies \mathbf{C} = -\dfrac{14}{3}\mathbf{j} + \dfrac{1}{3}\mathbf{k}$

$\mathbf{r}(t) = \left(\dfrac{t^3}{6} + \dfrac{9}{2}t - \dfrac{14}{3}\right)\mathbf{j} + \left(\dfrac{t^3}{6} - \dfrac{1}{2}t + \dfrac{1}{3}\right)\mathbf{k}$

$\mathbf{r}(2) = \dfrac{17}{3}\mathbf{j} + \dfrac{2}{3}\mathbf{k}$

22. $\mathbf{a}(t) = -\cos t\mathbf{i} - \sin t\mathbf{j}, \mathbf{v}(0) = \mathbf{j} + \mathbf{k}, \mathbf{r}(0) = \mathbf{i}$

$$\mathbf{v}(t) = \int(-\cos t\mathbf{i} - \sin t\mathbf{j})\,dt = -\sin t\mathbf{i} + \cos t\mathbf{j} + \mathbf{C}$$

$$\mathbf{v}(0) = \mathbf{j} + \mathbf{C} = \mathbf{j} + \mathbf{k} \implies \mathbf{C} = \mathbf{k}$$

$$\mathbf{v}(t) = -\sin t\mathbf{i} + \cos t\mathbf{j} + \mathbf{k}$$

$$\mathbf{r}(t) = \int(-\sin t\mathbf{i} + \cos t\mathbf{j} + \mathbf{k})\,dt = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k} + \mathbf{C}$$

$$\mathbf{r}(0) = \mathbf{i} + \mathbf{C} = \mathbf{i} \implies \mathbf{C} = \mathbf{0}$$

$$\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}$$

$$\mathbf{r}(2) = (\cos 2)\mathbf{i} + (\sin 2)\mathbf{j} + 2\mathbf{k}$$

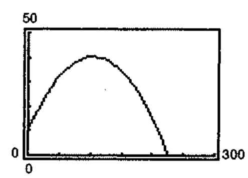
23. The velocity of an object involves both magnitude and direction of motion, whereas speed involves only magnitude.

24. (a) The speed is increasing.

(b) The speed is decreasing.

25. $\mathbf{r}(t) = (88\cos 30°)t\mathbf{i} + [10 + (88\sin 30°)t - 16t^2]\mathbf{j}$

$= 44\sqrt{3}t\mathbf{i} + (10 + 44t - 16t^2)\mathbf{j}$



26. $\mathbf{r}(t) = (900\cos 45°)t\mathbf{i} + [3 + (900\sin 45°)t - 16t^2]\mathbf{j}$

$= 450\sqrt{2}t\mathbf{i} + \left(3 + 450\sqrt{2}t - 16t^2\right)\mathbf{j}$

The maximum height occurs when $y'(t) = 450\sqrt{2} - 32t = 0$, which implies that $t = \left(225\sqrt{2}\right)/16$. The maximum height reached by the projectile is

$$y = 3 + 450\sqrt{2}\left(\frac{225\sqrt{2}}{16}\right) - 16\left(\frac{225\sqrt{2}}{16}\right)^2 = \frac{50{,}649}{8} = 6331.125 \text{ feet.}$$

The range is determined by setting $y(t) = 3 + 450\sqrt{2}t - 16t^2 = 0$ which implies that

$$t = \frac{-450\sqrt{2} - \sqrt{405{,}192}}{-32} \approx 39.779 \text{ seconds.}$$

$$\text{Range: } x = 450\sqrt{2}\left(\frac{-450\sqrt{2} - \sqrt{405{,}192}}{-32}\right) \approx 25{,}315.500 \text{ feet}$$

27. $\mathbf{r}(t) = (v_0\cos\theta)t\mathbf{i} + \left[h + (v_0\sin\theta)t - \frac{1}{2}gt^2\right]\mathbf{j} = \frac{v_0}{\sqrt{2}}t\mathbf{i} + \left(3 + \frac{v_0}{\sqrt{2}}t - 16t^2\right)\mathbf{j}$

$\frac{v_0}{\sqrt{2}}t = 300$ when $3 + \frac{v_0}{\sqrt{2}}t - 16t^2 = 3.$

$$t = \frac{300\sqrt{2}}{v_0},\ \frac{v_0}{\sqrt{2}}\left(\frac{300\sqrt{2}}{v_0}\right) - 16\left(\frac{300\sqrt{2}}{v_0}\right)^2 = 0,\ 300 - \frac{300^2(32)}{v_0^2} = 0$$

$$v_0^2 = 300(32),\ v_0 = \sqrt{9600} = 40\sqrt{6},\ v_0 = 40\sqrt{6} \approx 97.98 \text{ ft/sec}$$

The maximum height is reached when the derivative of the vertical component is zero.

$$y(t) = 3 + \frac{tv_0}{\sqrt{2}} - 16t^2 = 3 + \frac{40\sqrt{6}}{\sqrt{2}}t - 16t^2 = 3 + 40\sqrt{3}t - 16t^2$$

$$y'(t) = 40\sqrt{3} - 32t = 0$$

$$t = \frac{40\sqrt{3}}{32} = \frac{5\sqrt{3}}{4}$$

$$\text{Maximum height: } y\left(\frac{5\sqrt{3}}{4}\right) = 3 + 40\sqrt{3}\left(\frac{5\sqrt{3}}{4}\right) - 16\left(\frac{5\sqrt{3}}{4}\right)^2 = 78 \text{ feet}$$

28. 50 mph $= \frac{220}{3}$ ft/sec

$$\mathbf{r}(t) = \left(\frac{220}{3}\cos 15°\right)t\mathbf{i} + \left[5 + \left(\frac{220}{3}\sin 15°\right)t - 16t^2\right]\mathbf{j}$$

The ball is 90 feet from where it is thrown when

$$x = \frac{220}{3}\cos 15° t = 90 \implies t = \frac{27}{22\cos 15°} \approx 1.2706 \text{ seconds.}$$

The height of the ball at this time is

$$y = 5 + \left(\frac{220}{3}\sin 15°\right)\left(\frac{27}{22\cos 15°}\right) - 16\left(\frac{27}{22\cos 15°}\right)^2 \approx 3.286 \text{ feet.}$$

29. $x(t) = t(v_0\cos\theta)$ or $t = \frac{x}{v_0\cos\theta}$

$y(t) = t(v_0\sin\theta) - 16t^2 + h$

$$y = \frac{x}{v_0\cos\theta}(v_0\sin\theta) - 16\left(\frac{x^2}{v_0^2\cos^2\theta}\right) + h = (\tan\theta)x - \left(\frac{16}{v_0^2}\sec^2\theta\right)x^2 + h$$

30. $y = x - 0.005x^2$

From Exercise 29 we know that $\tan\theta$ is the coefficient of x. Therefore, $\tan\theta = 1$, $\theta = (\pi/4)$ rad $= 45°$. Also

$\frac{16}{v_0^2}\sec^2\theta =$ negative of coefficient of x^2

$\frac{16}{v_0^2}(2) = 0.005$ or $v_0 = 80$ ft/sec

$\mathbf{r}(t) = (40\sqrt{2}t)\mathbf{i} + (40\sqrt{2}t - 16t^2)\mathbf{j}$. Position function.

When $40\sqrt{2}t = 60$,

$$t = \frac{60}{40\sqrt{2}} = \frac{3\sqrt{2}}{4}$$

$$\mathbf{v}(t) = 40\sqrt{2}\mathbf{i} + (40\sqrt{2} - 32t)\mathbf{j}$$

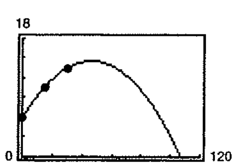
$$\mathbf{v}\left(\frac{3\sqrt{2}}{4}\right) = 40\sqrt{2}\mathbf{i} + (40\sqrt{2} - 24\sqrt{2})\mathbf{j} = 8\sqrt{2}(5\mathbf{i} + 2\mathbf{j}). \text{ direction}$$

$$\text{Speed} = \left\|\mathbf{v}\left(\frac{3\sqrt{2}}{4}\right)\right\| = 8\sqrt{2}\sqrt{25+4} = 8\sqrt{58} \text{ ft/sec}$$

31. $\mathbf{r}(t) = t\mathbf{i} + (-0.004t^2 + 0.3667t + 6)\mathbf{j}$

(a) $y = -0.004x^2 + 0.3667x + 6$

(b)



(c) $y' = -0.008x + 0.3667 = 0 \implies x = 45.8375$ and $y(45.8375) \approx 14.4$ feet.

(d) From Exercise 29,

$\tan\theta = 0.3667 \implies \theta \approx 20.14°$

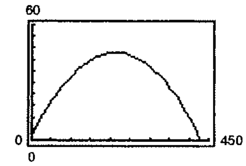
$$\frac{16\sec^2\theta}{v_0^2} = 0.004 \implies v_0^2 = \frac{16\sec^2\theta}{0.004} = \frac{4000}{\cos^2\theta}$$

$\implies v_0 \approx 67.4$ ft/sec.

32. $\mathbf{r}(t) = 140(\cos 22°)t\,\mathbf{i} + (2.5 + 140\,(\sin 22°)t - 16t^2)\mathbf{j}$

When $x = 375$, $t \approx 2.889$ and $y \approx 20.47$ feet.

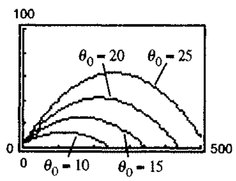
Thus, the ball clears the 10-foot fence.



33. $100 \text{ mph} = \left(100\,\dfrac{\text{miles}}{\text{hr}}\right)\left(5280\,\dfrac{\text{feet}}{\text{mile}}\right)/(3600 \text{ sec/hour}) = \dfrac{440}{3} \text{ ft/sec}$

(a) $\mathbf{r}(t) = \left(\dfrac{440}{3}\cos\theta_0\right)t\,\mathbf{i} + \left[3 + \left(\dfrac{440}{3}\sin\theta_0\right)t - 16t^2\right]\mathbf{j}$

(b)



Graphing these curves together with $y = 10$ shows that $\theta_0 = 20°$.

(c) We want

$$x(t) = \left(\frac{440}{3}\cos\theta\right)t \ge 400 \quad \text{and} \quad y(t) = 3 + \left(\frac{440}{3}\sin\theta\right)t - 16t^2 \ge 10.$$

From $x(t)$, the minimum angle occurs when $t = 30/(11\cos\theta)$. Substituting this for t in $y(t)$ yields:

$$3 + \left(\frac{440}{3}\sin\theta\right)\left(\frac{30}{11\cos\theta}\right) - 16\left(\frac{30}{11\cos\theta}\right)^2 = 10$$

$$400\tan\theta - \frac{14{,}400}{121}\sec^2\theta = 7$$

$$\frac{14{,}400}{121}(1 + \tan^2\theta) - 400\tan\theta + 7 = 0$$

$$14{,}400\tan^2\theta - 48{,}400\tan\theta + 15{,}247 = 0$$

$$\tan\theta = \frac{48{,}400 \pm \sqrt{48{,}400^2 - 4(14{,}400)(15{,}247)}}{2(14{,}400)}$$

$$\theta = \tan^{-1}\left(\frac{48{,}400 - \sqrt{1{,}464{,}332{,}800}}{28{,}800}\right) \approx 19.38°$$

34. $h = 7$ feet, $\theta = 35°$, 30 yards $= 90$ feet

$\mathbf{r}(t) = (v_0\cos 35°)t\mathbf{i} + [7 + (v_0\sin 35°)t - 16t^2]\mathbf{j}$

(a) $v_0\cos 35°t = 90$ when $7 + (v_0\sin 35°)t - 16t^2 = 4$

$$t = \frac{90}{v_0\cos 35°}$$

$$7 + (v_0\sin 35°)\left(\frac{90}{v_0\cos 35°}\right) - 16\left(\frac{90}{v_0\cos 35°}\right)^2 = 4$$

$$90\tan 35° + 3 = \frac{129{,}600}{v_0^2\cos^2 35°}$$

$$v_0^2 = \frac{129{,}600}{\cos^2 35°(90\tan 35° + 3)}$$

$$v_0 \approx 54.088 \text{ feet per second}$$

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34. —CONTINUED—

(b) The maximum height occurs when

$$y'(t) = v_0 \sin 35° - 32t = 0.$$

$$t = \frac{v_0 \sin 35}{32} \approx 0.969 \text{ second}$$

At this time, the height is $y(0.969) \approx 22.0$ feet.

(c) $x(t) = 90 \implies (v_0 \cos 35°)t = 90$

$$t = \frac{90}{54.088 \cos 35°} \approx 2.0 \text{ seconds}$$

35. $\mathbf{r}(t) = (v \cos \theta)t\mathbf{i} + [(v \sin \theta)t - 16t^2]\mathbf{j}$

(a) We want to find the minimum initial speed v as a function of the angle θ. Since the bale must be thrown to the position (16, 8), we have

$$16 = (v \cos \theta)t$$

$$8 = (v \sin \theta)t - 16t^2.$$

$t = 16/(v \cos \theta)$ from the first equation. Substituting into the second equation and solving for v, we obtain:

$$8 = (v \sin \theta)\left(\frac{16}{v \cos \theta}\right) - 16\left(\frac{16}{v \cos \theta}\right)^2$$

$$1 = 2\frac{\sin \theta}{\cos \theta} - 512\left(\frac{1}{v^2 \cos^2 \theta}\right)$$

$$512\frac{1}{v^2 \cos^2 \theta} = 2\frac{\sin \theta}{\cos \theta} - 1$$

$$\frac{1}{v^2} = \left(2\frac{\sin \theta}{\cos \theta} - 1\right)\frac{\cos^2 \theta}{512} = \frac{2 \sin \theta \cos \theta - \cos^2 \theta}{512}$$

$$v^2 = \frac{512}{2 \sin \theta \cos \theta - \cos^2 \theta}$$

We minimize $f(\theta) = \dfrac{512}{2 \sin \theta \cos \theta - \cos^2 \theta}$.

$$f'(\theta) = -512\frac{2 \cos^2 \theta - 2 \sin^2 \theta + 2 \sin \theta \cos \theta}{(2 \sin \theta \cos \theta - \cos^2 \theta)^2}$$

$$f'(\theta) = 0 \implies 2 \cos(2\theta) + \sin(2\theta) = 0$$

$$\tan(2\theta) = -2$$

$$\theta \approx 1.01722 \approx 58.28°$$

Substituting into the equation for v, $v \approx 28.78$ feet per second.

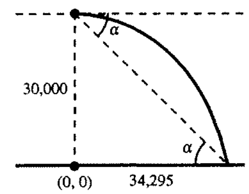
(b) If $\theta = 45°$,

$$16 = (v \cos \theta)t = v\frac{\sqrt{2}}{2}t$$

$$8 = (v \sin \theta)t - 16t^2 = v\frac{\sqrt{2}}{2}t - 16t^2$$

From part (a), $v^2 = \dfrac{512}{2\left(\sqrt{2}/2\right)\left(\sqrt{2}/2\right) - \left(\sqrt{2}/2\right)^2} = \dfrac{512}{1/2} = 1024 \implies v = 32$ ft/sec.

36. Place the origin directly below the plane. Then $\theta = 0$, $v_0 = 792$ and



$$\mathbf{r}(t) = (v_0 \cos \theta)t\mathbf{i} + (30{,}000 + (v_0 \sin \theta)t - 16t^2)\mathbf{j}$$
$$= 792t\mathbf{i} + (30{,}000 - 16t^2)\mathbf{j}$$
$$\mathbf{v}(t) = 792\mathbf{i} - 32t\mathbf{j}.$$

At time of impact, $30{,}000 - 16t^2 = 0 \implies t^2 = 1875 \implies t \approx 43.3$ seconds.

$$\mathbf{r}(43.3) = 34{,}294.6\mathbf{i}$$
$$\mathbf{v}(43.3) = 792\mathbf{i} - 1385.6\mathbf{j}$$
$$\|\mathbf{v}(43.3)\| = 1596 \text{ ft/sec} = 1088 \text{ mph}$$
$$\tan \alpha = \frac{30{,}000}{34{,}294.6} \approx 0.8748 \implies \alpha \approx 0.7187(41.18°)$$

37. $\mathbf{r}(t) = (v_0 \cos \theta)t\mathbf{i} + [(v_0 \sin \theta)t - 16t^2]\mathbf{j}$

$(v_0 \sin \theta)t - 16t^2 = 0$ when $t = 0$ and $t = \dfrac{v_0 \sin \theta}{16}$.

The range is

$$x = (v_0 \cos \theta)t = (v_0 \cos \theta)\frac{v_0 \sin \theta}{16} = \frac{v_0^2}{32} \sin 2\theta.$$

Hence,

$$x = \frac{1200^2}{32} \sin(2\theta) = 3000 \implies \sin 2\theta = \frac{1}{15} \implies \theta \approx 1.91°.$$

38. From Exercise 37, the range is

$$x = \frac{v_0^2}{32} \sin 2\theta.$$

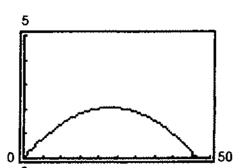
Hence, $x = 150 = \dfrac{v_0^2}{32} \sin(24°) \implies v_0^2 = \dfrac{4800}{\sin 24°} \implies v_0 \approx 108.6$ ft/sec.

39. (a) $\theta = 10°$, $v_0 = 66$ ft/sec

$$\mathbf{r}(t) = (66 \cos 10°)t\mathbf{i} + [0 + (66 \sin 10°)t - 16t^2]\mathbf{j}$$
$$\mathbf{r}(t) \approx (65t)\mathbf{i} + (11.46t - 16t^2)\mathbf{j}$$

Maximum height: 2.052 feet

Range: 46.557 feet

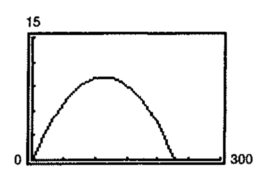


(b) $\theta = 10°$, $v_0 = 146$ ft/sec

$$\mathbf{r}(t) = (146 \cos 10°)t\mathbf{i} + [0 + (146 \sin 10°)t - 16t^2]\mathbf{j}$$
$$\mathbf{r}(t) \approx (143.78t)\mathbf{i} + (25.35t - 16t^2)\mathbf{j}$$

Maximum height: 10.043 feet

Range: 227.828 feet

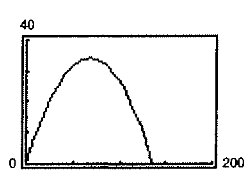


(c) $\theta = 45°$, $v_0 = 66$ ft/sec

$$\mathbf{r}(t) = (66 \cos 45°)t\mathbf{i} + [0 + (66 \sin 45°)t - 16t^2]\mathbf{j}$$
$$\mathbf{r}(t) \approx (46.67t)\mathbf{i} + (46.67t - 16t^2)\mathbf{j}$$

Maximum height: 34.031 feet

Range: 136.125 feet

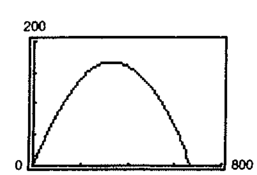


(d) $\theta = 45°$, $v_0 = 146$ ft/sec

$$\mathbf{r}(t) = (146 \cos 45°)t\mathbf{i} + [0 + (146 \sin 45°)t - 16t^2]\mathbf{j}$$
$$\mathbf{r}(t) \approx (103.24t)\mathbf{i} + (103.24t - 16t^2)\mathbf{j}$$

Maximum height: 166.531 feet

Range: 666.125 feet



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39. —CONTINUED—

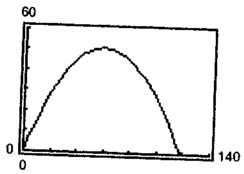
(e) $\theta = 60°$, $v_0 = 66$ ft/sec

$\mathbf{r}(t) = (66 \cos 60°)t\mathbf{i} + [0 + (66 \sin 60°)t - 16t^2]\mathbf{j}$

$\mathbf{r}(t) \approx (33t)\mathbf{i} + (57.16t - 16t^2)\mathbf{j}$

Maximum height: 51.047 feet

Range: 117.888 feet



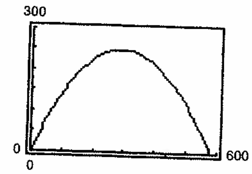
(f) $\theta = 60°$, $v_0 = 146$ ft/sec

$\mathbf{r}(t) = (146 \cos 60°)t\mathbf{i} + [0 + (146 \sin 60°)t - 16t^2]\mathbf{j}$

$\mathbf{r}(t) \approx (73t)\mathbf{i} + (126.44t - 16t^2)\mathbf{j}$

Maximum height: 249.797 feet

Range: 576.881 feet



40. (a) $\mathbf{r}(t) = t(v_0 \cos \theta)\mathbf{i} + (tv_0 \sin \theta - 16t^2)\mathbf{j}$

$t(v_0 \sin \theta - 16t) = 0$ when $t = \dfrac{v_0 \sin \theta}{16}$.

Range: $x = v_0 \cos \theta\left(\dfrac{v_0 \sin \theta}{32}\right) = \left(\dfrac{v_0^2}{32}\right) \sin 2\theta$

The range will be maximum when

$$\frac{dx}{dt} = \left(\frac{v_0^2}{32}\right) 2 \cos 2\theta = 0$$

or

$$2\theta = \frac{\pi}{2},\ \theta = \frac{\pi}{4} \text{ rad.}$$

(b) $y(t) = tv_0 \sin \theta - 16t^2$

$\dfrac{dy}{dt} = v_0 \sin \theta - 32t = 0$ when $t = \dfrac{v_0 \sin \theta}{32}$.

Maximum height:

$$y\left(\frac{v_0 \sin \theta}{32}\right) = \frac{v_0^2 \sin^2 \theta}{32} - 16\,\frac{v_0^2 \sin^2 \theta}{32^2} = \frac{v_0^2 \sin^2 \theta}{64}$$

Minimum height when $\sin \theta = 1$, or $\theta = \dfrac{\pi}{2}$.

41. $\mathbf{r}(t) = (v_0 \cos \theta)t\mathbf{i} + [h + (v_0 \sin \theta)t - 4.9t^2]\mathbf{j}$

$= (100 \cos 30°)t\mathbf{i} + [1.5 + (100 \sin 30°)t - 4.9t^2]\mathbf{j}$

The projectile hits the ground when $-4.9t^2 + 100(\frac{1}{2})t + 1.5 = 0 \implies t \approx 10.234$ seconds.

The range is therefore $(100 \cos 30°)(10.234) \approx 886.3$ meters.

The maximum height occurs when $dy/dt = 0$.

$100 \sin 30 = 9.8t \implies t \approx 5.102$ sec

The maximum height is

$$y = 1.5 + (100 \sin 30°)(5.102) - 4.9(5.102)^2 \approx 129.1 \text{ meters.}$$

42. $\mathbf{r}(t) = (v_0 \cos \theta)t\mathbf{i} + [h + (v_0 \sin \theta)t - 4.9t^2]\mathbf{j}$

$= (v_0 \cos 8°)t\mathbf{i} + [(v_0 \sin 8°)t - 4.9t^2]\mathbf{j}$

$x = 50$ when $(v_0 \cos 8°)t = 50 \implies t = \dfrac{50}{v_0 \cos 8°}$. For this value of t, $y = 0$:

$$(v_0 \sin 8°)\left(\frac{50}{v_0 \cos 8°}\right) - 4.9\left(\frac{50}{v_0 \cos 8°}\right)^2 = 0$$

$$50 \tan 8° = \frac{(4.9)(2500)}{v_0^2 \cos^2 8°} \implies v_0^2 = \frac{(4.9)50}{\tan 8° \cos^2 8°} \approx 1777.698$$

$$\implies v_0 \approx 42.2 \text{ m/sec}$$

43. $\mathbf{r}(t) = b(\omega t - \sin \omega t)\mathbf{i} + b(1 - \cos \omega t)\mathbf{j}$

$\mathbf{v}(t) = b(\omega - \omega \cos \omega t)\mathbf{i} + b\omega \sin \omega t\, \mathbf{j} = b\omega(1 - \cos \omega t)\mathbf{i} + b\omega \sin \omega t\mathbf{j}$

$\mathbf{a}(t) = (b\omega^2 \sin \omega t)\mathbf{i} + (b\omega^2 \cos \omega t)\mathbf{j} = b\omega^2[\sin(\omega t)\mathbf{i} + \cos(\omega t)\mathbf{j}]$

$\|\mathbf{v}(t)\| = \sqrt{2}\, b\omega\sqrt{1 - \cos(\omega t)}$

$\|\mathbf{a}(t)\| = b\omega^2$

(a) $\|\mathbf{v}(t)\| = 0$ when $\omega t = 0, 2\pi, 4\pi, \ldots$

(b) $\|\mathbf{v}(t)\|$ is maximum when $\omega t = \pi, 3\pi, \ldots,$ then $\|\mathbf{v}(t)\| = 2b\omega$.

44. $\mathbf{r}(t) = b(\omega t - \sin \omega t)\mathbf{i} + b(1 - \cos \omega t)\mathbf{j}$

$\mathbf{v}(t) = b\omega[(1 - \cos \omega t)\mathbf{i} + (\sin \omega t)\mathbf{j}]$

Speed $= \|\mathbf{v}(t)\| = \sqrt{2}b\omega\sqrt{1 - \cos \omega t}$ and has a maximum value of $2b\omega$ when $\omega t = \pi, 3\pi, \ldots$.

55 mph $= 80.67$ ft/sec $= 80.67$ rad/sec $= \omega$ since (since $b = 1$)

Therefore, the maximum speed of a point on the tire is twice the speed of the car:

$2(80.67)$ ft/sec $= 110$ mph

45. $\mathbf{v}(t) = -b\omega \sin(\omega t)\mathbf{i} + b\omega \cos(\omega t)\mathbf{j}$

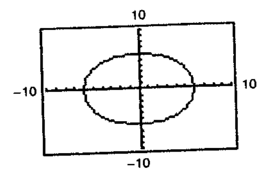
$\mathbf{r}(t) \cdot \mathbf{v}(t) = -b^2\omega \sin(\omega t) \cos(\omega t) + b^2\omega \sin(\omega t) \cos(\omega t) = 0$

Therefore, $\mathbf{r}(t)$ and $\mathbf{v}(t)$ are orthogonal.

46. (a) Speed $= \|\mathbf{v}\| = \sqrt{b^2\omega^2 \sin^2(\omega t) + b^2\omega^2 \cos^2(\omega t)}$

$= \sqrt{b^2\omega^2[\sin^2(\omega t) + \cos^2(\omega t)]} = b\omega$

(b)



The graphing utility draws the circle faster for greater values of ω.

47. $\mathbf{a}(t) = -b\omega^2 \cos(\omega t)\mathbf{i} - b\omega^2 \sin(\omega t)\mathbf{j} = -b\omega^2[\cos(\omega t)\mathbf{i} + \sin(\omega t)\mathbf{j}] = -\omega^2\mathbf{r}(t)$

$\mathbf{a}(t)$ is a negative multiple of a unit vector from $(0, 0)$ to $(\cos \omega t, \sin \omega t)$ and thus $\mathbf{a}(t)$ is directed toward the origin.

48. $\|\mathbf{a}(t)\| = b\omega^2\|\cos(\omega t)\mathbf{i} + \sin(\omega t)\mathbf{j}\| = b\omega^2$

49. $\|\mathbf{a}(t)\| = \omega^2 b,\ b = 2$

$1 = m(32)$

$\mathbf{F} = m(\omega^2 b) = \frac{1}{32}(2\omega^2) = 10$

$\omega = 4\sqrt{10}$ rad/sec

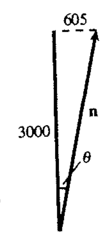
$\|\mathbf{v}(t)\| = b\omega = 8\sqrt{10}$ ft/sec

50. $\|\mathbf{v}(t)\| = 30$ mph $= 44$ ft/sec

$\omega = \frac{\|\mathbf{v}(t)\|}{b} = \frac{44}{300}$ rad/sec

$\|\mathbf{a}(t)\| = b\omega^2$

$\mathbf{F} = m(b\omega^2) = \frac{3000}{32}(300)\left(\frac{44}{300}\right)^2 = 605$ lb



Let $\mathbf{n}$ be normal to the road.

$\|\mathbf{n}\| \cos \theta = 3000$

$\|\mathbf{n}\| \sin \theta = 605$

Dividing the second equation by the first:

$\tan \theta = \frac{605}{3000}$

$\theta = \arctan\left(\frac{605}{3000}\right) \approx 11.4°.$

51. To find the range, set $y(t) = h + (v_0 \sin\theta)t - \frac{1}{2}gt^2 = 0$ then $0 = \left(\frac{1}{2}g\right)t^2 - (v_0\sin\theta)t - h$.
By the Quadratic Formula, (discount the negative value)

$$t = \frac{v_0\sin\theta + \sqrt{(-v_0\sin\theta)^2 - 4[(1/2)g](-h)}}{2[(1/2)g]} = \frac{v_0\sin\theta + \sqrt{v_0^2\sin^2\theta + 2gh}}{g} \text{ seconds}$$

At this time,

$$x(t) = v_0\cos\theta\left(\frac{v_0\sin\theta + \sqrt{v_0^2\sin^2\theta + 2gh}}{g}\right) = \frac{v_0\cos\theta}{g}\left(v_0\sin\theta + \sqrt{v_0^2\left(\sin^2\theta + \frac{2gh}{v_0^2}\right)}\right)$$

$$= \frac{v_0^2\cos\theta}{g}\left(\sin\theta + \sqrt{\sin^2\theta + \frac{2gh}{v_0^2}}\right) \text{ feet}$$

52. $h = 6$ feet, $v_0 = 45$ feet per second, $\theta = 42.5°$. From Exercise 47,

$$t = \frac{45\sin 42.5° + \sqrt{(45)^2\sin^2 42.5° + 2(32)(6)}}{32} \approx 2.08 \text{ seconds.}$$

At this time, $x(t) \approx 69.02$ feet.

53. $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ Position vector

$\mathbf{v}(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}$ Velocity vector

$\mathbf{a}(t) = x''(t)\mathbf{i} + y''(t)\mathbf{j} + z''(t)\mathbf{k}$ Acceleration vector

$$\text{Speed} = \|\mathbf{v}(t)\| = \sqrt{(x'(t)^2 + y'(t)^2 + z'(t)^2} = C, \ C \text{ is a constant.}$$

$$\frac{d}{dt}[x'(t)^2 + y'(t)^2 + z'(t)^2] = 0$$

$$2x'(t)x''(t) + 2y'(t)y''(t) + 2z'(t)z''(t) = 0$$

$$2[x'(t)x''(t) + y'(t)y''(t) + z'(t)z''(t)] = 0$$

$$\mathbf{v}(t)\cdot\mathbf{a}(t) = 0$$

Orthogonal

54. $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$

$y(t) = m(x(t)) + b$, m and b are constants.

$\mathbf{r}(t) = x(t)\mathbf{i} + [m(x(t)) + b]\mathbf{j}$

$\mathbf{v}(t) = x'(t)\mathbf{i} + mx'(t)\mathbf{j}$

$s(t) = \sqrt{[x'(t)]^2 + [mx'(t)]^2} = C$, C is a constant.

Thus, $x'(t) = \dfrac{C}{\sqrt{1+m^2}}$

$x''(t) = 0$

$\mathbf{a}(t) = x''(t)\mathbf{i} + mx''(t)\mathbf{j} = \mathbf{0}.$

55. $\mathbf{r}(t) = 6\cos t\,\mathbf{i} + 3\sin t\,\mathbf{j}$

(a) $\mathbf{v}(t) = \mathbf{r}'(t) = -6\sin t\,\mathbf{i} + 3\cos t\,\mathbf{j}$

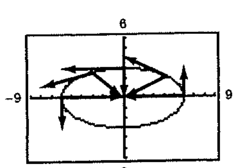
$$\|\mathbf{v}(t)\| = \sqrt{36\sin^2 t + 9\cos^2 t} = 3\sqrt{4\sin^2 t + \cos^2 t} = 3\sqrt{3\sin^2 t + 1}$$

$\mathbf{a}(t) = \mathbf{v}'(t) = -6\cos t\,\mathbf{i} - 3\sin t\,\mathbf{j}$

(b)

t	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π
Speed	3	$\frac{3}{2}\sqrt{10}$	6	$\frac{3}{2}\sqrt{13}$	3

(c)



(d) The speed is increasing when the angle between $\mathbf{v}$ and $\mathbf{a}$ is in the interval

$$\left[0, \frac{\pi}{2}\right).$$

The speed is decreasing when the angle is in the interval

$$\left(\frac{\pi}{2}, \pi\right].$$

56. (a) $\mathbf{r}_1(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$

$\mathbf{r}_2(t) = \mathbf{r}_1(2t)$

Velocity: $\mathbf{r}_2'(t) = 2\mathbf{r}_1'(2t)$

Acceleration: $\mathbf{r}_2''(t) = 4\mathbf{r}_1''(2t)$

(b) In general, if $\mathbf{r}_3(t) = \mathbf{r}_1(\omega t)$, then:

Velocity: $\mathbf{r}_3'(t) = \omega\mathbf{r}_1'(\omega t)$

Acceleration: $\mathbf{r}_3''(t) = \omega^2\mathbf{r}_1''(\omega t)$

57. False. The acceleration is the derivative of the velocity.

58. True

59. $\mathbf{a}(t) = \sin t\mathbf{i} - \cos t\mathbf{j}$

$$\mathbf{v}(t) = \int \mathbf{a}(t)\,dt = -\cos t\mathbf{i} - \sin t\mathbf{j} + \mathbf{C}_1$$

$$\mathbf{v}(0) = -\mathbf{i} = -\mathbf{i} + \mathbf{C}_1 \Rightarrow \mathbf{C}_1 = \mathbf{0}$$

$$\mathbf{v}(t) = -\cos t\mathbf{i} - \sin t\mathbf{j}$$

$$\mathbf{r}(t) = \int \mathbf{v}(t)\,dt = -\sin t\mathbf{i} + \cos t\mathbf{j} + \mathbf{C}_2$$

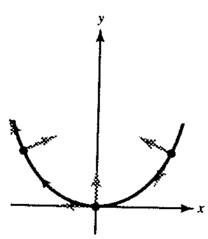
$$\mathbf{r}(0) = \mathbf{j} = \mathbf{j} + \mathbf{C}_2 \Rightarrow \mathbf{C}_2 = \mathbf{0}$$

$$\mathbf{r}(t) = -\sin t\mathbf{i} + \cos t\mathbf{j}$$

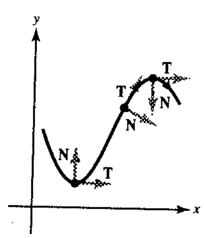
The path is a circle.

Section 12.4 Tangent Vectors and Normal Vectors

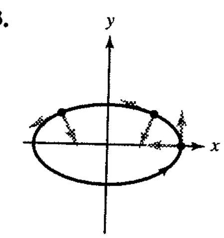
1.



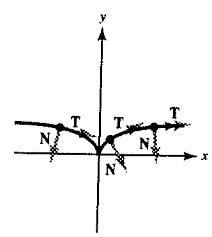
2.



3.



4.



5. $\mathbf{r}(t) = t^2\mathbf{i} + 2t\mathbf{j},\ t = 1$

$$\mathbf{r}'(t) = 2t\mathbf{i} + 2\mathbf{j},\ \|\mathbf{r}'(t)\| = \sqrt{4t^2 + 4} = 2\sqrt{t^2 + 1}$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{2t\mathbf{i} + 2\mathbf{j}}{2\sqrt{t^2 + 1}} = \frac{1}{\sqrt{t^2 + 1}}(t\mathbf{i} + \mathbf{j})$$

$$\mathbf{T}(1) = \frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j}) = \frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}$$

6. $\mathbf{r}(t) = t^3\mathbf{i} + 2t^2\mathbf{j}$

$$\mathbf{r}'(t) = 3t^2\mathbf{i} + 4t\mathbf{j}$$

$$\|\mathbf{r}'(t)\| = \sqrt{9t^4 + 16t^2}$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{1}{\sqrt{9t^4 + 16t^2}}(3t^2\mathbf{i} + 4t\mathbf{j})$$

$$\mathbf{T}(1) = \frac{1}{\sqrt{9 + 16}}(3\mathbf{i} + 4\mathbf{j}) = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$$

7. $\mathbf{r}(t) = 4\cos t\mathbf{i} + 4\sin t\mathbf{j},\ t = \frac{\pi}{4}$

$$\mathbf{r}'(t) = -4\sin t\mathbf{i} + 4\cos t\mathbf{j}$$

$$\|\mathbf{r}'(t)\| = \sqrt{16\sin^2 t + 16\cos^2 t} = 4$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = -\sin t\mathbf{i} + \cos t\mathbf{j}$$

$$\mathbf{T}\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}$$

8. $\mathbf{r}(t) = 6\cos t\mathbf{i} + 2\sin t\mathbf{j}$

$$\mathbf{r}'(t) = -6\sin t\mathbf{i} + 2\cos t\mathbf{j}$$

$$\|\mathbf{r}'(t)\| = \sqrt{36\sin^2 t + 4\cos^2 t}$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{-6\sin t\mathbf{i} + 2\cos t\mathbf{j}}{\sqrt{36\sin^2 t + 4\cos^2 t}}$$

$$\mathbf{T}\left(\frac{\pi}{3}\right) = \frac{-3\sqrt{3}\mathbf{i} + \mathbf{j}}{\sqrt{36(3/4) + (1/4)}} = \frac{1}{\sqrt{28}}(-3\sqrt{3}\mathbf{i} + \mathbf{j})$$

9. $\mathbf{r}(t) = \ln t\mathbf{i} + 2t\mathbf{j},\ t = e$

$$\mathbf{r}'(t) = \frac{1}{t}\mathbf{i} + 2\mathbf{j}$$

$$\mathbf{r}'(e) = \frac{1}{e}\mathbf{i} + 2\mathbf{j}$$

$$\mathbf{T}(e) = \frac{\mathbf{r}'(e)}{\|\mathbf{r}'(e)\|} = \frac{\frac{1}{e}\mathbf{i} + 2\mathbf{j}}{\sqrt{\frac{1}{e^2} + 4}} = \frac{\mathbf{i} + 2e\mathbf{j}}{\sqrt{1 + 4e^2}}$$

$$\approx 0.1809\mathbf{i} + 0.9835\mathbf{j}$$

10. $\mathbf{r}(t) = e^t\cos t\mathbf{i} + e^t\mathbf{j},\ t = 0$

$$\mathbf{r}'(t) = (e^t\cos t - e^t\sin t)\mathbf{i} + e^t\mathbf{j}$$

$$\mathbf{r}'(0) = \mathbf{i} + \mathbf{j}$$

$$\mathbf{T}(0) = \frac{\mathbf{r}'(0)}{\|\mathbf{r}'(0)\|} = \frac{\mathbf{i} + \mathbf{j}}{\sqrt{2}} = \frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}$$

11. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t\mathbf{k},\ P(0, 0, 0)$

$$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j} + \mathbf{k}$$

When $t = 0$, $\mathbf{r}'(0) = \mathbf{i} + \mathbf{k}$, $[t = 0 \text{ at } (0, 0, 0)]$.

$$\mathbf{T}(0) = \frac{\mathbf{r}'(0)}{\|\mathbf{r}'(0)\|} = \frac{\sqrt{2}}{2}(\mathbf{i} + \mathbf{k})$$

Direction numbers: $a = 1,\ b = 0,\ c = 1$

Parametric equations: $x = t,\ y = 0,\ z = t$

12. $\mathbf{r}(t) = t^2\mathbf{i} + t\mathbf{j} + \frac{4}{3}\mathbf{k},\ P\left(1, 1, \frac{4}{3}\right)$

$$\mathbf{r}'(t) = 2t\mathbf{i} + \mathbf{j}$$

When $t = 1$, $\mathbf{r}'(t) = \mathbf{r}'(1) = 2\mathbf{i} + \mathbf{j}$ $\left[t = 1 \text{ at } \left(1, 1, \frac{4}{3}\right)\right]$.

$$\mathbf{T}(1) = \frac{\mathbf{r}'(1)}{\|\mathbf{r}'(1)\|} = \frac{2\mathbf{i} + \mathbf{j}}{\sqrt{5}} = \frac{\sqrt{5}}{5}(2\mathbf{i} + \mathbf{j})$$

Direction numbers: $a = 2, b = 1, c = 0$

Parametric equations: $x = 2t + 1, y = t + 1, z = \frac{4}{3}$

13. $\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j} + t\mathbf{k},\ P(2, 0, 0)$

$$\mathbf{r}'(t) = -2\sin t\mathbf{i} + 2\cos t\mathbf{j} + \mathbf{k}$$

When $t = 0$, $\mathbf{r}'(0) = 2\mathbf{j} + \mathbf{k}$, $[t = 0 \text{ at } (2, 0, 0)]$.

$$\mathbf{T}(0) = \frac{\mathbf{r}'(0)}{\|\mathbf{r}'(0)\|} = \frac{\sqrt{5}}{5}(2\mathbf{j} + \mathbf{k})$$

Direction numbers: $a = 0,\ b = 2,\ c = 1$

Parametric equations: $x = 2,\ y = 2t,\ z = t$

14. $\mathbf{r}(t) = \left\langle t, t, \sqrt{4 - t^2}\right\rangle,\ P\left(1, 1, \sqrt{3}\right)$

$$\mathbf{r}'(t) = \left\langle 1, 1, -\frac{t}{\sqrt{4 - t^2}}\right\rangle$$

When $t = 1$, $\mathbf{r}'(1) = \left\langle 1, 1, -\frac{1}{\sqrt{3}}\right\rangle$, $t = 1$ at $\left(1, 1, \sqrt{3}\right)$].

$$\mathbf{T}(1) = \frac{\mathbf{r}'(1)}{\|\mathbf{r}'(1)\|} = \frac{\sqrt{21}}{7}\left\langle 1, 1, -\frac{1}{\sqrt{3}}\right\rangle$$

Direction numbers: $a = 1,\ b = 1,\ c = -\frac{1}{\sqrt{3}}$

Parametric equations: $x = t + 1,\ y = t + 1,$

$$z = -\frac{1}{\sqrt{3}}t + \sqrt{3}$$

15. $\mathbf{r}(t) = \langle 2\cos t, 2\sin t, 4\rangle,\ P\left(\sqrt{2}, \sqrt{2}, 4\right)$

$$\mathbf{r}'(t) = \langle -2\sin t, 2\cos t, 0\rangle$$

When $t = \frac{\pi}{4}$, $\mathbf{r}\left(\frac{\pi}{4}\right) = \left\langle -\sqrt{2}, \sqrt{2}, 0\right\rangle$, $\left[t = \frac{\pi}{4} \text{ at } \left(\sqrt{2}, \sqrt{2}, 4\right)\right]$.

$$\mathbf{T}\left(\frac{\pi}{4}\right) = \frac{\mathbf{r}'(\pi/4)}{\|\mathbf{r}'(\pi/4)\|} = \frac{1}{2}\left\langle -\sqrt{2}, \sqrt{2}, 0\right\rangle$$

Direction numbers: $a = -\sqrt{2},\ b = \sqrt{2},\ c = 0$

Parametric equations: $x = -\sqrt{2}t + \sqrt{2},\ y = \sqrt{2}t + \sqrt{2},\ z = 4$

16. $\mathbf{r}(t) = \langle 2\sin t, 2\cos t, 4\sin^2 t\rangle,\ P(1, \sqrt{3}, 1)$

$\mathbf{r}'(t) = \langle 2\cos t, -2\sin t, 8\sin t\cos t\rangle$

When $t = \frac{\pi}{6}$, $\mathbf{r}'\left(\frac{\pi}{6}\right) = \langle \sqrt{3}, -1, 2\sqrt{3}\rangle$, $\left[t = \frac{\pi}{6} \text{ at } (1, \sqrt{3}, 1)\right]$.

$$\mathbf{T}\left(\frac{\pi}{6}\right) = \frac{\mathbf{r}'(\pi/6)}{\|\mathbf{r}'(\pi/6)\|} = \frac{1}{4}\langle \sqrt{3}, -1, 2\sqrt{3}\rangle$$

Direction numbers: $a = \sqrt{3},\ b = -1,\ c = 2\sqrt{3}$

Parametric equations: $x = \sqrt{3}t + 1,\ y = -t + \sqrt{3},\ z = 2\sqrt{3}t + 1$

17. $\mathbf{r}(t) = \left\langle t, t^2, \frac{2}{3}t^3\right\rangle$

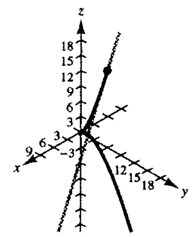
$\mathbf{r}'(t) = \langle 1, 2t, 2t^2\rangle$

When $t = 3$, $\mathbf{r}'(3) = \langle 1, 6, 18\rangle$, $[t = 3 \text{ at } (3, 9, 18)]$.

$$\mathbf{T}(3) = \frac{\mathbf{r}'(3)}{\|\mathbf{r}'(3)\|} = \frac{1}{19}\langle 1, 6, 18\rangle$$

Direction numbers: $a = 1,\ b = 6,\ c = 18$

Parametric equations: $x = t + 3,\ y = 6t + 9,\ z = 18t + 18$



18. $\mathbf{r}(t) = 3\cos t\mathbf{i} + 4\sin t\mathbf{j} + \frac{1}{2}\mathbf{k}$

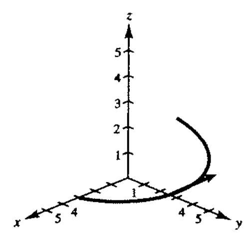
$\mathbf{r}'(t) = -3\sin t\mathbf{i} + 4\cos t\mathbf{j} + \frac{1}{2}\mathbf{k}$

When $t = \frac{\pi}{2}$, $\mathbf{r}'\left(\frac{\pi}{2}\right) = -3\mathbf{i} + \frac{1}{2}\mathbf{k}$, $\left[t = \frac{\pi}{2} \text{ at } \left(0, 4, \frac{\pi}{4}\right)\right]$.

$$\mathbf{T}\left(\frac{\pi}{2}\right) = \frac{\mathbf{r}'(\pi/2)}{\|\mathbf{r}'(\pi/2)\|} = \frac{2}{\sqrt{37}}\left(-3\mathbf{i} + \frac{1}{2}\mathbf{k}\right) = \frac{1}{\sqrt{37}}(-6\mathbf{i} + \mathbf{k})$$

Direction numbers: $a = -6,\ b = 0,\ c = 1$

Parametric equations: $x = -6t,\ y = 4,\ z = t + \frac{\pi}{4}$



19. $\mathbf{r}(t) = t\mathbf{i} + \ln t\mathbf{j} + \sqrt{t}\mathbf{k},\quad t_0 = 1$

$\mathbf{r}'(t) = \mathbf{i} + \frac{1}{t}\mathbf{j} + \frac{1}{2\sqrt{t}}\mathbf{k};\ \mathbf{r}'(1) = \mathbf{i} + \mathbf{j} + \frac{1}{2}\mathbf{k}$

$$\mathbf{T}(1) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{\mathbf{i} + \mathbf{j} + (1/2)\mathbf{k}}{\sqrt{1 + 1 + (1/4)}} = \frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{1}{3}\mathbf{k}$$

Tangent line: $x = 1 + t,\ y = t,\ z = 1 + \frac{1}{2}t$

$$\mathbf{r}(t_0 + 0.1) = \mathbf{r}(1.1) \approx 1.1\mathbf{i} + 0.1\mathbf{j} + 1.05\mathbf{k}$$
$$= \langle 1.1, 0.1, 1.05\rangle$$

20. $\mathbf{r}(t) = e^{-t}\mathbf{i} + 2\cos t\mathbf{j} + 2\sin t\mathbf{k},\ t_0 = 0$

$\mathbf{r}'(t) = -e^{-t}\mathbf{i} - 2\sin t\mathbf{j} + 2\cos t\mathbf{k}$

$\mathbf{r}(0) = \mathbf{i} + 2\mathbf{j},\ \mathbf{r}'(0) = -\mathbf{i} + 2\mathbf{k},\ \|\mathbf{r}'(0)\| = \sqrt{5}$

$$\mathbf{T}(0) = \frac{\mathbf{r}'(0)}{\|\mathbf{r}'(0)\|} = \frac{-\mathbf{i} + 2\mathbf{k}}{\sqrt{5}}$$

Parametric equations: $x(s) = 1 - s,\ y(s) = 2,\ z(s) = 2s$

$$\mathbf{r}(t_0 + 0.1) = \mathbf{r}(0 + 0.1) \approx \langle 1 - 0.1, 2, 2(0.1)\rangle$$
$$= \langle 0.9, 2, 0.2\rangle$$

21. $\mathbf{r}(4) = \langle 2, 16, 2\rangle$

$\mathbf{u}(8) = \langle 2, 16, 2\rangle$

Hence the curves intersect.

$$\mathbf{r}'(t) = \left\langle 1, 2t, \frac{1}{2}\right\rangle,\ \mathbf{r}'(4) = \left\langle 1, 8, \frac{1}{2}\right\rangle$$

$$\mathbf{u}'(s) = \left\langle \frac{1}{4}, 2, \frac{1}{3}s^{-2/3}\right\rangle,\ \mathbf{u}'(8) = \left\langle \frac{1}{4}, 2, \frac{1}{12}\right\rangle$$

$$\cos\theta = \frac{\mathbf{r}'(4)\cdot\mathbf{u}'(8)}{\|\mathbf{r}'(4)\|\,\|\mathbf{u}'(8)\|} \approx \frac{16.29167}{16.29513} \Rightarrow \theta \approx 1.2°$$

22. $\mathbf{r}(0) = \langle 0, 1, 0\rangle$

$\mathbf{u}(0) = \langle 0, 1, 0\rangle$

Hence the curves intersect.

$$\mathbf{r}'(t) = \langle 1, -\sin t, \cos t\rangle,\ \mathbf{r}'(0) = \langle 1, 0, 1\rangle$$

$$\mathbf{u}'(s) = \left\langle -\sin s\cos s - \cos s, -\sin s\cos s - \cos s, \frac{1}{2}\cos 2s + \frac{1}{2}\right\rangle$$

$$\mathbf{u}'(0) = \langle -1, 0, 1\rangle$$

$$\cos\theta = \frac{\mathbf{r}'(0)\cdot\mathbf{u}'(0)}{\|\mathbf{r}'(0)\|\,\|\mathbf{u}'(0)\|} = 0 \Rightarrow \theta = \frac{\pi}{2}$$

23. $\mathbf{r}(t) = t\mathbf{i} + \frac{1}{2}t^2\mathbf{j},\ t = 2$

$$\mathbf{r}'(t) = \mathbf{i} + t\mathbf{j}$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{\mathbf{i} + t\mathbf{j}}{\sqrt{1+t^2}}$$

$$\mathbf{T}'(t) = \frac{-t}{(t^2+1)^{3/2}}\mathbf{i} + \frac{1}{(t^2+1)^{3/2}}\mathbf{j}$$

$$\mathbf{T}'(2) = \frac{-2}{5^{3/2}}\mathbf{i} + \frac{1}{5^{3/2}}\mathbf{j}$$

$$\mathbf{N}(2) = \frac{\mathbf{T}'(2)}{\|\mathbf{T}'(2)\|} = \frac{1}{\sqrt{5}}(-2\mathbf{i} + \mathbf{j}) = \frac{-2\sqrt{5}}{5}\mathbf{i} + \frac{\sqrt{5}}{5}\mathbf{j}$$

24. $\mathbf{r}(t) = t\mathbf{i} + \frac{6}{t}\mathbf{j},\ t = 3$

$$\mathbf{r}'(t) = \mathbf{i} - \frac{6}{t^2}\mathbf{j}$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{1}{\sqrt{1+(36/t^4)}}\left(\mathbf{i} - \frac{6}{t^2}\mathbf{j}\right)$$

$$= \frac{t^2}{\sqrt{t^4+36}}\left(\mathbf{i} - \frac{6}{t^2}\mathbf{j}\right)$$

$$\mathbf{T}'(t) = \frac{72t}{(t^4+36)^{3/2}}\mathbf{i} + \frac{12t^3}{(t^4+36)^{3/2}}\mathbf{j}$$

$$\mathbf{N}(2) = \frac{\mathbf{T}'(2)}{\|\mathbf{T}'(2)\|} + \frac{1}{\sqrt{13}}(3\mathbf{i} + 2\mathbf{j})$$

25. $\mathbf{r}(t) = \ln t\,\mathbf{i} + (t+1)\mathbf{j},\ t = 2$

$$\mathbf{r}'(t) = \frac{1}{t}\mathbf{i} + \mathbf{j}$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{\frac{1}{t}\mathbf{i} + \mathbf{j}}{\sqrt{\frac{1}{t^2}+1}} = \frac{\mathbf{i} + t\mathbf{j}}{\sqrt{1+t^2}}$$

$$\mathbf{T}'(t) = \frac{-t}{(1+t^2)^{3/2}}\mathbf{i} + \frac{1}{(1+t^2)^{3/2}}\mathbf{j}$$

$$\mathbf{T}'(2) = \frac{-2}{5^{3/2}}\mathbf{i} + \frac{1}{5^{3/2}}\mathbf{j}$$

$$\mathbf{N}(2) = \frac{\mathbf{T}'(2)}{\|\mathbf{T}'(2)\|} = \frac{-2\sqrt{5}}{5}\mathbf{i} + \frac{\sqrt{5}}{5}\mathbf{j}$$

26. $\mathbf{r}(t) = 3\cos t\,\mathbf{i} + 3\sin t\,\mathbf{j},\ t = \frac{\pi}{4}$

$$\mathbf{r}'(t) = -3\sin t\,\mathbf{i} + 3\cos t\,\mathbf{j},\ \|\mathbf{r}'(t)\| = 3$$

$$\mathbf{T}(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j}$$

$$\mathbf{T}'(t) = -\cos t\,\mathbf{i} - \sin t\,\mathbf{j}$$

$$\mathbf{T}'\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}$$

$$\mathbf{N}\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}$$

27. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \ln t\mathbf{k}, t = 1$

$$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j} + \frac{1}{t}\mathbf{k}$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{\mathbf{i} + 2t\mathbf{j} + \frac{1}{t}\mathbf{k}}{\sqrt{1 + 4t^2 + \frac{1}{t^2}}} = \frac{t\mathbf{i} + 2t^2\mathbf{j} + \mathbf{k}}{\sqrt{4t^4 + t^2 + 1}}$$

$$\mathbf{T}'(t) = \frac{1 - 4t^4}{(4t^4 + t^2 + 1)^{3/2}}\mathbf{i} + \frac{2t^3 + 4t}{(4t^4 + t^2 + 1)^{3/2}}\mathbf{j} + \frac{-8t^3 - t}{(4t^4 + t^2 + 1)^{3/2}}\mathbf{k}$$

$$\mathbf{T}'(1) = \frac{-3}{6^{3/2}}\mathbf{i} + \frac{6}{6^{3/2}}\mathbf{j} + \frac{-9}{6^{3/2}}\mathbf{k} = \frac{3}{6^{3/2}}[-\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}]$$

$$\mathbf{N}(1) = \frac{-\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}}{\sqrt{14}} = \frac{-\sqrt{14}}{14}\mathbf{i} + \frac{2\sqrt{14}}{14}\mathbf{j} - \frac{3\sqrt{14}}{14}\mathbf{k}$$

28. $\mathbf{r}(t) = \sqrt{2}t\mathbf{i} + e^t\mathbf{j} + e^{-t}\mathbf{k}, t = 0$

$$\mathbf{r}'(t) = \sqrt{2}\mathbf{i} + e^t\mathbf{j} - e^{-t}\mathbf{k}$$

$$\|\mathbf{r}'(t)\| = \sqrt{2 + e^{2t} + e^{-2t}} = \sqrt{(e^t + e^{-t})^2} = e^t + e^{-t}$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{\sqrt{2}\mathbf{i} + e^t\mathbf{j} - e^{-t}\mathbf{k}}{e^t + e^{-t}}$$

$$\mathbf{T}'(t) = \frac{\sqrt{2}(e^{-t} - e^t)}{(e^t + e^{-t})^2}\mathbf{i} + \frac{2}{(e^t + e^{-t})^2}\mathbf{j} + \frac{2}{(e^t + e^{-t})^2}\mathbf{k}$$

$$\mathbf{T}'(0) = \frac{1}{2}\mathbf{j} + \frac{1}{2}\mathbf{k}$$

$$\mathbf{N}(0) = \frac{\sqrt{2}}{2}\mathbf{j} + \frac{\sqrt{2}}{2}\mathbf{k}$$

29. $\mathbf{r}(t) = 6\cos t\mathbf{i} + 6\sin t\mathbf{j} + \mathbf{k}, t = \frac{3\pi}{4}$

$$\mathbf{r}'(t) = -6\sin t\mathbf{i} + 6\cos t\mathbf{j}$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = -\sin t\mathbf{i} + \cos t\mathbf{j}$$

$$\mathbf{T}'(t) = -\cos t\mathbf{i} - \sin t\mathbf{j}, \|\mathbf{T}'(t)\| = 1$$

$$\mathbf{N}\left(\frac{3\pi}{4}\right) = \frac{\mathbf{T}'(3\pi/4)}{\|\mathbf{T}'(3\pi/4)\|} = \frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}$$

30. $\mathbf{r}(t) = \cos t\mathbf{i} + 2\sin t\mathbf{j} + \mathbf{k}, t = -\frac{\pi}{4}$

$$\mathbf{r}'(t) = -\sin t\mathbf{i} + 2\cos t\mathbf{j}$$

$$\mathbf{T}(t) = \frac{\mathbf{r}(t)}{\|\mathbf{r}'(t)\|} = \frac{-\sin t\mathbf{i} + 2\cos t\mathbf{j}}{\sqrt{\sin^2 t + 4\cos^2 t}}$$

The unit normal vector is perpendicular to this vector and points toward the z-axis:

$$\mathbf{N}(t) = \frac{-2\cos t\mathbf{i} - \sin t\mathbf{j}}{\sqrt{\sin^2 t + 4\cos^2 t}}.$$

$$\mathbf{N}\left(\frac{-\pi}{4}\right) = \left(\frac{-2\sqrt{5}}{5}, \frac{\sqrt{5}}{5}, 0\right)$$

31. $\mathbf{r}(t) = 4t\mathbf{i}$

$$\mathbf{v}(t) = 4\mathbf{i}$$

$$\mathbf{a}(t) = \mathbf{0}$$

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \frac{4\mathbf{i}}{4} = \mathbf{i}$$

$$\mathbf{T}'(t) = \mathbf{0}$$

$\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$ is undefined.

The path is a line and the speed is constant.

32. $\mathbf{r}(t) = 4t\mathbf{i} - 2t\mathbf{j}$

$$\mathbf{v}(t) = 4\mathbf{i} - 2\mathbf{j}$$

$$\mathbf{a}(t) = \mathbf{0}$$

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \frac{1}{\sqrt{5}}(2\mathbf{i} - \mathbf{j})$$

$$\mathbf{T}'(t) = \mathbf{0}$$

$\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$ is undefined.

The path is a line and the speed is constant.

33. $\mathbf{r}(t) = 4t^2\mathbf{i}$

$$\mathbf{v}(t) = 8t\mathbf{i}$$

$$\mathbf{a}(t) = 8\mathbf{i}$$

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \frac{8t\mathbf{i}}{8t} = \mathbf{i}$$

$$\mathbf{T}'(t) = \mathbf{0}$$

$\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$ is undefined.

The path is a line and the speed is variable.

34. $\mathbf{r}(t) = t^2\mathbf{j} + \mathbf{k}$

$\mathbf{v}(t) = 2t\mathbf{j}$

$\mathbf{a}(t) = 2\mathbf{j}$

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \frac{2t\mathbf{j}}{2t} = \mathbf{j}$$

$\mathbf{T}'(t) = \mathbf{0}$

$\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$ is undefined.

The path is a line and the speed is variable.

35. $\mathbf{r}(t) = t\mathbf{i} + \dfrac{1}{t}\mathbf{j},\ \mathbf{v}(t) = \mathbf{i} - \dfrac{1}{t^2}\mathbf{j},\ \mathbf{v}(1) = \mathbf{i} - \mathbf{j},$

$\mathbf{a}(t) = \dfrac{2}{t^3}\mathbf{j},\ \mathbf{a}(1) = 2\mathbf{j}$

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \frac{t^2}{\sqrt{t^4+1}}\left(\mathbf{i} - \frac{1}{t^2}\mathbf{j}\right) = \frac{1}{\sqrt{t^4+1}}(t^2\mathbf{i} - \mathbf{j})$$

$$\mathbf{T}(1) = \frac{1}{\sqrt{2}}(\mathbf{i} - \mathbf{j}) = \frac{\sqrt{2}}{2}(\mathbf{i} - \mathbf{j})$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} = \frac{\dfrac{2t}{(t^4+1)^{3/2}}\mathbf{i} + \dfrac{2t^3}{(t^4+1)^{3/2}}\mathbf{j}}{\dfrac{2t}{(t^4+1)}}$$

$$= \frac{1}{\sqrt{t^4+1}}(\mathbf{i} + t^2\mathbf{j})$$

$$\mathbf{N}(1) = \frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j}) = \frac{\sqrt{2}}{2}(\mathbf{i} + \mathbf{j})$$

$a_{\mathbf{T}} = \mathbf{a} \cdot \mathbf{T} = -\sqrt{2}$

$a_{\mathbf{N}} = \mathbf{a} \cdot \mathbf{N} = \sqrt{2}$

36. $\mathbf{r}(t) = t^2\mathbf{i} + 2t\mathbf{j},\ t = 1$

$\mathbf{v}(t) = 2t\mathbf{i} + 2\mathbf{j},\ \mathbf{v}(1) = 2\mathbf{i} + 2\mathbf{j}$

$\mathbf{a}(t) = 2\mathbf{i},\ \mathbf{a}(1) = 2\mathbf{i}$

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \frac{1}{\sqrt{4t^2+4}}(2t\mathbf{i} + 2\mathbf{j}) = \frac{1}{\sqrt{t^2+1}}(t\mathbf{i} + \mathbf{j})$$

$$\mathbf{T}(1) = \frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j}) = \frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} = \frac{\dfrac{1}{(t^2+1)^{3/2}}\mathbf{i} + \dfrac{-t}{(t^2+1)^{3/2}}\mathbf{j}}{\dfrac{1}{t^2+1}}$$

$$= \frac{1}{\sqrt{t^2+1}}(\mathbf{i} + t\mathbf{j})$$

$$\mathbf{N}(1) = \frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}$$

$a_{\mathbf{T}} = \mathbf{a} \cdot \mathbf{T} = \sqrt{2}$

$a_{\mathbf{N}} = \mathbf{a} \cdot \mathbf{N} = \sqrt{2}$

37. $\mathbf{r}(t) = (t - t^3)\mathbf{i} + 2t^2\mathbf{j},\ t = 1$

$\mathbf{v}(t) = (1 - 3t^2)\mathbf{i} + 4t\mathbf{j},\ \mathbf{v}(1) = -2\mathbf{i} + 4\mathbf{j}$

$\mathbf{a}(t) = -6t\mathbf{i} + 4\mathbf{j},\ \mathbf{a}(1) = -6\mathbf{i} + 4\mathbf{j}$

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \frac{(1 - 3t^2)\mathbf{i} + 4t\mathbf{j}}{\sqrt{9t^4 + 10t^2 + 1}}$$

$$\mathbf{T}(1) = \frac{-2\mathbf{i} + 4\mathbf{j}}{\sqrt{20}} = \frac{-\mathbf{i} + 2\mathbf{j}}{\sqrt{5}} = \frac{-\sqrt{5}}{5}(\mathbf{i} - 2\mathbf{j})$$

$$\mathbf{T}'(t) = \frac{-16t(3t^2+1)}{(9t^4 + 10t^2 + 1)^{3/2}}\mathbf{i} + \frac{4 - 36t^4}{(9t^4 + 10t^2 + 1)^{3/2}}\mathbf{j}$$

$$\mathbf{T}'(1) = \frac{-64}{20^{3/2}}\mathbf{i} + \frac{-32}{20^{3/2}}\mathbf{j}$$

$$\mathbf{N}(1) = \frac{-2\mathbf{i} - \mathbf{j}}{\sqrt{5}}$$

$$a_{\mathbf{T}} = \mathbf{a} \cdot \mathbf{T} = \frac{1}{\sqrt{5}}(6 + 8) = \frac{14\sqrt{5}}{5}$$

$$a_{\mathbf{N}} = \mathbf{a} \cdot \mathbf{N} = \frac{1}{\sqrt{5}}(12 - 4) = \frac{8\sqrt{5}}{5}$$

38. $\mathbf{r}(t) = (t^3 - 4t)\mathbf{i} + (t^2 - 1)\mathbf{j},\ t = 0$

$\mathbf{v}(t) = (3t^2 - 4)\mathbf{i} + 2t\mathbf{j},\ \mathbf{v}(0) = -4\mathbf{i}$

$\mathbf{a}(t) = 6t\mathbf{i} + 2\mathbf{j},\ \mathbf{a}(0) = 2\mathbf{j}$

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \frac{(3t^2 - 4)\mathbf{i} + 2t\mathbf{j}}{\sqrt{9t^4 - 20t^2 + 16}}$$

$$\mathbf{T}(0) = \frac{-4\mathbf{i}}{\sqrt{16}} = -\mathbf{i}$$

$$\mathbf{T}'(t) = \frac{4t(3t^2 + 4)}{(9t^4 - 20t^2 + 16)^{3/2}}\mathbf{i} + \frac{32 - 18t^4}{(9t^4 - 20t^2 + 16)^{3/2}}\mathbf{j}$$

$$\mathbf{T}'(1) = \frac{32}{16^{3/2}}\mathbf{j} = \frac{1}{2}\mathbf{j}$$

$\mathbf{N}(1) = \mathbf{j}$

$a_{\mathbf{T}} = \mathbf{a} \cdot \mathbf{T} = 0$

$a_{\mathbf{N}} = \mathbf{a} \cdot \mathbf{N} = 2$

39. $\mathbf{r}(t) = e^t\mathbf{i} + e^{-2t}\mathbf{j},\ t = 0$

$\mathbf{v}(t) = e^t\mathbf{i} - 2e^{-2t}\mathbf{j},\ \mathbf{v}(0) = \mathbf{i} - 2\mathbf{j}$

$\mathbf{a}(t) = e^t\mathbf{i} + 4e^{-2t}\mathbf{j},\ \mathbf{a}(0) = \mathbf{i} + 4\mathbf{j}$

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \frac{e^t\mathbf{i} - 2e^{-2t}\mathbf{j}}{\sqrt{4e^{-4t} + e^{2t}}}$$

$$\mathbf{T}(0) = \frac{\mathbf{i} - 2\mathbf{j}}{\sqrt{5}}$$

$$\mathbf{N}(0) = \frac{2\mathbf{i} + \mathbf{j}}{\sqrt{5}}$$

$$a_{\mathbf{T}} = \mathbf{a} \cdot \mathbf{T} = \frac{1}{\sqrt{5}}(1 - 8) = \frac{-7\sqrt{5}}{5}$$

$$a_{\mathbf{N}} = \mathbf{a} \cdot \mathbf{N} = \frac{1}{\sqrt{5}}(2 + 4) = \frac{6\sqrt{5}}{5}$$

40. $\mathbf{r}(t) = e^t\mathbf{i} + e^{-t}\mathbf{j} + t\mathbf{k},\ t = 0$

$\mathbf{v}(t) = e^t\mathbf{i} - e^{-t}\mathbf{j} + \mathbf{k},\ \mathbf{v}(0) = \mathbf{i} - \mathbf{j} + \mathbf{k}$

$\mathbf{a}(t) = e^t\mathbf{i} + e^{-t}\mathbf{j},\ \mathbf{a}(0) = \mathbf{i} + \mathbf{j}$

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \frac{e^t\mathbf{i} - e^{-t}\mathbf{j} + \mathbf{k}}{\sqrt{e^{2t} + e^{-2t} + 1}}$$

$$\mathbf{T}(0) = \frac{\mathbf{i} - \mathbf{j} + \mathbf{k}}{\sqrt{3}}$$

$$\mathbf{T}'(t) = \frac{e^{2t}(e^{2t} + 2)}{(e^{4t} + e^{2t} + 1)^{3/2}}\mathbf{i} + \frac{e^{2t}(2e^{2t} + 1)}{(e^{4t} + e^{2t} + 1)^{3/2}}\mathbf{j} + \frac{e^t(1 - e^{4t})}{(e^{4t} + e^{2t} + 1)^{3/2}}\mathbf{k}$$

$$\mathbf{T}'(0) = \frac{3}{3^{3/2}}\mathbf{i} + \frac{3}{3^{3/2}}\mathbf{j}$$

$$\mathbf{N}(0) = \frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}$$

$a_{\mathbf{T}} = \mathbf{a} \cdot \mathbf{T} = 0$

$a_{\mathbf{N}} = \mathbf{a} \cdot \mathbf{N} = \sqrt{2}$

41. $\mathbf{r}(t) = (e^t \cos t)\mathbf{i} + (e^t \sin t)\mathbf{j}$

$\mathbf{v}(t) = e^t(\cos t - \sin t)\mathbf{i} + e^t(\cos t + \sin t)\mathbf{j}$

$\mathbf{a}(t) = e^t(-2 \sin t)\mathbf{i} + e^t(2 \cos t)\mathbf{j}$

At $t = \dfrac{\pi}{2}$, $\mathbf{T} = \dfrac{\mathbf{v}}{\|\mathbf{v}\|} = \dfrac{1}{\sqrt{2}}(-\mathbf{i} + \mathbf{j}) = \dfrac{\sqrt{2}}{2}(-\mathbf{i} + \mathbf{j})$.

Motion along $\mathbf{r}$ is counterclockwise. Therefore,

$$\mathbf{N} = \frac{1}{\sqrt{2}}(-\mathbf{i} - \mathbf{j}) = -\frac{\sqrt{2}}{2}(\mathbf{i} + \mathbf{j}).$$

$a_{\mathbf{T}} = \mathbf{a} \cdot \mathbf{T} = \sqrt{2}e^{\pi/2}$

$a_{\mathbf{N}} = \mathbf{a} \cdot \mathbf{N} = \sqrt{2}e^{\pi/2}$

42. $\mathbf{r}(t) = a\cos(\omega t)\mathbf{i} + b\sin(\omega t)\mathbf{j}$

$\mathbf{v}(t) = -a\omega\sin(\omega t)\mathbf{i} + b\omega\cos(\omega t)\mathbf{j}$

$\mathbf{v}(0) = b\omega\mathbf{j}$

$\mathbf{a}(t) = -a\omega^2\cos(\omega t)\mathbf{i} - b\omega^2\sin(\omega t)\mathbf{j}$

$\mathbf{a}(0) = -a\omega^2\mathbf{i}$

$$\mathbf{T}(0) = \frac{\mathbf{v}(0)}{\|\mathbf{v}(0)\|} = \mathbf{j}$$

Motion along $\mathbf{r}(t)$ is counterclockwise. Therefore,

$\mathbf{N}(0) = -\mathbf{i}.$

$a_{\mathbf{T}} = \mathbf{a} \cdot \mathbf{T} = 0$

$a_{\mathbf{N}} = \mathbf{a} \cdot \mathbf{N} = a\omega^2$

43. $\mathbf{r}(t_0) = (\cos \omega t_0 + \omega t_0 \sin \omega t_0)\mathbf{i} + (\sin \omega t_0 - \omega t_0 \cos \omega t_0)\mathbf{j}$

$\mathbf{v}(t_0) = (\omega^2 t_0 \cos \omega t_0)\mathbf{i} + (\omega^2 t_0 \sin \omega t_0)\mathbf{j}$

$\mathbf{a}(t_0) = \omega^2[(\cos \omega t_0 - \omega t_0 \sin \omega t_0)\mathbf{i} + (\omega t_0 \cos \omega t_0 + \sin \omega t_0)\mathbf{j}]$

$\mathbf{T}(t_0) = \dfrac{\mathbf{v}}{\|\mathbf{v}\|} = (\cos \omega t_0)\mathbf{i} + (\sin \omega t_0)\mathbf{j}$

Motion along $\mathbf{r}$ is counterclockwise. Therefore

$\mathbf{N}(t_0) = (-\sin \omega t_0)\mathbf{i} + (\cos \omega t_0)\mathbf{j}.$

$a_{\mathbf{T}} = \mathbf{a} \cdot \mathbf{T} = \omega^2$

$a_{\mathbf{N}} = \mathbf{a} \cdot \mathbf{N} = \omega^2(\omega t_0) = \omega^3 t_0$

44. $\mathbf{r}(t_0) = (\omega t_0 - \sin \omega t_0)\mathbf{i} + (1 - \cos \omega t_0)\mathbf{j}$

$\mathbf{v}(t_0) = \omega[(1 - \cos \omega t_0)\mathbf{i} + (\sin \omega t_0)\mathbf{j}]$

$\mathbf{a}(t_0) = \omega^2[(\sin \omega t_0)\mathbf{i} + (\cos \omega t_0)\mathbf{j}]$

$\mathbf{T} = \dfrac{\mathbf{v}}{\|\mathbf{v}\|} = \dfrac{(1 - \cos \omega t_0)\mathbf{i} + (\sin \omega t_0)\mathbf{j}}{\sqrt{2}\sqrt{1 - \cos \omega t_0}}$

Motion along $\mathbf{r}$ is clockwise. Therefore, $\mathbf{N} = \dfrac{(\sin \omega t_0)\mathbf{i} - (1 - \cos \omega t_0)\mathbf{j}}{\sqrt{2}\sqrt{1 - \cos \omega t_0}}.$

$a_{\mathbf{T}} = \mathbf{a} \cdot \mathbf{T} = \dfrac{\omega^2 \sin \omega t_0}{\sqrt{2}\sqrt{1 - \cos \omega t_0}} = \dfrac{\omega^2}{\sqrt{2}}\sqrt{1 + \cos \omega t_0}$

$a_{\mathbf{N}} = \mathbf{a} \cdot \mathbf{N} = \dfrac{\omega^2}{\sqrt{2}}\sqrt{1 - \cos \omega t_0}$

45. $\mathbf{r}(t) = a \cos \omega t\, \mathbf{i} + a \sin \omega t\, \mathbf{j}$

$\mathbf{v}(t) = -a\omega \sin \omega t\, \mathbf{i} + a\omega \cos \omega t\, \mathbf{j}$

$\mathbf{a}(t) = -a\omega^2 \cos \omega t\, \mathbf{i} - a\omega^2 \sin \omega t\, \mathbf{j}$

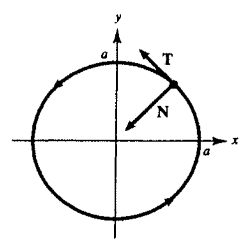
$\mathbf{T}(t) = \dfrac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = -\sin \omega t\, \mathbf{i} + \cos \omega t\, \mathbf{j}$

$\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} = -\cos \omega t\, \mathbf{i} - \sin \omega t\, \mathbf{j}$

$a_{\mathbf{T}} = \mathbf{a} \cdot \mathbf{T} = 0$

$a_{\mathbf{N}} = \mathbf{a} \cdot \mathbf{N} = a\omega^2$

46. $\mathbf{T}(t)$ points in the direction that $\mathbf{r}$ is moving. $\mathbf{N}(t)$ points in the direction that $\mathbf{r}$ is turning, toward the concave side of the curve.



47. Speed: $\|\mathbf{v}(t)\| = a\omega$

The speed is constant since $a_{\mathbf{T}} = 0$.

48. If the angular velocity ω is halved,

$$a_{\mathbf{N}} = a\left(\frac{\omega}{2}\right)^2 = \frac{a\omega^2}{4}.$$

$a_{\mathbf{N}}$ is changed by a factor of $\frac{1}{4}$.

49. $\mathbf{r}(t) = t\mathbf{i} + \frac{1}{t}\mathbf{j},\ t_0 = 2$

$x = t,\ y = \frac{1}{t} \Rightarrow xy = 1$

$\mathbf{r}'(t) = \mathbf{i} - \frac{1}{t^2}\mathbf{j}$

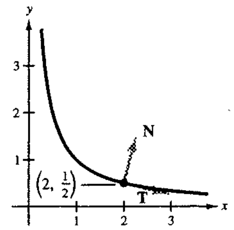
$\mathbf{T}(t) = \frac{t^2\mathbf{i} - \mathbf{j}}{\sqrt{t^4 + 1}}$

$\mathbf{N}(t) = \frac{\mathbf{i} + t^2\mathbf{j}}{\sqrt{t^4 + 1}}$

$\mathbf{r}(2) = 2\mathbf{i} + \frac{1}{2}\mathbf{j}$

$\mathbf{T}(2) = \frac{\sqrt{17}}{17}(4\mathbf{i} - \mathbf{j})$

$\mathbf{N}(2) = \frac{\sqrt{17}}{17}(\mathbf{i} + 4\mathbf{j})$



50. $\mathbf{r}(t) = t^3\mathbf{i} + t\mathbf{j},\ t_0 = 1$

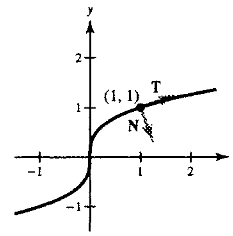
$x = t^3,\ y = t \Rightarrow x = y^3 \quad \text{or} \quad y = x^{1/3}$

$\mathbf{r}'(t) = 3t^2\mathbf{i} + \mathbf{j}$

$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{3t^2\mathbf{i} + \mathbf{j}}{\sqrt{9t^4 + 1}}$

$\mathbf{T}(1) = \frac{3\mathbf{i} + \mathbf{j}}{\sqrt{10}} = \frac{3\sqrt{10}}{10}\mathbf{i} + \frac{\sqrt{10}}{10}\mathbf{j}$

$\mathbf{N}(1) = \frac{\sqrt{10}}{10}\mathbf{i} - \frac{3\sqrt{10}}{10}\mathbf{j}$



51. $\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j},\ t_0 = \frac{\pi}{4}$

$x = 2\cos t,\ y = 2\sin t \Rightarrow x^2 + y^2 = 4$

$\mathbf{r}'(t) = -2\sin t\mathbf{i} + 2\cos t\mathbf{j}$

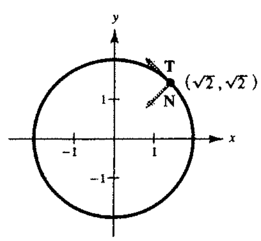
$\mathbf{T}(t) = \frac{1}{2}(-2\sin t\mathbf{i} + 2\cos t\mathbf{j}) = -\sin t\mathbf{i} + \cos t\mathbf{j}$

$\mathbf{N}(t) = -\cos t\mathbf{i} - \sin t\mathbf{j}$

$\mathbf{r}\left(\frac{\pi}{4}\right) = \sqrt{2}\mathbf{i} + \sqrt{2}\mathbf{j}$

$\mathbf{T}\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}(-\mathbf{i} + \mathbf{j})$

$\mathbf{N}\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}(-\mathbf{i} - \mathbf{j})$



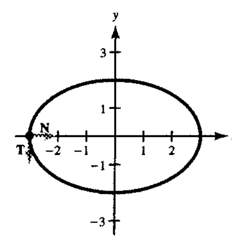
52. $\mathbf{r}(t) = 3\cos t\mathbf{i} + 2\sin t\mathbf{j},\ t_0 = \pi$

$\frac{x}{3} = \cos t,\ \frac{y}{2} = \sin t \Rightarrow \frac{x^2}{9} + \frac{y^2}{4} = 1$ Ellipse

$\mathbf{r}'(t) = -3\sin t\mathbf{i} + 2\cos t\mathbf{j}$

$\mathbf{r}'(\pi) = -2\mathbf{j} \Rightarrow \mathbf{T}(\pi) = -\mathbf{j}$

$\mathbf{N}(\pi) = \mathbf{i}$



53. $\mathbf{r}(t) = t\mathbf{i} + 2t\mathbf{j} - 3t\mathbf{k},\ t = 1$

$\mathbf{v}(t) = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$

$\mathbf{a}(t) = \mathbf{0}$

$\mathbf{T}(t) = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\sqrt{14}}(\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}) = \frac{\sqrt{14}}{14}(\mathbf{i} + 2\mathbf{j} - 3\mathbf{k})$

$= \mathbf{T}(1)$

$\mathbf{N}(t) = \frac{\mathbf{T}'}{\|\mathbf{T}'\|}$ is undefined.

$a_\mathbf{T}, a_\mathbf{N}$ are not defined.

54. $\mathbf{r}(t) = 4t\mathbf{i} - 4t\mathbf{j} + 2t\mathbf{k}$

$\mathbf{v}(t) = 4\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$

$\mathbf{a}(t) = \mathbf{0}$

$\mathbf{T}(t) = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{3}(2\mathbf{i} - 2\mathbf{j} + \mathbf{k})$

$\mathbf{N}(t) = \frac{\mathbf{T}'}{\|\mathbf{T}'\|}$ is undefined.

$a_\mathbf{T}, a_\mathbf{N}$ are not defined.

55. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{t^2}{2}\mathbf{k},\ t = 1$

$\mathbf{v}(t) = \mathbf{i} + 2t\mathbf{j} + t\mathbf{k}$

$\mathbf{v}(1) = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$

$\mathbf{a}(t) = 2\mathbf{j} + \mathbf{k}$

$\mathbf{T}(t) = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\sqrt{1+5t^2}}(\mathbf{i} + 2t\mathbf{j} + t\mathbf{k})$

$\mathbf{T}(1) = \frac{\sqrt{6}}{6}(\mathbf{i} + 2\mathbf{j} + \mathbf{k})$

$$\mathbf{N}(t) = \frac{\mathbf{T}'}{\|\mathbf{T}'\|} = \frac{\dfrac{-5t\mathbf{i} + 2\mathbf{j} + \mathbf{k}}{(1+5t^2)^{3/2}}}{\dfrac{\sqrt{5}}{1+5t^2}} = \frac{-5t\mathbf{i} + 2\mathbf{j} + \mathbf{k}}{\sqrt{5}\sqrt{1+5t^2}}$$

$\mathbf{N}(1) = \frac{\sqrt{30}}{30}(-5\mathbf{i} + 2\mathbf{j} + \mathbf{k})$

$a_{\mathbf{T}} = \mathbf{a} \cdot \mathbf{T} = \frac{5\sqrt{6}}{6}$

$a_{\mathbf{N}} = \mathbf{a} \cdot \mathbf{N} = \frac{\sqrt{30}}{6}$

56. $\mathbf{r}(t) = e^t \sin t\,\mathbf{i} + e^t \cos t\mathbf{j} + e^t\mathbf{k}$

$\mathbf{v}(t) = (e^t \cos t + e^t \sin t)\mathbf{i} + (-e^t \sin t + e^t \cos t)\mathbf{j} + e^t\mathbf{k}$

$\mathbf{v}(0) = \mathbf{i} + \mathbf{j} + \mathbf{k}$

$\mathbf{a}(t) = 2e^t \cos t\mathbf{i} - 2e^t \sin t\mathbf{j} + e^t\mathbf{k}$

$\mathbf{a}(0) = 2\mathbf{i} + \mathbf{k}$

$\mathbf{T}(t) = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\sqrt{3}}[(\cos t + \sin t)\mathbf{i} + (-\sin t + \cos t)\mathbf{j} + \mathbf{k}]$

$\mathbf{T}(0) = \frac{1}{\sqrt{3}}[\mathbf{i} + \mathbf{j} + \mathbf{k}]$

$\mathbf{N}(t) = \frac{1}{\sqrt{2}}[(-\sin t + \cos t)\mathbf{i} + (-\cos t - \sin t)\mathbf{j}]$

$\mathbf{N}(0) = \frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}$

$a_{\mathbf{T}} = \mathbf{a} \cdot \mathbf{T} = \sqrt{3}$

$a_{\mathbf{N}} = \mathbf{a} \cdot \mathbf{N} = \sqrt{2}$

57. $\mathbf{r}(t) = 4t\mathbf{i} + 3\cos t\mathbf{j} + 3\sin t\mathbf{k},\ t = \frac{\pi}{2}$

$\mathbf{v}(t) = 4\mathbf{i} - 3\sin t\mathbf{j} + 3\cos t\mathbf{k}$

$\mathbf{v}\left(\frac{\pi}{2}\right) = 4\mathbf{i} - 3\mathbf{j}$

$\mathbf{a}(t) = -3\cos t\mathbf{j} - 3\sin t\mathbf{k}$

$\mathbf{a}\left(\frac{\pi}{2}\right) = -3\mathbf{k}$

$\mathbf{T}(t) = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{5}(4\mathbf{i} - 3\sin t\mathbf{j} + 3\cos t\mathbf{k})$

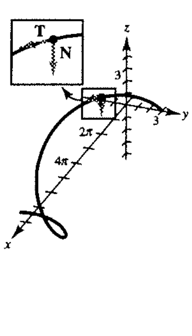
$\mathbf{T}\left(\frac{\pi}{2}\right) = \frac{1}{5}(4\mathbf{i} - 3\mathbf{j})$

$\mathbf{N}(t) = \frac{\mathbf{T}'}{\|\mathbf{T}'\|} = -\cos t\mathbf{j} - \sin t\mathbf{k}$

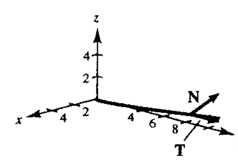
$\mathbf{N}\left(\frac{\pi}{2}\right) = -\mathbf{k}$

$a_{\mathbf{T}} = \mathbf{a} \cdot \mathbf{T} = 0$

$a_{\mathbf{N}} = \mathbf{a} \cdot \mathbf{N} = 3$



58. $\mathbf{r}(t) = t\mathbf{i} + 3t^2\mathbf{j} + \frac{t^2}{2}\mathbf{k}$



$\mathbf{v}(t) = \mathbf{i} + 6t\mathbf{j} + t\mathbf{k}$

$\mathbf{v}(2) = \mathbf{i} + 12\mathbf{j} + 2\mathbf{k}$

$\mathbf{a}(t) = 6\mathbf{j} + \mathbf{k}$

$\mathbf{T}(t) = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\sqrt{1+37t^2}}(\mathbf{i} + 6t\mathbf{j} + t\mathbf{k})$

$\mathbf{T}(2) = \frac{1}{\sqrt{149}}(\mathbf{i} + 12\mathbf{j} + 2\mathbf{k})$

$$\mathbf{N}(t) = \frac{\mathbf{T}'}{\|\mathbf{T}'\|} = \frac{\dfrac{1}{(1+37t^2)^{3/2}}[-37t\mathbf{i} + 6\mathbf{j} + \mathbf{k}]}{\dfrac{\sqrt{37}}{1+37t^2}}$$

$$= \frac{1}{\sqrt{37}\sqrt{1+37t^2}}[-37t\mathbf{i} + 6\mathbf{j} + \mathbf{k}]$$

$\mathbf{N}(2) = \frac{1}{\sqrt{37}\sqrt{149}}[-74\mathbf{i} + 6\mathbf{j} + \mathbf{k}]$

$= \frac{1}{\sqrt{5513}}(-74\mathbf{i} + 6\mathbf{j} + \mathbf{k})$

$a_{\mathbf{T}} = \mathbf{a} \cdot \mathbf{T} = \frac{74}{\sqrt{149}}$

$a_{\mathbf{N}} = \mathbf{a} \cdot \mathbf{N} = \frac{37}{\sqrt{5513}} = \frac{\sqrt{37}}{\sqrt{149}}$

59. $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$

$\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$

If $a(t) = a_{\mathbf{T}}\mathbf{T}(t) + a_{\mathbf{N}}\mathbf{N}(t)$, then $a_{\mathbf{T}}$ is the tangential component of acceleration and $a_{\mathbf{N}}$ is the normal component of acceleration.

60. The unit tangent vector points in the direction of motion.

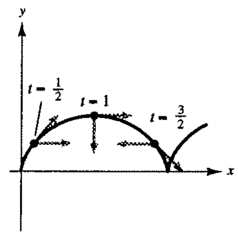
61. If $a_{\mathbf{N}} = 0$, then the motion is in a straight line.

62. If $a_{\mathbf{T}} = 0$, then the speed is constant.

63. $\mathbf{r}(t) = \langle \pi t - \sin \pi t, 1 - \cos \pi t\rangle$

The graph is a cycloid.

(a) $\mathbf{r}(t) = \langle \pi t - \sin \pi t, 1 - \cos \pi t\rangle$



$\mathbf{v}(t) = \langle \pi - \pi\cos \pi t, \pi \sin \pi t\rangle$

$\mathbf{a}(t) = \langle \pi^2 \sin \pi t, \pi^2 \cos \pi t\rangle$

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \frac{1}{\sqrt{2(1 - \cos \pi t)}}\langle 1 - \cos \pi t, \sin \pi t\rangle$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} = \frac{1}{\sqrt{2(1 - \cos \pi t)}}\langle \sin \pi t, -1 + \cos \pi t\rangle$$

$$a_{\mathbf{T}} = \mathbf{a}\cdot\mathbf{T} = \frac{1}{\sqrt{2(1 - \cos \pi t)}}[\pi^2 \sin \pi t(1 - \cos \pi t) + \pi^2 \cos \pi t \sin \pi t] = \frac{\pi^2 \sin \pi t}{\sqrt{2(1 - \cos \pi t)}}$$

$$a_{\mathbf{N}} = \mathbf{a}\cdot\mathbf{N} = \frac{1}{\sqrt{2(1 - \cos \pi t)}}[\pi^2 \sin^2 \pi t + \pi^2 \cos \pi t(-1 + \cos \pi t)] = \frac{\pi^2(1 - \cos \pi t)}{\sqrt{2(1 - \cos \pi t)}} = \frac{\pi^2\sqrt{2(1 - \cos \pi t)}}{2}$$

When $t = \frac{1}{2}$: $a_{\mathbf{T}} = \dfrac{\pi^2}{\sqrt{2}} = \dfrac{\sqrt{2}\pi^2}{2}$, $a_{\mathbf{N}} = \dfrac{\sqrt{2}\pi^2}{2}$

When $t = 1$: $a_{\mathbf{T}} = 0$, $a_{\mathbf{N}} = \pi^2$

When $t = \frac{3}{2}$: $a_{\mathbf{T}} = -\dfrac{\sqrt{2}\pi^2}{2}$, $a_{\mathbf{N}} = \dfrac{\sqrt{2}\pi^2}{2}$

(b) Speed: $s = \|\mathbf{v}(t)\| = \pi\sqrt{2(1 - \cos \pi t)}$

$$\frac{ds}{dt} = \frac{\pi^2 \sin \pi t}{\sqrt{2(1 - \cos \pi t)}} = a_{\mathbf{T}}$$

When $t = \frac{1}{2}$: $a_{\mathbf{T}} = \dfrac{\sqrt{2}\pi^2}{2} > 0 \implies$ the speed in increasing.

When $t = 1$: $a_{\mathbf{T}} = 0 \implies$ the height is maximum.

When $t = \frac{3}{2}$: $a_{\mathbf{T}} = -\dfrac{\sqrt{2}\pi^2}{2} < 0 \implies$ the speed is decreasing.

64. (a) $\mathbf{r}(t) = \langle \cos \pi t + \pi t \sin \pi t, \sin \pi t - \pi t \cos \pi t \rangle$

$\mathbf{v}(t) = \langle -\pi \sin \pi t + \pi \sin \pi t + \pi^2 t \cos \pi t, \pi \cos \pi t - \pi \cos \pi t + \pi^2 t \sin \pi t \rangle = \langle \pi^2 t \cos \pi t, \pi^2 t \sin \pi t \rangle$

$\mathbf{a}(t) = \langle \pi^2 \cos \pi t - \pi^3 t \sin \pi t, \pi^2 \sin \pi t + \pi^3 t \cos \pi t \rangle$

$\mathbf{T}(t) = \dfrac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \langle \cos \pi t, \sin \pi t \rangle$

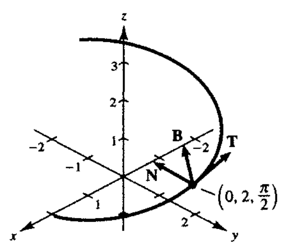
$a_{\mathbf{T}} = \mathbf{a} \cdot \mathbf{T} = \cos \pi t(\pi^2 \cos \pi t - \pi^3 t \sin \pi t) + \sin \pi t(\pi^2 \sin \pi t + \pi^3 t \cos \pi t) = \pi^2$

$a_{\mathbf{N}} = \sqrt{\|\mathbf{a}\|^2 - a_{\mathbf{T}}^2} = \sqrt{\pi^4(1 + \pi^2 t^2) - \pi^4} = \pi^3 t$

When $t = 1$, $a_{\mathbf{T}} = \pi^2$, $a_{\mathbf{N}} = \pi^3$. When $t = 2$, $a_{\mathbf{T}} = \pi^2$, $a_{\mathbf{N}} = 2\pi^3$.

(b) Since $a_{\mathbf{T}} = \pi^2 > 0$ for all values of t, the speed is increasing when $t = 1$ and $t = 2$.

65. $\mathbf{r}(t) = 2 \cos t\mathbf{i} + 2 \sin t\mathbf{j} + \dfrac{t}{2}\mathbf{k},\ t_0 = \dfrac{\pi}{2}$



$\mathbf{r}'(t) = -2 \sin t\mathbf{i} + 2 \cos t\mathbf{j} + \dfrac{1}{2}\mathbf{k}$

$\mathbf{T}(t) = \dfrac{2\sqrt{17}}{17}\left(-2 \sin t\mathbf{i} + 2 \cos t\mathbf{j} + \dfrac{1}{2}\mathbf{k}\right)$

$\mathbf{N}(t) = -\cos t\mathbf{i} - \sin t\mathbf{j}$

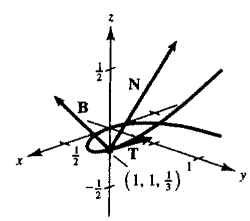
$\mathbf{r}\left(\dfrac{\pi}{2}\right) = 2\mathbf{j} + \dfrac{\pi}{4}\mathbf{k}$

$\mathbf{T}\left(\dfrac{\pi}{2}\right) = \dfrac{2\sqrt{17}}{17}\left(-2\mathbf{i} + \dfrac{1}{2}\mathbf{k}\right) = \dfrac{\sqrt{17}}{17}(-4\mathbf{i} + \mathbf{k})$

$\mathbf{N}\left(\dfrac{\pi}{2}\right) = -\mathbf{j}$

$$\mathbf{B}\left(\frac{\pi}{2}\right) = \mathbf{T}\left(\frac{\pi}{2}\right) \times \mathbf{N}\left(\frac{\pi}{2}\right) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\frac{4\sqrt{17}}{17} & 0 & \frac{\sqrt{17}}{17} \\ 0 & -1 & 0 \end{vmatrix} = \frac{\sqrt{17}}{17}\mathbf{i} + \frac{4\sqrt{17}}{17}\mathbf{k} = \frac{\sqrt{17}}{17}(\mathbf{i} + 4\mathbf{k})$$

66. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \dfrac{t^3}{3}\mathbf{k},\ t_0 = 1$



$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j} + t^2\mathbf{k}$

$\mathbf{T}(t) = \dfrac{1}{\sqrt{1 + 4t^2 + t^4}}(\mathbf{i} + 2t\mathbf{j} + t^2\mathbf{k})$

$\mathbf{N}(t) = \dfrac{1}{\sqrt{1 + 4t^2 + t^4}\sqrt{1 + t^2 + t^4}}[(-2t - t^3)\mathbf{i} + (1 - t^4)\mathbf{j} + (t + 2t^3)\mathbf{k}]$

$\mathbf{r}(1) = \mathbf{i} + \mathbf{j} + \dfrac{1}{3}\mathbf{k}$

$\mathbf{T}(1) = \dfrac{1}{\sqrt{6}}(\mathbf{i} + 2\mathbf{j} + \mathbf{k})$

$\mathbf{N}(1) = \dfrac{1}{\sqrt{6}\sqrt{3}}(-3\mathbf{i} + 3\mathbf{k}) = \dfrac{\sqrt{2}}{2}(-\mathbf{i} + \mathbf{k})$

$$\mathbf{B}(1) = \mathbf{T}(1) \times \mathbf{N}(1) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\sqrt{6}}{6} & \frac{\sqrt{6}}{3} & \frac{\sqrt{6}}{6} \\ -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{vmatrix} = \frac{\sqrt{3}}{3}\mathbf{i} - \frac{\sqrt{3}}{3}\mathbf{j} + \frac{\sqrt{3}}{3}\mathbf{k} = \frac{\sqrt{3}}{3}(\mathbf{i} - \mathbf{j} + \mathbf{k})$$

67. $\mathbf{r}(t) = \mathbf{i} + \sin t\mathbf{j} + \cos t\mathbf{k},\ t_0 = \dfrac{\pi}{4}$

$\mathbf{r}'(t) = \cos t\mathbf{j} - \sin t\mathbf{k},$

$\|\mathbf{r}'(t)\| = 1$

$\mathbf{r}'\left(\dfrac{\pi}{4}\right) = \mathbf{T}\left(\dfrac{\pi}{4}\right) = \dfrac{\sqrt{2}}{2}\mathbf{j} - \dfrac{\sqrt{2}}{2}\mathbf{k}$

$\mathbf{T}'(t) = -\sin t\mathbf{j} - \cos t\mathbf{k},$

$\mathbf{N}\left(\dfrac{\pi}{4}\right) = -\dfrac{\sqrt{2}}{2}\mathbf{j} - \dfrac{\sqrt{2}}{2}\mathbf{k}$

$$\mathbf{B}\left(\frac{\pi}{4}\right) = \mathbf{T}\left(\frac{\pi}{4}\right) \times \mathbf{N}\left(\frac{\pi}{4}\right) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 0 & -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{vmatrix} = -\mathbf{i}$$

68. $\mathbf{r}(t) = 2e^t\mathbf{i} + e^t\cos t\mathbf{j} + e^t\sin t\mathbf{k},\ t_0 = 0$

$\mathbf{r}'(t) = 2e^t\mathbf{i} + (e^t\cos t - e^t\sin t)\mathbf{j} + (e^t\sin t + e^t\cos t)\mathbf{k}$

$\mathbf{r}'(0) = 2\mathbf{i} + \mathbf{j} + \mathbf{k} \implies \mathbf{T}(0) = \dfrac{1}{\sqrt{6}}(2\mathbf{i} + \mathbf{j} + \mathbf{k})$

$$\|\mathbf{r}'(t)\|^2 = 4e^{2t} + e^{2t}\cos^2 t + e^{2t}\sin^2 t - 2e^{2t}\cos t\sin t + e^{2t}\sin^2 t + e^{2t}\cos^2 t + 2e^{2t}\sin t\cos t$$

$$= 4e^{2t} + 2e^{2t}(\cos^2 t + \sin^2 t) = 6e^{2t}$$

$\|\mathbf{r}'(t)\| = \sqrt{6}e^t$

$\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \dfrac{1}{\sqrt{6}}[2\mathbf{i} + (\cos t - \sin t)\mathbf{j} + (\sin t + \cos t)\mathbf{k}]$

$\mathbf{T}'(t) = \dfrac{1}{\sqrt{6}}[(-\sin t - \cos t)\mathbf{j} + (\cos t - \sin t)\mathbf{k}]$

$\mathbf{T}'(0) = \dfrac{1}{\sqrt{6}}[-\mathbf{j} + \mathbf{k}] \implies \mathbf{N}(0) = \dfrac{-\sqrt{2}}{2}\mathbf{j} + \dfrac{\sqrt{2}}{2}\mathbf{k}$

$$\mathbf{B}(0) = \mathbf{T}(0) \times \mathbf{N}(0) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{vmatrix} = \frac{\sqrt{3}}{3}\mathbf{i} - \frac{\sqrt{3}}{3}\mathbf{j} - \frac{\sqrt{3}}{3}\mathbf{k}$$

69. $\mathbf{r}(t) = 4\sin t\mathbf{i} + 4\cos t\mathbf{j} + 2t\mathbf{k},\ t_0 = \dfrac{\pi}{3}$

$$\mathbf{r}'(t) = 4\cos t\mathbf{i} - 4\sin t\mathbf{j} + 2\mathbf{k},$$

$$\|\mathbf{r}'(t)\| = \sqrt{16\cos^2 t + 16\sin^2 t + 4} = \sqrt{20} = 2\sqrt{5}$$

$$\mathbf{r}'\left(\frac{\pi}{3}\right) = 2\mathbf{i} - 2\sqrt{3}\mathbf{j} + 2\mathbf{k}$$

$$\mathbf{T}\left(\frac{\pi}{3}\right) = \frac{1}{2\sqrt{5}}\left(2\mathbf{i} - 2\sqrt{3}\mathbf{j} + 2\mathbf{k}\right) = \frac{\sqrt{5}}{5}\mathbf{i} - \frac{\sqrt{15}}{5}\mathbf{j} + \frac{\sqrt{5}}{5}\mathbf{k} = \frac{\sqrt{5}}{5}\left(\mathbf{i} - \sqrt{3}\mathbf{j} + \mathbf{k}\right)$$

$$\mathbf{T}'(t) = \frac{1}{2\sqrt{5}}(-4\sin t\mathbf{i} - 4\cos t\mathbf{j})$$

$$\mathbf{N}\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}\mathbf{i} - \frac{1}{2}\mathbf{j}$$

$$\mathbf{B}\left(\frac{\pi}{3}\right) = \mathbf{T}\left(\frac{\pi}{3}\right) \times \mathbf{N}\left(\frac{\pi}{3}\right) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\sqrt{5}}{5} & -\frac{\sqrt{15}}{5} & \frac{\sqrt{5}}{5} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \end{vmatrix} = \frac{\sqrt{5}}{10}\mathbf{i} - \frac{\sqrt{15}}{10}\mathbf{j} - \frac{4\sqrt{5}}{10}\mathbf{k} = \frac{\sqrt{5}}{10}\left(\mathbf{i} - \sqrt{3}\mathbf{j} - 4\mathbf{k}\right)$$

70. $\mathbf{r}(t) = 2\cos 2t\mathbf{i} + 2\sin 2t\mathbf{j} + t\mathbf{k},\ t_0 = \dfrac{\pi}{4}$

$$\mathbf{r}'(t) = -4\sin 2t\mathbf{i} + 4\cos 2t\mathbf{j} + \mathbf{k}$$

$$\|\mathbf{r}'(t)\| = \sqrt{17}$$

$$\mathbf{r}'\left(\frac{\pi}{4}\right) = -4\mathbf{i} + \mathbf{k}$$

$$\mathbf{T}\left(\frac{\pi}{4}\right) = \frac{\sqrt{17}}{17}(-4\mathbf{i} + \mathbf{k})$$

$$\mathbf{T}'(t) = \frac{1}{\sqrt{17}}(-8\cos 2t\mathbf{i} - 8\sin 2t\mathbf{j})$$

$$\mathbf{N}\left(\frac{\pi}{4}\right) = -\mathbf{j}$$

$$\mathbf{B}\left(\frac{\pi}{4}\right) = \mathbf{T}\left(\frac{\pi}{4}\right) \times \mathbf{N}\left(\frac{\pi}{4}\right) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\frac{4\sqrt{17}}{17} & 0 & \frac{\sqrt{17}}{17} \\ 0 & -1 & 0 \end{vmatrix} = \frac{\sqrt{17}}{17}\mathbf{i} + \frac{4\sqrt{17}}{17}\mathbf{k}$$

71. From Theorem 12.3 we have:

$$\mathbf{r}(t) = (v_0 t\cos\theta)\mathbf{i} + (h + v_0 t\sin\theta - 16t^2)\mathbf{j}$$

$$\mathbf{v}(t) = v_0\cos\theta\mathbf{i} + (v_0\sin\theta - 32t)\mathbf{j}$$

$$\mathbf{a}(t) = -32\mathbf{j}$$

$$\mathbf{T}(t) = \frac{(v_0\cos\theta)\mathbf{i} + (v_0\sin\theta - 32t)\mathbf{j}}{\sqrt{v_0^2\cos^2\theta + (v_0\sin\theta - 32t)^2}}$$

$$\mathbf{N}(t) = \frac{(v_0\sin\theta - 32t)\mathbf{i} - v_0\cos\theta\mathbf{j}}{\sqrt{v_0^2\cos^2\theta + (v_0\sin\theta - 32t)^2}} \quad \text{(Motion is clockwise.)}$$

$$a_{\mathbf{T}} = \mathbf{a}\cdot\mathbf{T} = \frac{-32(v_0\sin\theta - 32t)}{\sqrt{v_0^2\cos^2\theta + (v_0\sin\theta - 32t)^2}}$$

$$a_{\mathbf{N}} = \mathbf{a}\cdot\mathbf{N} = \frac{32v_0\cos\theta}{\sqrt{v_0^2\cos^2\theta + (v_0\sin\theta - 32t)^2}}$$

Maximum height when $v_0\sin\theta - 32t = 0$; (vertical component of velocity)

At maximum height, $a_{\mathbf{T}} = 0$ and $a_{\mathbf{N}} = 32$.

72. $\theta = 45°$, $v_0 = 150$

$$v_0 \cos\theta = 150 \cdot \frac{\sqrt{2}}{2} = 75\sqrt{2}$$

$$v_0 \sin\theta - 32t = 150 \cdot \frac{\sqrt{2}}{2} - 32t = 75\sqrt{2} - 32t$$

$$a_{\mathbf{T}} = \frac{-32(75\sqrt{2} - 32t)}{\sqrt{11250 + (75\sqrt{2} - 32t)^2}} = \frac{16(32t - 75\sqrt{2})}{\sqrt{256t^2 - 1200\sqrt{2}t + 5625}}$$

$$a_{\mathbf{N}} = \frac{32(75\sqrt{2})}{\sqrt{11250 + (75\sqrt{2} - 32t)^2}} = \frac{1200\sqrt{2}}{\sqrt{256t^2 - 1200\sqrt{2}t + 5625}}$$

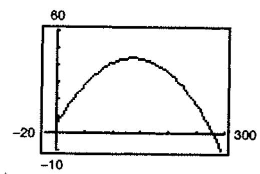
At the maximum height, $a_{\mathbf{T}} = 0$ and $a_{\mathbf{N}} = 32$.

73. (a) $\mathbf{r}(t) = (v_0 \cos\theta)t\mathbf{i} + \left[h + (v_0 \sin\theta)t - \frac{1}{2}gt^2\right]\mathbf{j}$

$= (100 \cos 30°)t\mathbf{i} + [5 + (100 \sin 30°)t - 16t^2]\mathbf{j}$

$= 50\sqrt{3}t\mathbf{i} + [5 + 50t - 16t^2]\mathbf{j}$

(b)



Maximum height ≈ 44.0625
Range ≈ 279.0325

(c) $\mathbf{v}(t) = 50\sqrt{3}\mathbf{i} + (50 - 32t)\mathbf{j}$

Speed $= \|\mathbf{v}(t)\| = \sqrt{2500(3) + (50 - 32t)^2}$

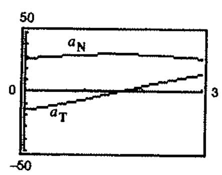
$= 4\sqrt{64t^2 - 200t + 625}$

$\mathbf{a}(t) = -32\mathbf{j}$

(d)

t	0.5	1.0	1.5	2.0	2.5	3.0
Speed	93.04	88.45	86.63	87.73	91.65	98.06

(e)



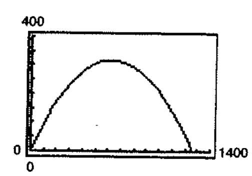
The speed is increasing when $a_{\mathbf{T}}$ and $a_{\mathbf{N}}$ have opposite signs.

74. (a) $\mathbf{r}(t) = (v_0 \cos\theta)t\mathbf{i} + \left[h + (v_0 \sin\theta)t - \frac{1}{2}gt^2\right]\mathbf{j}$

$= (200 \cos 45°)t\mathbf{i} + [4 + (200 \sin 45°)t - 16t^2]\mathbf{j}$

$= 100\sqrt{2}t\mathbf{i} + [5 + 100\sqrt{2}t - 16t^2]\mathbf{j}$

(b)



Maximum height ≈ 317.5 ft
Range ≈ 1258 ft

(c) $\mathbf{v}(t) = 100\sqrt{2}\mathbf{i} + (100\sqrt{2} - 32t)\mathbf{j}$

$\|\mathbf{v}(t)\| = \sqrt{20{,}000 + (100\sqrt{2} - 32t)^2}$

$\mathbf{a}(t) = -32\mathbf{j}$

(d)

t	0.5	1.0	1.5	2.0	2.5	3.0
Speed	189.03	178.81	169.49	161.23	154.18	148.54

75. $\mathbf{r}(t) = \langle 10\cos 10\pi t, 10\sin 10\pi t, 4 + 4t\rangle$, $0 \le t \le \frac{1}{20}$

(a) $\mathbf{r}'(t) = \langle -100\pi \sin(10\pi t), 100\pi \cos(10\pi t), 4\rangle$

$\|\mathbf{r}'(t)\| = \sqrt{(100\pi)^2 \sin^2(10\pi t) + (100\pi)^2 \cos^2(10\pi t) + 16}$

$= \sqrt{(100\pi)^2 + 16} = 4\sqrt{625\pi^2 + 1} \approx 314$ mi/hr

(b) $a_{\mathbf{T}} = 0$ and $a_{\mathbf{N}} = 1000\pi^2$

$a_{\mathbf{T}} = 0$ because the speed is constant.

76. 600 mph = 880 ft/sec

$\mathbf{r}(t) = 880t\mathbf{i} + (-16t^2 + 36{,}000)\mathbf{j}$

$\mathbf{v}(t) = 880\mathbf{i} - 32t\mathbf{j}$

$\mathbf{a}(t) = -32\mathbf{j}$

$$\mathbf{T}(t) = \frac{880\mathbf{i} - 32t\mathbf{j}}{16\sqrt{4t^2 + 3025}} = \frac{55\mathbf{i} - 2t\mathbf{j}}{\sqrt{4t^2 + 3025}}$$

Motion along **r** is clockwise, therefore

$$\mathbf{N}(t) = \frac{-2t\mathbf{i} - 55\mathbf{j}}{\sqrt{4t^2 + 3025}}$$

$$a_{\mathbf{T}} = \mathbf{a} \cdot \mathbf{T} = \frac{64t}{\sqrt{4t^2 + 3025}}$$

$$a_{\mathbf{N}} = \mathbf{a} \cdot \mathbf{N} = \frac{1760}{\sqrt{4t^2 + 3025}}$$

77. $\mathbf{r}(t) = (a\cos\omega t)\mathbf{i} + (a\sin\omega t)\mathbf{j}$

From Exercise 45, we know $\mathbf{a} \cdot \mathbf{T} = 0$ and $\mathbf{a} \cdot \mathbf{N} = a\omega^2$.

(a) Let $\omega_0 = 2\omega$. Then

$$\mathbf{a} \cdot \mathbf{N} = a\omega_0^2 = a(2\omega)^2 = 4a\omega^2$$

or the centripetal acceleration is increased by a factor of 4 when the velocity is doubled.

(b) Let $a_0 = a/2$. Then

$$\mathbf{a} \cdot \mathbf{N} = a_0\omega^2 = \left(\frac{a}{2}\right)\omega^2 = \left(\frac{1}{2}\right)a\omega^2$$

or the centripetal acceleration is halved when the radius is halved.

78. $\mathbf{r}(t) = (r\cos\omega t)\mathbf{i} + (r\sin\omega t)\mathbf{j}$

$\mathbf{v}(t) = (-r\omega\sin\omega t)\mathbf{i} + (r\omega\cos\omega t)\mathbf{j}$

$\|\mathbf{v}(t)\| = r\omega\sqrt{1} = r\omega = v$

$\mathbf{a}(t) = (-r\omega^2\cos\omega t)\mathbf{i} - (r\omega^2\sin\omega t)\mathbf{j}$

$\|\mathbf{a}(t)\| = r\omega^2$

(a) $F = m\|\mathbf{a}(t)\| = m(r\omega^2) = \dfrac{m}{r}(r^2\omega^2) = \dfrac{mv^2}{r}$

(b) By Newton's Law:

$$\frac{mv^2}{r} = \frac{GMm}{r^2},\ v^2 = \frac{GM}{r},\ v = \sqrt{\frac{GM}{r}}$$

79. $v = \sqrt{\dfrac{9.56 \times 10^4}{4100}} \approx 4.83$ mi/sec

80. $v = \sqrt{\dfrac{9.56 \times 10^4}{4200}} \approx 4.77$ mi/sec

81. $v = \sqrt{\dfrac{9.56 \times 10^4}{4385}} \approx 4.67$ mi/sec

82. Let x = distance from the satellite to the center of the earth ($x = r + 4000$). Then:

$$v = \frac{2\pi x}{t} = \frac{2\pi x}{24(3600)} = \sqrt{\frac{9.56 \times 10^4}{x}}$$

$$\frac{4\pi^2x^2}{(24)^2(3600)^2} = \frac{9.56 \times 10^4}{x}$$

$$x^3 = \frac{(9.56 \times 10^4)(24)^2(3600)^2}{4\pi^2} \Longrightarrow x \approx 26{,}245 \text{ mi}$$

$$v \approx \frac{2\pi(26{,}245)}{24(3600)} \approx 1.92 \text{ mi/sec} \approx 6871 \text{ mph}$$

83. False. You could be turning.

84. True. All the motion is in the tangential direction.

85. (a) $\mathbf{r}(t) = \cosh(bt)\mathbf{i} + \sinh(bt)\mathbf{j},\ b > 0$

$x = \cosh(bt),\ y = \sinh(bt)$

$x^2 - y^2 = \cosh^2(bt) - \sinh^2(bt) = 1$, hyperbola

(b) $\mathbf{v}(t) = b\sinh(bt)\mathbf{i} + b\cosh(bt)\mathbf{j}$

$\mathbf{a}(t) = b^2\cosh(bt)\mathbf{i} + b^2\sinh(bt)\mathbf{j}$

$= b^2\mathbf{r}(t)$

86. Let $\mathbf{T}(t) = \cos\phi\mathbf{i} + \sin\phi\mathbf{j}$ be the unit tangent vector. Then

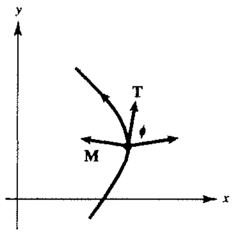
$$\mathbf{T}'(t) = \frac{d\mathbf{T}}{dt} = \frac{d\mathbf{T}}{d\phi}\frac{d\phi}{dt} = -(\sin\phi\mathbf{i} - \cos\phi\mathbf{j})\frac{d\phi}{dt} = \mathbf{M}\frac{d\phi}{dt}.$$

$\mathbf{M} = -\sin\phi\mathbf{i} + \cos\phi\mathbf{j} = \cos[\phi + (\pi/2)]\mathbf{i} + \sin[\phi + (\pi/2)]\mathbf{j}$ and is rotated counterclockwise through an angle of $\pi/2$ from $\mathbf{T}$.

If $d\phi/dt > 0$, then the curve bends to the left and $\mathbf{M}$ has the same direction as $\mathbf{T}'$. Thus, $\mathbf{M}$ has the same direction as

$$\mathbf{N} = \frac{\mathbf{T}'}{\|\mathbf{T}'\|},$$

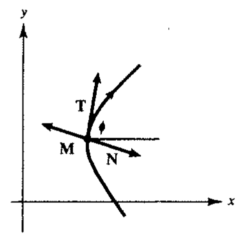
which is toward the concave side of the curve.



If $d\phi/dt < 0$, then the curve bends to the right and $\mathbf{M}$ has the opposite direction as $\mathbf{T}'$. Thus,

$$\mathbf{N} = \frac{\mathbf{T}'}{\|\mathbf{T}'\|}$$

again points to the concave side of the curve.



87. $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$

$y(t) = m(x(t)) + b$, m and b are constants.

$\mathbf{r}(t) = x(t)\mathbf{i} + [m(x(t)) + b]\mathbf{j}$

$\mathbf{v}(t) = x'(t)\mathbf{i} + mx'(t)\mathbf{j}$

$\|\mathbf{v}(t)\| = \sqrt{[x'(t)]^2 + [mx'(t)]^2} = |x'(t)|\sqrt{1 + m^2}$

$\mathbf{T}(t) = \dfrac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \dfrac{\pm(\mathbf{i} + m\mathbf{j})}{\sqrt{1 + m^2}}$, constant

Hence, $\mathbf{T}'(t) = \mathbf{0}$.

88. Using $\mathbf{a} = a_{\mathbf{T}}\mathbf{T} + a_{\mathbf{N}}\mathbf{N}$, $\mathbf{T} \times \mathbf{T} = \mathbf{O}$, and $\|\mathbf{T} \times \mathbf{N}\| = 1$, we have:

$$\begin{aligned}\mathbf{v} \times \mathbf{a} &= \|\mathbf{v}\|\mathbf{T} \times (a_{\mathbf{T}}\mathbf{T} + a_{\mathbf{N}}\mathbf{N}) \\ &= \|\mathbf{v}\|a_{\mathbf{T}}(\mathbf{T} \times \mathbf{T}) + \|\mathbf{v}\|a_{\mathbf{N}}(\mathbf{T} \times \mathbf{N}) \\ &= \|\mathbf{v}\|a_{\mathbf{N}}(\mathbf{T} \times \mathbf{N}) \\ \|\mathbf{v} \times \mathbf{a}\| &= \|\mathbf{v}\|a_{\mathbf{N}}\|\mathbf{T} \times \mathbf{N}\| \\ &= \|\mathbf{v}\|a_{\mathbf{N}}\end{aligned}$$

Thus, $a_{\mathbf{N}} = \dfrac{\|\mathbf{v} \times \mathbf{a}\|}{\|\mathbf{v}\|}$.

89. $\|\mathbf{a}\|^2 = \mathbf{a} \cdot \mathbf{a}$

$$\begin{aligned}&= (a_{\mathbf{T}}\mathbf{T} + a_{\mathbf{N}}\mathbf{N}) \cdot (a_{\mathbf{T}}\mathbf{T} + a_{\mathbf{N}}\mathbf{N}) \\ &= a_{\mathbf{T}}^2\|\mathbf{T}\|^2 + 2a_{\mathbf{T}}a_{\mathbf{N}}\mathbf{T} \cdot \mathbf{N} + a_{\mathbf{N}}^2\|\mathbf{N}\|^2 \\ &= a_{\mathbf{T}}^2 + a_{\mathbf{N}}^2\end{aligned}$$

$a_{\mathbf{N}}^2 = \|\mathbf{a}\|^2 - a_{\mathbf{T}}^2$

Since $a_{\mathbf{N}} > 0$, we have $a_{\mathbf{N}} = \sqrt{\|\mathbf{a}\|^2 - a_{\mathbf{T}}^2}$.

90. $F = ma = (1)\dfrac{dv}{dt} = \dfrac{dv}{dt}$ Force

$x = at + bt^2 + ct^3$

$v = \dfrac{dx}{dt} = a + 2bt + 3ct^2$

$\dfrac{dv}{dt} = 2b + 6ct$

$$\begin{aligned}F^2 &= 4b^2 + 24bct + 36c^2t^2 \\ &= 4b^2 + 12c + (2bt + 3ct^2) \\ &= 4b^2 + 12c + (v - a)\end{aligned}$$

$F = f(v) = \pm\sqrt{4b^2 - 12ac + 12cv}$

The sign of the radical is the sign of $2b + 6ct$, which cannot change.

Section 12.5 Arc Length and Curvature

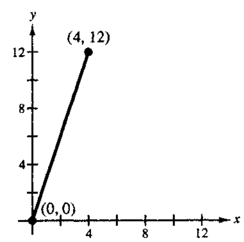
1. $\mathbf{r}(t) = t\mathbf{i} + 3t\mathbf{j}$

$$\frac{dx}{dt} = 1, \quad \frac{dy}{dt} = 3, \quad \frac{dz}{dt} = 0$$

$$s = \int_0^4 \sqrt{1 + 9}\,dt$$

$$= \sqrt{10}\int_0^4 dt$$

$$= \left[\sqrt{10}t\right]_0^4 = 4\sqrt{10}$$



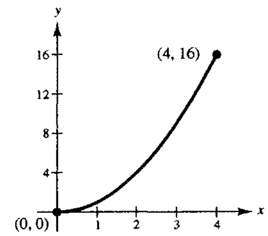
2. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{k}$

$$\frac{dx}{dt} = 1, \quad \frac{dy}{dt} = 0, \quad \frac{dz}{dt} = 2t$$

$$s = \int_0^4 \sqrt{1 + 4t^2}\,dt$$

$$= \frac{1}{4}\left[2t\sqrt{1 + 4t^2} + \ln|2t + \sqrt{1 + 4t}|\right]_0^4$$

$$= \frac{1}{4}\left[8\sqrt{65} + \ln\left(8 + \sqrt{65}\right)\right] \approx 16.819$$



3. $\mathbf{r}(t) = t^3\mathbf{i} + t^2\mathbf{j}, \quad 0 \le t \le 2$

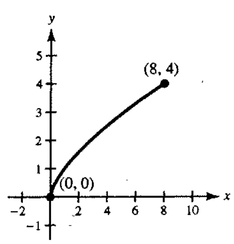
$$\frac{dx}{dt} = 3t^2, \quad \frac{dy}{dt} = 2t$$

$$s = \int_0^2 \sqrt{9t^4 + 4t^2}\,dt = \int_0^2 \sqrt{9t^2 + 4}\,t\,dt$$

$$= \frac{1}{27}(9t^2 + 4)^{3/2}\Big]_0^2$$

$$= \frac{1}{27}(40^{3/2} - 4^{3/2})$$

$$= \frac{1}{27}\left[80\sqrt{10} - 8\right]$$



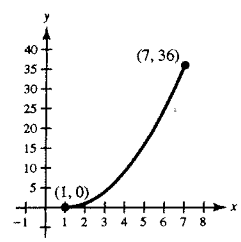
4. $\mathbf{r}(t) = (t + 1)\mathbf{i} + t^2\mathbf{j}, \quad 0 \le t \le 6$

$$\frac{dx}{dt} = 1, \quad \frac{dy}{dt} = 2t$$

$$s = \int_0^6 \sqrt{1 + 4t^2}\,dt$$

$$= \left[\frac{1}{4}\ln\left(\sqrt{4t^2 + 1} + 2t\right) + \frac{1}{2}t\sqrt{4t^2 + 1}\right]_0^6$$

$$= \frac{1}{4}\ln\left(\sqrt{145} + 12\right) + 3\sqrt{145}$$



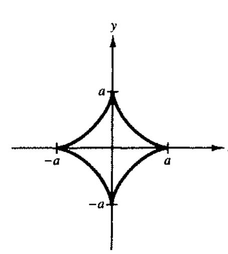
5. $\mathbf{r}(t) = a\cos^3 t\mathbf{i} + a\sin^3 t\mathbf{j}$

$$\frac{dx}{dt} = -3a\cos^2 t\sin t, \quad \frac{dy}{dt} = 3a\sin^2 t\cos t$$

$$s = 4\int_0^{\pi/2} \sqrt{\left[-3a\cos^2 t\sin t\right]^2 + \left[3a\sin^2 t\cos t\right]^2}\,dt$$

$$= 12a\int_0^{\pi/2} \sin t\cos t\,dt$$

$$= 3a\int_0^{\pi/2} 2\sin 2t\,dt = \left[-3a\cos 2t\right]_0^{\pi/2} = 6a$$

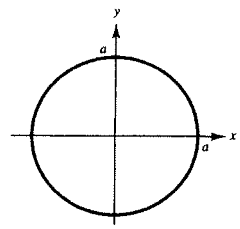


6. $\mathbf{r}(t) = a\cos t\mathbf{i} + a\sin t\mathbf{j}$

$\dfrac{dx}{dt} = -a\sin t,\ \dfrac{dy}{dt} = a\cos t$

$$s = \int_0^{2\pi} \sqrt{a^2\sin^2 t + a^2\cos^2 t}\,dt$$

$$= \int_0^{2\pi} a\,dt = \Big[at\Big]_0^{2\pi} = 2\pi a$$



7. (a) $\mathbf{r}(t) = (v_0\cos\theta)t\mathbf{i} + \left[h + (v_0\sin\theta)t - \frac{1}{2}gt^2\right]\mathbf{j}$

$$= (100\cos 45°)t\mathbf{i} + \left[3 + (100\sin 45°)t - \frac{1}{2}(32)t^2\right]\mathbf{j}$$

$$= 50\sqrt{2}t\mathbf{i} + \left[3 + 50\sqrt{2}t - 16t^2\right]\mathbf{j}$$

(b) $\mathbf{v}(t) = 50\sqrt{2}\mathbf{i} + (50\sqrt{2} - 32t)\mathbf{j}$

$50\sqrt{2} - 32t = 0 \implies t = \dfrac{25\sqrt{2}}{16}$

Maximum height: $3 + 50\sqrt{2}\left(\dfrac{25\sqrt{2}}{16}\right) - 16\left(\dfrac{25\sqrt{2}}{16}\right)^2 = 81.125$ ft

(c) $3 + 50\sqrt{2}t - 16t^2 = 0 \implies t \approx 4.4614$

Range: $50\sqrt{2}(4.4614) \approx 315.5$ feet

(d) $s = \displaystyle\int_0^{4.4614} \sqrt{(50\sqrt{2})^2 + (50\sqrt{2} - 32t)^2}\,dt \approx 362.9$ feet

8. (a) $\mathbf{r}(t) = (v_0\cos\theta)t\mathbf{i} + \left[(v_0\sin\theta)t - \frac{1}{2}gt^2\right]\mathbf{j}$

$y(t) = (v_0\sin\theta)t - \frac{1}{2}gt^2$

$y'(t) = v_0\sin\theta - gt = 0$ when $t = \dfrac{v_0\sin\theta}{g}$.

Maximum height when $\sin\theta = 1$, or $\theta = \dfrac{\pi}{2}$.

(b) $y(t) = (v_0\sin\theta)t - \frac{1}{2}gt^2 = 0 \implies t = \dfrac{2v_0\sin\theta}{g}$

Range: $x(t) = (v_0\cos\theta)\left(\dfrac{2v_0\sin\theta}{g}\right) = \dfrac{v_0^2}{g}\sin^2\theta$

The range $x(t)$ is a maximum for $\sin 2\theta = 1$, or $\theta = \dfrac{\pi}{4}$.

(c) $x'(t) = v_0\cos\theta$

$y'(t) = v_0\sin\theta - gt$

$$x'(t)^2 + y'(t)^2 = v_0^2\cos^2\theta + (v_0\sin\theta - gt)^2$$

$$= v_0^2\cos^2\theta + v_0^2\sin^2\theta - 2v_0^2 g\sin\theta t + g^2t^2$$

$$= v_0^2 - 2v_0 g\sin\theta t + g^2t^2$$

$$s(\theta) = \int_0^{2v_0\sin\theta/g}\left[v_0^2 - 2v_0 g\sin\theta t + g^2t^2\right]^{1/2}_{dt}$$

Since $v_0 = 96$ ft/sec, we have

$$s(\theta) = \int_0^{6\sin\theta}\left[96^2 - (6144\sin\theta)t + 1024t^2\right]^{1/2}_{dt}.$$

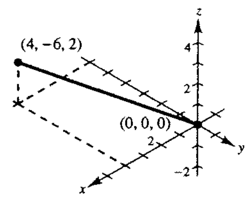
Using a computer algebra system, $s(\theta)$ is a maximum for $\theta \approx 0.9855 \approx 56.5°$.

9. $\mathbf{r}(t) = 2t\mathbf{i} - 3t\mathbf{j} + t\mathbf{k}$

$$\frac{dx}{dt} = 2, \ \frac{dy}{dt} = -3, \ \frac{dz}{dt} = 1$$

$$s = \int_0^2 \sqrt{2^2 + (-3)^2 + 1^2}\, dt$$

$$= \int_0^2 \sqrt{14}\, dt = \Big[\sqrt{14}t\Big]_0^2 = 2\sqrt{14}$$



10. $\mathbf{r}(t) = \mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}, \quad [0, 2]$

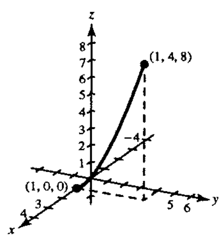
$$\frac{dx}{dt} = 0, \ \frac{dy}{dt} = 2t, \ \frac{dz}{dt} = 3t^2$$

$$s = \int_0^2 \sqrt{4t^2 + 9t^4}\, dt = \int_0^2 \sqrt{4 + 9t^2}\, t\, dt$$

$$= \frac{1}{27}(4 + 9t^2)^{3/2}\Big]_0^2$$

$$= \frac{1}{27}(40^{3/2} - 4^{3/2})$$

$$= \frac{1}{27}\left[80\sqrt{10} - 8\right]$$

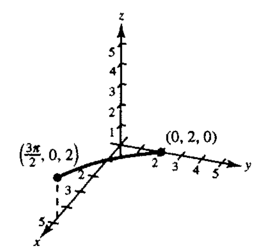


11. $\mathbf{r}(t) = \langle 3t, 2\cos t, 2\sin t\rangle$

$$\frac{dx}{dt} = 3, \frac{dy}{dt} = -2\sin t, \frac{dz}{dt} = 2\cos t$$

$$s = \int_0^{\pi/2} \sqrt{3^2 + (-2\sin t)^2 + (2\cos t)^2}\, dt$$

$$= \int_0^{\pi/2} \sqrt{13}\, dt = \sqrt{13}t\Big]_0^{\pi/2} = \frac{\sqrt{13}\pi}{2}$$

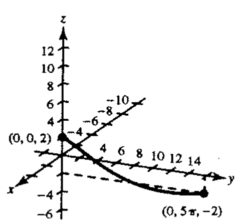


12. $\mathbf{r}(t) = \langle 2\sin t, 5t, 2\cos t\rangle, \quad [0, \pi]$

$$\frac{dx}{dt} = 2\cos t, \ \frac{dy}{dt} = 5, \ \frac{dz}{dt} = -2\sin t$$

$$s = \int_0^{\pi} \sqrt{4\cos^2 t + 25 + 4\sin^2 t}\, dt$$

$$= \int_0^{\pi} \sqrt{29}\, dt = \sqrt{29}\pi$$

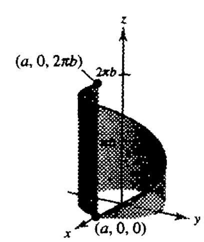


13. $\mathbf{r}(t) = a\cos t\mathbf{i} + a\sin t\mathbf{j} + bt\mathbf{k}$

$$\frac{dx}{dt} = -a\sin t, \ \frac{dy}{dt} = a\cos t, \ \frac{dz}{dt} = b$$

$$s = \int_0^{2\pi} \sqrt{a^2\sin^2 t + a^2\cos^2 t + b^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{a^2 + b^2}\, dt = \Big[\sqrt{a^2 + b^2}\,t\Big]_0^{2\pi} = 2\pi\sqrt{a^2 + b^2}$$

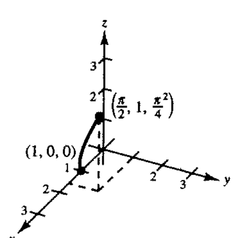


14. $\mathbf{r}(t) = \langle \cos t + t\sin t, \sin t - t\cos t, t^2\rangle$

$$\frac{dx}{dt} = t\cos t, \frac{dy}{dt} = t\sin t, \frac{dz}{dt} = 2t$$

$$s = \int_0^{\pi/2} \sqrt{(t\cos t)^2 + (t\sin t)^2 + (2t)^2}\, dt$$

$$= \int_0^{\pi/2} \sqrt{5t^2}\, dt = \sqrt{5}\frac{t^2}{2}\Big]_0^{\pi/2} = \frac{\sqrt{5}\pi^2}{8}$$



15. $\mathbf{r}(t) = t^2\mathbf{i} + t\mathbf{j} + \ln t\mathbf{k}$

$$\frac{dx}{dt} = 2t, \ \frac{dy}{dt} = 1, \ \frac{dz}{dt} = \frac{1}{t}$$

$$s = \int_1^3 \sqrt{(2t)^2 + (1)^2 + \left(\frac{1}{t}\right)^2}\, dt$$

$$= \int_1^3 \sqrt{\frac{4t^4 + t^2 + 1}{t^2}}\, dt$$

$$= \int_1^3 \frac{\sqrt{4t^4 + t^2 + 1}}{t}\, dt \approx 8.37$$

16. $\mathbf{r}(t) = \sin \pi t\mathbf{i} + \cos \pi t\mathbf{j} + t^3\mathbf{k}$

$$\frac{dx}{dt} = \pi \cos \pi t, \ \frac{dy}{dt} = -\pi \sin \pi t, \ \frac{dz}{dt} = 3t^2$$

$$s = \int_0^2 \sqrt{(\pi \cos \pi t)^2 + (-\pi \sin \pi t)^2 + (3t^2)^2}\, dt$$

$$= \int_0^2 \sqrt{\pi^2 + 9t^4}\, dt \approx 11.15$$

17. $\mathbf{r}(t) = t\mathbf{i} + (4 - t^2)\mathbf{j} + t^3\mathbf{k}, \quad 0 \le t \le 2$

(a) $\mathbf{r}(0) = \langle 0, 4, 0\rangle, \ \mathbf{r}(2) = \langle 2, 0, 8\rangle$

distance $= \sqrt{2^2 + 4^2 + 8^2} = \sqrt{84} = 2\sqrt{21} \approx 9.165$

(b) $\mathbf{r}(0) = \langle 0, 4, 0\rangle$

$\mathbf{r}(0.5) = \langle 0.5, 3.75, 0.125\rangle$

$\mathbf{r}(1) = \langle 1, 3, 1\rangle$

$\mathbf{r}(1.5) = \langle 1.5, 1.75, 3.375\rangle$

$\mathbf{r}(2) = \langle 2, 0, 8\rangle$

distance $\approx \sqrt{(0.5)^2 + (0.25)^2 + (0.125)^2} + \sqrt{(0.5)^2 + (0.75)^2 + (0.875)^2} + \sqrt{(0.5)^2 + (1.25)^2 + (2.375)^2} + \sqrt{(0.5)^2 + (1.75)^2 + (4.625)^2}$

$\approx 0.5728 + 1.2562 + 2.7300 + 4.9702 \approx 9.529$

(c) Increase the number of line segments.

(d) Using a graphing utility, you obtain 9.57057.

18. $\mathbf{r}(t) = 6\cos\left(\frac{\pi t}{4}\right)\mathbf{i} + 2\sin\left(\frac{\pi t}{4}\right)\mathbf{j} + t\mathbf{k}, \ 0 \le t \le 2$

(a) $\mathbf{r}(0) = 6\mathbf{i} = \langle 6, 0, 0\rangle$

$\mathbf{r}(2) = 2\mathbf{j} + 2\mathbf{k} = \langle 0, 2, 2\rangle$

distance $= \sqrt{6^2 + 2^2 + 2^2} = \sqrt{44} = 2\sqrt{11} \approx 6.633$

(b) $\mathbf{r}(0) = \langle 6, 0, 0\rangle$

$\mathbf{r}(0.5) = \langle 5.543, 0.765, 0.5\rangle$

$\mathbf{r}(1.0) = \langle 4.243, 1.414, 1.0\rangle$

$\mathbf{r}(1.5) = \langle 2.296, 1.848, 1.5\rangle$

$\mathbf{r}(2.0) = \langle 0, 2, 2\rangle$

distance ≈ 6.9698

(c) Increase the number of line segments.

(d) Using a graphing utility, you obtain

$$s = \int_0^2 \|\mathbf{r}'(t)\|\, dt \approx 7.0105.$$

19. $\mathbf{r}(t) = \langle 2\cos t, 2\sin t, t\rangle$

(a) $s = \int_0^t \sqrt{[x'(u)]^2 + [y'(u)]^2 + [z'(u)]^2}\, du$

$= \int_0^t \sqrt{(-2\sin u)^2 + (2\cos u)^2 + (1)^2}\, du$

$= \int_0^t \sqrt{5}\, du = \Big[\sqrt{5}u\Big]_0^t = \sqrt{5}t$

(b) $\frac{s}{\sqrt{5}} = t$

$x = 2\cos\left(\frac{s}{\sqrt{5}}\right), \ y = 2\sin\left(\frac{s}{\sqrt{5}}\right), \ z = \frac{s}{\sqrt{5}}$

$\mathbf{r}(s) = 2\cos\left(\frac{s}{\sqrt{5}}\right)\mathbf{i} + 2\sin\left(\frac{s}{\sqrt{5}}\right)\mathbf{j} + \frac{s}{\sqrt{5}}\mathbf{k}$

—CONTINUED—

19. —CONTINUED—

(c) When $s = \sqrt{5}$:

$x = 2\cos 1 \approx 1.081$

$y = 2\sin 1 \approx 1.683$

$z = 1$

$(1.081, 1.683, 1.000)$

When $s = 4$:

$x = 2\cos\frac{4}{\sqrt{5}} \approx -0.433$

$y = 2\sin\frac{4}{\sqrt{5}} \approx 1.953$

$z = \frac{4}{\sqrt{5}} \approx 1.789$

$(-0.433, 1.953, 1.789)$

(d) $\|\mathbf{r}'(s)\| = \sqrt{\left(-\frac{2}{\sqrt{5}}\sin\left(\frac{s}{\sqrt{5}}\right)\right)^2 + \left(\frac{2}{\sqrt{5}}\cos\left(\frac{s}{\sqrt{5}}\right)\right)^2 + \left(\frac{1}{\sqrt{5}}\right)^2} = \sqrt{\frac{4}{5} + \frac{1}{5}} = 1$

20. $\mathbf{r}(t) = \left\langle 4(\sin t - t\cos t), 4(\cos t + t\sin t), \frac{3}{2}t^2 \right\rangle$

(a) $s = \int_0^t \sqrt{[x'(u)]^2 + [y'(u)]^2 + [z'(u)]^2}\, du$

$= \int_0^t \sqrt{(4u\sin u)^2 + (4u\cos u)^2 + (3u)^2}\, du = \int_0^t \sqrt{16u + 9u^2}\, du = \int_0^t 5u\, du = \frac{5}{2}t^2$

(b) $t = \sqrt{\frac{2s}{5}}$

$x = 4\left(\sin\sqrt{\frac{2s}{5}} - \sqrt{\frac{2s}{5}}\cos\sqrt{\frac{2s}{5}}\right)$

$y = 4\left(\cos\sqrt{\frac{2s}{5}} + \sqrt{\frac{2s}{5}}\sin\sqrt{\frac{2s}{5}}\right)$

$z = \frac{3}{2}\left(\sqrt{\frac{2s}{5}}\right)^2 = \frac{3s}{5}$

$\mathbf{r}(s) = 4\left(\sin\sqrt{\frac{2s}{5}} - \sqrt{\frac{2s}{5}}\cos\sqrt{\frac{2s}{5}}\right)\mathbf{i} + 4\left(\cos\sqrt{\frac{2s}{5}} + \sqrt{\frac{2s}{5}}\sin\sqrt{\frac{2s}{5}}\right)\mathbf{j} + \frac{3s}{5}\mathbf{k}$

(c) When $s = \sqrt{5}$:

$x = 4\left(\sin\sqrt{\frac{2\sqrt{5}}{5}} - \sqrt{\frac{2\sqrt{5}}{5}}\cos\sqrt{\frac{2\sqrt{5}}{5}}\right) \approx -1.030$

$y = 4\left(\cos\sqrt{\frac{2\sqrt{5}}{5}} + \sqrt{\frac{2\sqrt{5}}{5}}\sin\sqrt{\frac{2\sqrt{5}}{5}}\right) \approx 5.408$

$z = \frac{3\sqrt{5}}{5} \approx 1.342$

$(-1.030, 5.408, 1.342)$

When $s = 4$:

$x = 4\left(\sin\sqrt{\frac{8}{5}} - \sqrt{\frac{8}{5}}\cos\sqrt{\frac{8}{5}}\right) \approx 2.291$

$y = 4\left(\cos\sqrt{\frac{8}{5}} + \sqrt{\frac{8}{5}}\sin\sqrt{\frac{8}{5}}\right) \approx 6.029$

$z = \frac{12}{5} = 2.4$

$(2.291, 6.029, 2.400)$

(d) $\|\mathbf{r}'(s)\| = \sqrt{\left(\frac{4}{5}\sin\sqrt{\frac{2s}{5}}\right)^2 + \left(\frac{4}{5}\cos\sqrt{\frac{2s}{5}}\right)^2 + \left(\frac{3}{5}\right)^2} = \sqrt{\frac{16}{25} + \frac{9}{25}} = 1$

21. $\mathbf{r}(s) = \left(1 + \frac{\sqrt{2}}{2}s\right)\mathbf{i} + \left(1 - \frac{\sqrt{2}}{2}s\right)\mathbf{j}$

$\mathbf{r}'(s) = \frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}$ and $\|\mathbf{r}'(s)\| = \sqrt{\frac{1}{2} + \frac{1}{2}} = 1$

$\mathbf{T}(s) = \frac{\mathbf{r}'(s)}{\|\mathbf{r}'(s)\|} = \mathbf{r}'(s)$

$\mathbf{T}'(s) = \mathbf{0} \implies K = \|\mathbf{T}'(s)\| = 0$ (The curve is a line.)

22. $\mathbf{r}(s) = (3 + s)\mathbf{i} + \mathbf{j}$

$\mathbf{r}'(s) = \mathbf{i}$ and $\|\mathbf{r}'(s)\| = 1$

$\mathbf{T}(s) = \mathbf{r}'(s)$

$\mathbf{T}'(s) = \mathbf{0} \implies K = \|\mathbf{T}'(s)\| = 0$ (The curve is a line.)

23. $\mathbf{r}(s) = 2\cos\left(\frac{s}{\sqrt{5}}\right)\mathbf{i} + 2\sin\left(\frac{s}{\sqrt{5}}\right)\mathbf{j} + \frac{s}{\sqrt{5}}\mathbf{k}$

$\mathbf{T}(s) = \mathbf{r}'(s) = -\frac{2}{\sqrt{5}}\sin\left(\frac{s}{\sqrt{5}}\right)\mathbf{i} + \frac{2}{\sqrt{5}}\cos\left(\frac{s}{\sqrt{5}}\right)\mathbf{j} + \frac{1}{\sqrt{5}}\mathbf{k}$

$\mathbf{T}'(s) = -\frac{2}{5}\cos\left(\frac{s}{\sqrt{5}}\right)\mathbf{i} - \frac{2}{5}\sin\left(\frac{s}{\sqrt{5}}\right)\mathbf{j}$

$K = \|\mathbf{T}'(s)\| = \frac{2}{5}$

24. $\mathbf{r}(s) = 4\left(\sin\sqrt{\frac{2s}{5}} - \sqrt{\frac{2s}{5}}\cos\sqrt{\frac{2s}{5}}\right)\mathbf{i} + 4\left(\cos\sqrt{\frac{2s}{5}} + \sqrt{\frac{2s}{5}}\sin\sqrt{\frac{2s}{5}}\right)\mathbf{j} + \frac{3s}{5}\mathbf{k}$

$\mathbf{T}(s) = \mathbf{r}'(s) = \frac{4}{5}\sin\sqrt{\frac{2s}{5}}\mathbf{i} + \frac{4}{5}\cos\sqrt{\frac{2s}{5}}\mathbf{j} + \frac{3}{5}\mathbf{k}$

$\mathbf{T}'(s) = \frac{4}{25}\sqrt{\frac{5}{2s}}\cos\sqrt{\frac{2s}{5}}\mathbf{i} - \frac{4}{25}\sqrt{\frac{5}{2s}}\sin\sqrt{\frac{2s}{5}}\mathbf{j}$

$K = \|\mathbf{T}'(s)\| = \frac{4}{25}\sqrt{\frac{5}{2s}} = \frac{2\sqrt{10s}}{25s}$

25. $\mathbf{r}(t) = 4t\mathbf{i} - 2t\mathbf{j}$

$\mathbf{v}(t) = 4\mathbf{i} - 2\mathbf{j}$

$\mathbf{T}(t) = \frac{1}{\sqrt{5}}(2\mathbf{i} - \mathbf{j})$

$\mathbf{T}'(t) = \mathbf{0}$

$K = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = 0$

(The curve is a line.)

26. $\mathbf{r}(t) = t^2\mathbf{j} + \mathbf{k}$

$\mathbf{v}(t) = 2t\mathbf{j}$

$\mathbf{T}(t) = \mathbf{j}$

$\mathbf{T}'(t) = 0$

$K = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = 0$

27. $\mathbf{r}(t) = t\mathbf{i} + \frac{1}{t}\mathbf{j}$

$\mathbf{v}(t) = \mathbf{i} - \frac{1}{t^2}\mathbf{j}$

$\mathbf{v}(1) = \mathbf{i} - \mathbf{j}$

$\mathbf{a}(t) = \frac{2}{t^3}\mathbf{j}$

$\mathbf{a}(1) = 2\mathbf{j}$

$\mathbf{T}(t) = \frac{t^2\mathbf{i} - \mathbf{j}}{\sqrt{t^4 + 1}}$

$\mathbf{N}(t) = \frac{1}{(t^4 + 1)^{1/2}}(\mathbf{i} + t^2\mathbf{j})$

$\mathbf{N}(1) = \frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j})$

$K = \frac{\mathbf{a} \cdot \mathbf{N}}{\|\mathbf{v}\|^2} = \frac{\sqrt{2}}{2}$

28. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}$

$\mathbf{v}(t) = \mathbf{i} + 2t\mathbf{j}$

$\mathbf{v}(1) = \mathbf{i} + 2\mathbf{j}$

$\mathbf{a}(t) = 2\mathbf{j}$

$\mathbf{a}(1) = 2\mathbf{j}$

$$\mathbf{T}(t) = \frac{\mathbf{i} + 2t\mathbf{j}}{\sqrt{1 + 4t^2}}$$

$$\mathbf{N}(t) = \frac{1}{\sqrt{1 + 4t^2}}(-2t\mathbf{i} + \mathbf{j})$$

$$\mathbf{N}(1) = \frac{1}{\sqrt{5}}(-2\mathbf{i} + \mathbf{j})$$

$$K = \frac{\mathbf{a} \cdot \mathbf{N}}{\|\mathbf{v}\|^2} = \frac{2}{5\sqrt{5}}$$

29. $\mathbf{r}(t) = t\mathbf{i} + \cos t\mathbf{j},\ t = 0$

$\mathbf{r}'(t) = \mathbf{i} - \sin t\mathbf{j},\ \|\mathbf{r}'(t)\| = \sqrt{1 + \sin^2 t}$

$$\mathbf{T}(t) = \frac{\mathbf{i} - \sin t\mathbf{j}}{\sqrt{1 + \sin^2 t}}$$

$$\mathbf{T}'(t) = \frac{-\sin t \cos t}{(1 + \sin^2 t)^{3/2}}\mathbf{i} + \frac{-\cos t}{(1 + \sin^2 t)^{3/2}}\mathbf{j}$$

$\mathbf{T}'(0) = -\mathbf{j},\ \mathbf{r}'(0) = \mathbf{i}$

$$K = \frac{\|\mathbf{T}'(0)\|}{\|\mathbf{r}'(0)\|} = 1$$

Alternate Solution: $x(t) = t,\ y(t) = \cos t$

$$K = \frac{|x'y'' - y'x''|}{[(x')^2 + (y')^2]^{3/2}} = \frac{|-\cos t|}{[1 + \sin^2 t]^{3/2}}$$

$K(0) = 1$

30. $\mathbf{r}(t) = 5\cos t\mathbf{i} + 4\sin t\mathbf{j},\ t = \frac{\pi}{3}$

$x(t) = 5\cos t,\ y(t) = 4\sin t$

$$K = \frac{|x'y'' - y'x''|}{[(x')^2 + (y')^2]^{3/2}}$$

$$= \frac{|(-5\sin t)(-4\sin t) - (4\cos t)(-5\cos t)|}{[25\sin^2 t + 16\cos^2 t]^{3/2}}$$

$$= \frac{20}{[25\sin^2 t + 16\cos^2 t]^{3/2}}$$

$$K\left(\frac{\pi}{3}\right) = \frac{20}{[25(3/4) + 16(1/4)]^{3/2}}$$

$$= \frac{160\sqrt{91}}{8281}$$

31. $\mathbf{r}(t) = 4\cos 2\pi t\mathbf{i} + 4\sin 2\pi t\mathbf{j}$

$\mathbf{r}'(t) = -8\pi\sin 2\pi t\mathbf{i} + 8\pi\cos 2\pi t\mathbf{j}$

$\mathbf{T}(t) = -\sin 2\pi t\mathbf{i} + \cos 2\pi t\mathbf{j}$

$\mathbf{T}'(t) = -2\pi\cos 2\pi t\mathbf{i} - 2\pi\sin 2\pi t\mathbf{j}$

$$K = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{2\pi}{8\pi} = \frac{1}{4}$$

32. $\mathbf{r}(t) = 2\cos \pi t\mathbf{i} + \sin \pi t\mathbf{j}$

$\mathbf{r}'(t) = -2\pi\sin \pi t\mathbf{i} + \pi\cos \pi t\mathbf{j}$

$\|\mathbf{r}'(t)\| = \pi\sqrt{4\sin^2 \pi t + \cos^2 \pi t}$

$$\mathbf{T}(t) = \frac{-2\sin \pi t\mathbf{i} + \cos \pi t\mathbf{j}}{\sqrt{4\sin^2 \pi t + \cos^2 \pi t}}$$

$$\mathbf{T}'(t) = \frac{-2\pi\cos \pi t\mathbf{i} - 4\pi\sin \pi t\mathbf{j}}{(4\sin^2 \pi t + \cos^2 \pi t)^{3/2}}$$

$$K = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\dfrac{2\pi}{4\sin^2 \pi t + \cos^2 \pi t}}{\pi\sqrt{4\sin^2 \pi t + \cos^2 \pi t}}$$

$$= \frac{2}{(4\sin^2 \pi t + \cos^2 \pi t)^{3/2}}$$

33. $\mathbf{r}(t) = a\cos \omega t\,\mathbf{i} + a\sin \omega t\,\mathbf{j}$

$\mathbf{r}'(t) = -a\omega\sin \omega t\,\mathbf{i} + a\omega\cos \omega t\,\mathbf{j}$

$\mathbf{T}(t) = -\sin \omega t\,\mathbf{i} + \cos \omega t\,\mathbf{j}$

$\mathbf{T}'(t) = -\omega\cos \omega t\,\mathbf{i} - \omega\sin \omega t\,\mathbf{j}$

$$K = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\omega}{a\omega} = \frac{1}{a}$$

34. $\mathbf{r}(t) = a\cos(\omega t)\mathbf{i} + b\sin(\omega t)\mathbf{j}$

$$\mathbf{r}'(t) = -a\omega\sin(\omega t)\mathbf{i} + b\omega\cos(\omega t)\mathbf{j}$$

$$\mathbf{T}(t) = \frac{-a\sin(\omega t)\mathbf{i} + b\cos(\omega t)\mathbf{j}}{\sqrt{a^2\sin^2(\omega t) + b^2\cos^2(\omega t)}}$$

$$\mathbf{T}'(t) = \frac{-ab^2\omega\cos(\omega t)\mathbf{i} - a^2b\omega\sin(\omega t)\mathbf{j}}{[a^2\sin^2(\omega t) + b^2\cos^2(\omega t)]^{3/2}}$$

$$K = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\dfrac{ab\omega}{a^2\sin^2(\omega t) + b^2\cos^2(\omega t)}}{\omega\sqrt{a^2\sin^2(\omega t) + b^2\cos^2(\omega t)}}$$

$$= \frac{ab}{[a^2\sin^2(\omega t) + b^2\cos^2(\omega t)]^{3/2}}$$

35. $\mathbf{r}(t) = \langle a(\omega t - \sin\omega t),\ a(1 - \cos\omega t)\rangle$

From Exercise 44, Section 12.4, we have:

$$\mathbf{a}\cdot\mathbf{N} = \frac{a\omega^2}{\sqrt{2}}\cdot\sqrt{1 - \cos\omega t}$$

$$K = \frac{\mathbf{a}(t)\cdot\mathbf{N}(t)}{\|\mathbf{v}(t)\|^2}$$

$$= \frac{\left(\dfrac{a\omega^2}{\sqrt{2}}\right)\sqrt{1 - \cos\omega t}}{2a^2\omega^2(1 - \cos\omega t)} = \frac{\sqrt{2}}{4a\sqrt{1 - \cos\omega t}}$$

36. $\mathbf{r}(t) = \langle\cos\omega t + \omega t\sin\omega t,\ \sin\omega t - \omega t\cos\omega t\rangle$

From Exercise 43, Section 12.4, we have:

$$\mathbf{a}\cdot\mathbf{N} = \omega^3 t$$

$$K = \frac{\mathbf{a}(t)\cdot\mathbf{N}(t)}{\|\mathbf{v}\|^2} = \frac{\omega^3 t}{\omega^4 t^2} = \frac{1}{\omega t}$$

37. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \dfrac{t^2}{2}\mathbf{k}$

$$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j} + t\mathbf{k}$$

$$\mathbf{T}(t) = \frac{\mathbf{i} + 2t\mathbf{j} + t\mathbf{k}}{\sqrt{1 + 5t^2}}$$

$$\mathbf{T}'(t) = \frac{-5t\mathbf{i} + 2\mathbf{j} + \mathbf{k}}{(1 + 5t^2)^{3/2}}$$

$$K = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|}$$

$$= \frac{\dfrac{\sqrt{5}}{(1 + 5t^2)}}{\sqrt{1 + 5t^2}} = \frac{\sqrt{5}}{(1 + 5t^2)^{3/2}}$$

38. $\mathbf{r}(t) = 2t^2\mathbf{i} + t\mathbf{j} + \dfrac{1}{2}t^2\mathbf{k}$

$$\mathbf{r}'(t) = 4t\mathbf{i} + \mathbf{j} + t\mathbf{k}$$

$$\mathbf{T}(t) = \frac{4t\mathbf{i} + \mathbf{j} + t\mathbf{k}}{\sqrt{1 + 17t^2}}$$

$$\mathbf{T}'(t) = \frac{4\mathbf{i} - 17t\mathbf{j} + \mathbf{k}}{(1 + 17t^2)^{3/2}}$$

$$K = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\sqrt{289t^2 + 17}}{(1 + 17t^2)^{3/2}}/(1 + 17t^2)^{1/2}$$

$$= \frac{\sqrt{17}}{(1 + 17t^2)^{3/2}}$$

39. $\mathbf{r}(t) = 4t\mathbf{i} + 3\cos t\mathbf{j} + 3\sin t\mathbf{k}$

$$\mathbf{r}'(t) = 4\mathbf{i} - 3\sin t\mathbf{j} + 3\cos t\mathbf{k}$$

$$\mathbf{T}(t) = \frac{1}{5}[4\mathbf{i} - 3\sin t\mathbf{j} + 3\cos t\mathbf{k}]$$

$$\mathbf{T}'(t) = \frac{1}{5}[-3\cos t\mathbf{j} - 3\sin t\mathbf{k}]$$

$$K = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{3/5}{5} = \frac{3}{25}$$

40. $\mathbf{r}(t) = e^t\cos t\mathbf{i} + e^t\sin t\mathbf{j} + e^t\mathbf{k}$

$$\mathbf{r}'(t) = (-e^t\sin t + e^t\cos t)\mathbf{i} + (e^t\cos t + e^t\sin t)\mathbf{j} + e^t\mathbf{k}$$

$$\mathbf{T}(t) = \frac{1}{\sqrt{3}}[(-\sin t + \cos t)\mathbf{i} + (\cos t + \sin t)\mathbf{j} + \mathbf{k}]$$

$$\mathbf{T}'(t) = \frac{1}{\sqrt{3}}[(-\cos t - \sin t)\mathbf{i} + (-\sin t + \cos t)\mathbf{j}]$$

$$K = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\left(1/\sqrt{3}\right)\sqrt{(-\cos t - \sin t)^2 + (-\sin t + \cos t)^2}}{\sqrt{3}e^t} = \frac{\sqrt{2}}{3e^t}$$

41. $y = 3x - 2$

Since $y'' = 0$, $K = 0$, and the radius of curvature is undefined.

42. $y = mx + b$

Since $y'' = 0$, $K = 0$, and the radius of curvature is undefined.

43. $y = 2x^2 + 3,\ x = -1$

$y' = 4x$

$y'' = 4$

$$K = \frac{4}{[1 + (-4)^2]^{3/2}} = \frac{4}{17^{3/2}} \approx 0.057$$

$$\frac{1}{K} = \frac{17^{3/2}}{4} \approx 17.523 \quad \text{(radius of curvature)}$$

44. $y = 2x + \frac{4}{x}, x = 1$

$$y' = 2 - \frac{4}{x^2}, y'(1) = -2$$

$$y'' = \frac{8}{x^3}, y''(1) = 8$$

$$K = \frac{|y''|}{[1 + (y')^2]^{3/2}} = \frac{8}{(1 + 4)^{3/2}} = \frac{8}{5^{3/2}}$$

$$\frac{1}{K} = \frac{5^{3/2}}{8} \quad \text{(radius of curvature)}$$

45. $y = \sqrt{a^2 - x^2},\ x = 0$

$$y' = \frac{-x}{\sqrt{a^2 - x^2}}$$

$$y'' = \frac{a^2}{(a^2 - x^2)^{3/2}}$$

At $x = 0$: $y' = 0$

$$y'' = \frac{1}{a}$$

$$K = \frac{1/a}{(1 + 0^2)^{3/2}} = \frac{1}{a}$$

$$\frac{1}{K} = a \quad \text{(radius of curvature)}$$

46. $y = \frac{3}{4}\sqrt{16 - x^2}$

$$y' = \frac{-9x}{16y}$$

$$y'' = \frac{-[9 + (16y')^2]}{16y}$$

At $x = 0$: $y' = 0$

$$y'' = -\frac{3}{16}$$

$$K = \left|\frac{-3/16}{(1 + 0^2)^{3/2}}\right| = \frac{3}{16}$$

$$\frac{1}{K} = \frac{16}{3} \quad \text{(radius of curvature)}$$

47. (a) Point on circle: $\left(\frac{\pi}{2}, 1\right)$

Center: $\left(\frac{\pi}{2}, 0\right)$

Equation: $\left(x - \frac{\pi}{2}\right)^2 + y^2 = 1$

(b) The circles have different radii since the curvature is different and

$$r = \frac{1}{K}.$$

48. (a) $y = \frac{4x^2}{x^2 + 3}$

$y' = \frac{24x}{(x^2 + 3)^2}$

$y'' = \frac{72(1 - x^2)}{(x^2 + 3)^3}$

At $x = 0$: $y' = 0$

$y'' = \frac{72}{27} = \frac{8}{3}$

$K = \frac{8/3}{(1 + 0^2)^{3/2}} = \frac{8}{3}$

$r = \frac{1}{K} = \frac{3}{8}$

Center: $\left(0, \frac{3}{8}\right)$

Equation: $x^2 + \left(y - \frac{3}{8}\right)^2 = \frac{9}{64}$

(b) The circles have different radii since the curvature is different and

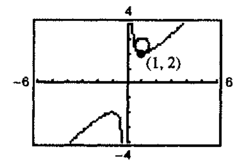
$r = \frac{1}{K}$.

49. $y = x + \frac{1}{x}, y' = 1 - \frac{1}{x^2}, y'' = \frac{2}{x^3}$

$K = \frac{2}{(1 + 0^2)^{3/2}} = 2$ at $(1, 2)$

Radius of curvature $= 1/2$. Since the tangent line is horizontal at $(1, 2)$, the normal line is vertical. The center of the circle is $1/2$ unit above the point $(1, 2)$ at $(1, 5/2)$.

Circle: $(x - 1)^2 + \left(y - \frac{5}{2}\right)^2 = \frac{1}{4}$



50. $y = \ln x, \quad x = 1$

$y' = \frac{1}{x}, \quad y'' = -\frac{1}{x^2}$

$y'(1) = 1, \quad y''(1) = -1$

$K = \frac{|-1|}{(1 + (1)^2)^{3/2}} = \frac{1}{2^{3/2}}, \; r = \frac{1}{K} = 2^{3/2} = 2\sqrt{2}$

The slope of the tangent line at $(1, 0)$ is $y'(1) = 1$.

The slope of the normal line is -1.

Equation of normal line: $y = -(x - 1) = -x + 1$

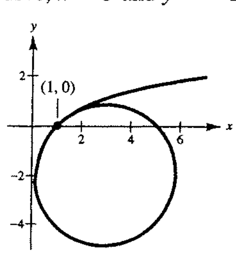
The center of the circle is on the normal line $2\sqrt{2}$ units away from the point $(1, 0)$.

$$\sqrt{(1 - x)^2 + (0 - y)^2} = 2\sqrt{2}$$
$$(1 - x)^2 + (x - 1)^2 = 8$$
$$2x^2 - 4x + 2 = 8$$
$$2(x^2 - 2x - 3) = 0$$
$$2(x - 3)(x + 1) = 0$$
$$x = 3 \text{ or } x = -1$$

Since the circle is below the curve, $x = 3$ and $y = -2$.

Center of circle: $(3, -2)$

Equation of circle:
$(x - 3)^2 + (y + 2)^2 = 8$



51. $y = e^x, \quad x = 0$

$y' = e^x, \quad y'' = e^x$

$y'(0) = 1, \quad y''(0) = 1$

$K = \frac{1}{(1 + 1^2)^{3/2}} = \frac{1}{2^{3/2}} = \frac{1}{2\sqrt{2}}, \; r = \frac{1}{K} = 2\sqrt{2}$

The slope of the tangent line at $(0, 1)$ is $y'(0) = 1$.

The slope of the normal line is -1.

Equation of normal line: $y - 1 = -x$ or $y = -x + 1$

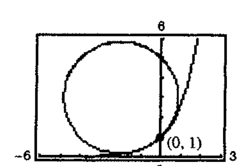
The center of the circle is on the normal line $2\sqrt{2}$ units away from the point $(0, 1)$.

$$\sqrt{(0 - x)^2 + (1 - y)^2} = 2\sqrt{2}$$
$$x^2 + x^2 = 8$$
$$x^2 = 4$$
$$x = \pm 2$$

Since the circle is above the curve, $x = -2$ and $y = 3$.

Center of circle: $(-2, 3)$

Equation of circle:
$(x + 2)^2 + (y - 3)^2 = 8$



52. $y = \frac{1}{3}x^3, \qquad x = 1$

$y' = x^2, \qquad y''(1) = 2x$

$y'(1) = 1, \qquad y''(1) = 2$

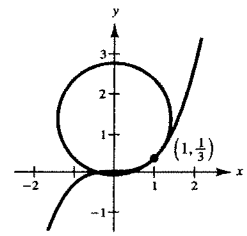
$K = \frac{2}{(1+1)^{3/2}} = \frac{1}{\sqrt{2}}, \; r = \frac{1}{K} = \sqrt{2}$

The slope of the tangent line at $\left(1, \frac{1}{3}\right)$ is $y'(1) = 1$.

The slope of the normal line is -1.

Equation of normal line: $y - \frac{1}{3} = -(x - 1)$ or $y = -x + \frac{4}{3}$

The center of the circle is on the normal line $\sqrt{2}$ units away from the point $\left(1, \frac{1}{3}\right)$.



$$\sqrt{(1-x)^2 + \left(\tfrac{1}{3} - y\right)^2} = \sqrt{2}$$

$$(1-x)^2 + (x-1)^2 = 2$$

$$(x-1)^2 = 1$$

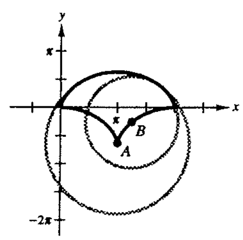
$$x = 0 \text{ or } x = 2$$

Since the circle is above the curve, $x = 0$ and $y = \frac{4}{3}$.

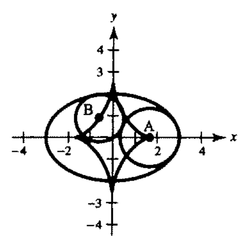
Center of circle: $\left(0, \frac{4}{3}\right)$

Equation of circle: $x^2 + \left(y - \frac{4}{3}\right)^2 = 2$

53.



54.



55. $y = (x-1)^2 + 3, \; y' = 2(x-1), \; y'' = 2$

$$K = \frac{2}{(1 + [2(x-1)]^2)^{3/2}} = \frac{2}{[1 + 4(x-1)^2]^{3/2}}$$

(a) K is maximum when $x = 1$ or at the vertex $(1, 3)$.

(b) $\lim_{x\to\infty} K = 0$

56. $y = x^3, \; y' = 3x^2, \; y'' = 6x$

$$K = \left|\frac{6x}{(1 + 9x^4)^{3/2}}\right|$$

(a) K is maximum at $\left(\frac{1}{\sqrt[4]{45}}, \frac{1}{\sqrt[4]{45^3}}\right), \left(\frac{-1}{\sqrt[4]{45}}, \frac{-1}{\sqrt[4]{45^3}}\right)$.

(b) $\lim_{x\to\infty} K = 0$

57. $y = x^{2/3}, \; y' = \frac{2}{3}x^{-1/3}, \; y'' = -\frac{2}{9}x^{-4/3}$

$$K = \left|\frac{(-2/9)x^{-4/3}}{[1 + (4/9)x^{-2/3}]^{3/2}}\right| = \left|\frac{6}{x^{1/3}(9x^{2/3} + 4)^{3/2}}\right|$$

(a) $K \implies \infty$ as $x \implies 0$. No maximum

(b) $\lim_{x\to\infty} K = 0$

58. $y = \frac{1}{x}, \; y' = -\frac{1}{x^2}, \; y'' = \frac{2}{x^3}.$ Assume $x > 0$.

$$K = \frac{|y''|}{[1 + (y')^2]^{3/2}} = \frac{|2/x^3|}{(1 + 1/x^4)^{3/2}} = \frac{2x^3}{(x^4 + 1)^{3/2}}$$

$$\frac{dK}{dx} = \frac{6x^2(1 - x^4)}{(x^4 + 1)^{5/2}}$$

(a) K has a maximum at $x = 1$ (and $x = -1$ by symmetry).

(b) $\lim_{x\to\infty} K = 0$

59. $y = \ln x,\ y' = \frac{1}{x},\ y'' = -\frac{1}{x^2}$

$$K = \left|\frac{-1/x^2}{[1 + (1/x)^2]^{3/2}}\right| = \frac{x}{(x^2+1)^{3/2}}$$

$$\frac{dK}{dx} = \frac{-2x^2+1}{(x^2+1)^{5/2}}$$

(a) K has a maximum when $x = \frac{1}{\sqrt{2}}$.

(b) $\lim_{x\to\infty} K = 0$

60. $y = e^x,\ y' = y'' = e^x$

$$K = \frac{|y''|}{[1 + (y')^2]^{3/2}} = \frac{e^x}{(1 + e^{2x})^{3/2}}$$

$$\frac{dK}{dx} = \frac{e^x(1 - 2e^{2x})}{(1 + e^{2x})^{5/2}}$$

(a) $1 - 2e^{2x} = 0 \Rightarrow e^{2x} = \frac{1}{2} \Rightarrow x = \frac{1}{2}\ln\left(\frac{1}{2}\right) = -\frac{1}{2}\ln 2$

K has maximum curvature at $x = -\frac{1}{2}\ln 2$.

(b) $\lim_{x\to\infty} K = 0$

61. $y = 1 - x^3,\ y' = -3x^2,\ y'' = -6x$

$$K = \frac{|-6x|}{[1 + 9x^4]^{3/2}}$$

Curvature is 0 at $x = 0$: $(0, 1)$.

62. $y = (x-1)^3 + 3,\ y' = 3(x-1)^2,\ y'' = 6(x-1)$

$$K = \frac{|y''|}{[1 + (y')^2]^{3/2}} = \frac{|6(x-1)|}{[1 + 9(x-1)^4]^{3/2}} = 0 \text{ at } x = 1.$$

Curvature is 0 at $(1, 3)$.

63. $y = \cos x,\ y' = -\sin x,\ y'' = -\cos x$

$$K = \frac{|y''|}{[1 + (y')^2]^{3/2}} = \frac{|-\cos x|}{(1 + \sin^2 x)^{3/2}} = 0 \text{ for } x = \frac{\pi}{2} + K\pi.$$

Curvature is 0 at $\left(\frac{\pi}{2} + K\pi, 0\right)$.

64. $y = \sin x,\ y' = \cos x,\ y'' = -\sin x$

$$K = \frac{|-\sin x|}{(1 + \cos^2 x)^{3/2}} = 0 \text{ for } x = n\pi.$$

Curvature is 0 for $x = n\pi$: $(n\pi, 0)$

65. The curve is a line.

66. $K = \frac{|y''|}{[1 + (y')^2]^{3/2}}$

At the smooth relative extremum $y' = 0$, so $K = |y''|$. Yes, for example, $y = x^4$ has a curvature of 0 at its relative minimum $(0, 0)$. The curvature is positive for any other point of the curvature.

67. Endpoints of the major axis: $(\pm 2, 0)$
Endpoints of the minor axis: $(0, \pm 1)$

$$x^2 + 4y^2 = 4$$

$$2x + 8yy' = 0$$

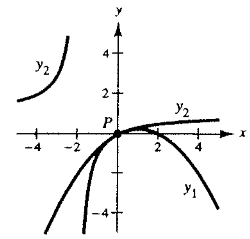
$$y' = -\frac{x}{4y}$$

$$y'' = \frac{(4y)(-1) - (-x)(4y')}{16y^2} = \frac{-4y - (x^2/y)}{16y^2} = \frac{-(4y^2 + x^2)}{16y^3} = \frac{-1}{4y^3}$$

$$K = \frac{|-1/4y^3|}{[1 + (-x/4y)^2]^{3/2}} = \frac{|-16|}{(16y^2 + x^2)^{3/2}} = \frac{16}{(12y^2 + 4)^{3/2}} = \frac{16}{(16 - 3x^2)^{3/2}}$$

Therefore, since $-2 \le x \le 2$, K is largest when $x = \pm 2$ and smallest when $x = 0$.

68. $y_1 = ax(b - x),\ y_2 = \dfrac{x}{x + 2}$



We observe that (0, 0) is a solution point to both equations. Therefore, the point P is the origin.

$$y_1 = ax(b - x), \quad y_1' = a(b - 2x), \quad y_1'' = -2a$$

$$y_2 = \frac{x}{x + 2}, \quad y_2' = \frac{2}{(x + 2)^2}, \quad y_2'' = \frac{-4}{(x + 2)^3}$$

At P,

$$y_1'(0) = ab \text{ and } y_2'(0) = \frac{2}{(0 + 2)^2} = \frac{1}{2}.$$

Since the curves have a common tangent at P, $y_1'(0) = y_2'(0)$ or $ab = \frac{1}{2}$. Therefore, $y_1'(0) = \frac{1}{2}$. Since the curves have the same curvature at P, $K_1(0) = K_2(0)$.

$$K_1(0) = \left|\frac{y_1''(0)}{[1 + (y_1'(0))^2]^{3/2}}\right| = \left|\frac{-2a}{[1 + (1/2)^2]^{3/2}}\right|$$

$$K_2(0) = \left|\frac{y_2''(0)}{[1 + (y_2'(0))^2]^{3/2}}\right| = \left|\frac{-1/2}{[1 + (1/2)^2]^{3/2}}\right|$$

Therefore, $2a = \pm\frac{1}{2}$ or $a = \pm\frac{1}{4}$. In order that the curves intersect at only one point, the parabola must be concave downward. Thus,

$$a = \frac{1}{4} \quad \text{and} \quad b = \frac{1}{2a} = 2.$$

$$y_1 = \frac{1}{4}x(2 - x) \quad \text{and} \quad y_2 = \frac{x}{x + 2}$$

69. $f(x) = x^4 - x^2$

(a) $K = \dfrac{2|6x^2 - 1|}{[16x^6 - 16x^4 + 4x^2 + 1]^{3/2}}$

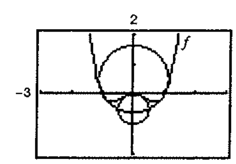
(b) For $x = 0$, $K = 2$. $f(0) = 0$. At (0, 0), the circle of curvature has radius $\frac{1}{2}$. Using the symmetry of the graph of f, you obtain

$$x^2 + \left(y + \frac{1}{2}\right)^2 = \frac{1}{4}.$$

For $x = 1$, $K = (2\sqrt{5})/5$. $f(1) = 0$. At (1, 0), the circle of curvature has radius

$$\frac{\sqrt{5}}{2} = \frac{1}{K}.$$

Using the graph of f, you see that the center of curvature is $\left(0, \frac{1}{2}\right)$. Thus,

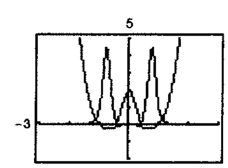


$$x^2 + \left(y - \frac{1}{2}\right)^2 = \frac{5}{4}.$$

To graph these circles, use

$$y = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - x^2} \quad \text{and} \quad y = \frac{1}{2} \pm \sqrt{\frac{5}{4} - x^2}.$$

(c) The curvature tends to be greatest near the extrema of f, and K decreases as $x \to \pm\infty$. However, f and K do not have the same critical numbers.

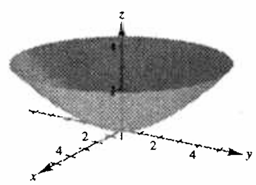


Critical numbers of f: $x = 0,\ \pm\dfrac{\sqrt{2}}{2} \approx \pm 0.7071$

Critical numbers of K: $x = 0,\ \pm 0.7647,\ \pm 0.4082$

70. $y = \frac{1}{4}x^{8/5}$, $0 \le x \le 5$

(a)



(rotated about y-axis)

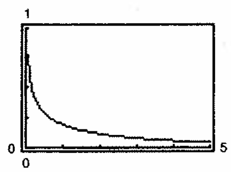
(b) $V = \int_0^5 2\pi x\left(\frac{5^{8/5}}{4} - \frac{x^{8/5}}{4}\right) dx$

$= \frac{125\pi 5^{3/5}}{9}$

$\approx 114.6 \text{ cm}^3$

(c) $y' = \frac{2}{5}x^{3/5}$, $y'' = \frac{6}{25}x^{-2/5} = \frac{6}{25x^{2/5}}$

$$K = \frac{\frac{6}{25x^{2/5}}}{\left[1 + \frac{4}{25}x^{6/5}\right]^{3/2}} = \frac{6}{25x^{2/5}\left[1 + \frac{4}{25}x^{6/5}\right]^{3/2}}$$



(d) No, the curvature approaches ∞ as $x \to 0^+$. Hence, any spherical object will hit the sides of the goblet before touching the bottom $(0, 0)$.

71. (a) Imagine dropping the circle $x^2 + (y - k)^2 = 16$ into the parabola $y = x^2$. The circle will drop to the point where the tangents to the circle and parabola are equal.

$y = x^2$ and $x^2 + (y - k)^2 = 16 \implies x^2 + (x^2 - k)^2 = 16$

Taking derivatives, $2x + 2(y - k)y' = 0$ and $y' = 2x$. Hence,

$$(y - k)y' = -x \implies y' = \frac{-x}{y - k}.$$

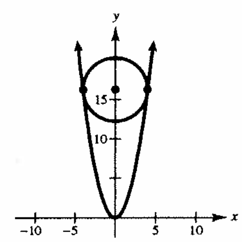
Thus,

$$\frac{-x}{y - k} = 2x \implies -x = 2x(y - k) \implies -1 = 2(x^2 - k) \implies x^2 - k = -\frac{1}{2}.$$

Thus,

$$x^2 + (x^2 - k)^2 = x^2 + \left(-\frac{1}{2}\right)^2 = 16 \implies x^2 = 15.75.$$

Finally, $k = x^2 + \frac{1}{2} = 16.25$, and the center of the circle is 16.25 units from the vertex of the parabola. Since the radius of the circle is 4, the circle is 12.25 units from the vertex.



(b) In 2-space, the parabola $z = y^2$ (or $z = x^2$) has a curvature of $K = 2$ at $(0, 0)$. The radius of the largest sphere that will touch the vertex has radius $= 1/K = \frac{1}{2}$.

72. $s = \frac{c}{\sqrt{K}}$

$y = \frac{1}{3}x^3$

$y' = x^2$

$y'' = 2x$

$K = \left|\frac{2x}{(1 + x^4)^{3/2}}\right|$

When $x = 1$: $K = \frac{1}{\sqrt{2}}$

$s = \frac{c}{\sqrt{1/\sqrt{2}}} = \sqrt[4]{2}c$

$30 = \sqrt[4]{2}c \implies c = \frac{30}{\sqrt[4]{2}}$

At $x = \frac{3}{2}$, $K = \frac{3}{[1 + (81/16)]^{3/2}} \approx 0.201$

$s = \left(\frac{3}{2}\right) = \frac{c}{\sqrt{K}} = \frac{30/\sqrt[4]{2}}{\sqrt{K}} \approx 56.27 \text{ mi/hr}.$

73. $P(x_0, y_0)$ point on curve $y = f(x)$. Let (α, β) be the center of curvature. The radius of curvature is $\frac{1}{K}$.

$y' = f'(x)$. Slope of normal line at (x_0, y_0) is $\frac{-1}{f'(x_0)}$.

Equation of normal line: $y - y_0 = \frac{-1}{f'(x_0)}(x - x_0)$

(α, β) is on the normal line: $-f'(x_0)(\beta - y_0) = \alpha - x_0$ Equation 1

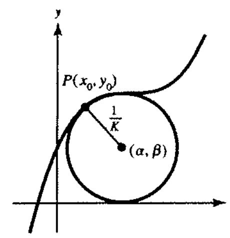
(x_0, y_0) lies on the circle: $(x_0 - \alpha)^2 + (y_0 - \beta)^2 = \left(\frac{1}{K}\right)^2 = \left[\frac{(1 + f'(x_0)^2)^{3/2}}{|f''(x_0)|}\right]^2$ Equation 2

Substituting Equation 1 into Equation 2:

$$[f'(x_0)(\beta - y_0)]^2 + (y_0 - \beta)^2 = \left(\frac{1}{K}\right)^2$$

$$(\beta - y_0)^2 + [1 + f'(x_0)^2] = \frac{(1 + f'(x_0)^2)^3}{(f''(x_0))^2}$$

$$(\beta - y_0)^2 = \frac{[1 + f'(x_0)^2]^2}{f''(x_0)^2}$$



When $f''(x_0) > 0$, $\beta - y_0 > 0$, and if $f''(x_0) < 0$, then $\beta - y_0 < 0$.

Hence

$$\beta - y_0 = \frac{1 + f'(x_0)^2}{f''(x_0)}$$

$$\beta = y_0 + \frac{1 + f'(x_0)^2}{f''(x_0)} = y_0 + z$$

Similarly, $\alpha = x_0 - f'(x_0)z$.

74. (a) $y = f(x) = e^x, f'(x) = f''(x) = e^x$, $(0, 1)$

$$z = \frac{1 + f'(0)^2}{f''(0)} = 2$$

$$(\alpha, \beta) = (0 - 2, 1 + 2) = (-2, 3)$$

(b) $y = \frac{x^2}{2}, y' = x, y'' = 1, \left(1, \frac{1}{2}\right)$

$$z = \frac{1 + f'(1)^2}{f''(1)} = 2$$

$$(\alpha, \beta) = \left(1 - 2, \frac{1}{2} + 2\right) = \left(-1, \frac{5}{2}\right)$$

(c) $y = x^2, y' = 2x, y'' = 2$, $(0, 0)$

$$z = \frac{1 + f'(0)^2}{f''(0)} = \frac{1}{2}$$

$$(\alpha, \beta) = \left(0, 0 + \frac{1}{2}\right) = \left(0, \frac{1}{2}\right)$$

75. $\mathbf{r}(\theta) = r\cos\theta\mathbf{i} + r\sin\theta\mathbf{j} = f(\theta)\cos\theta\mathbf{i} + f(\theta)\sin\theta\mathbf{j}$

$x(\theta) = f(\theta)\cos\theta$

$y(\theta) = f(\theta)\sin\theta$

$x'(\theta) = -f(\theta)\sin\theta + f'(\theta)\cos\theta$

$y'(\theta) = f(\theta)\cos\theta + f'(\theta)\sin\theta$

$x''(\theta) = -f(\theta)\cos\theta - f'(\theta)\sin\theta - f'(\theta)\sin\theta + f''(\theta)\cos\theta = -f(\theta)\cos\theta - 2f'(\theta)\sin\theta + f''(\theta)\cos\theta$

$y''(\theta) = -f(\theta)\sin\theta + f'(\theta)\cos\theta + f'(\theta)\cos\theta + f''(\theta)\sin\theta = -f(\theta)\sin\theta + 2f'(\theta)\cos\theta + f''(\theta)\sin\theta$

$$K = \frac{|x'y'' - y'x''|}{[(x')^2 + (y')^2]^{3/2}} = \frac{|f^2(\theta) - f(\theta)f''(\theta) + 2(f'(\theta))^2|}{[f^2(\theta) + (f'(\theta))^2]^{3/2}} = \frac{|r^2 - rr'' + 2(r')^2|}{[r^2 + (r')^2]^{3/2}}$$

76. (a) $r = 1 + \sin\theta$

$r' = \cos\theta$

$r'' = -\sin\theta$

$$K = \frac{|2(r')^2 - rr'' + r^2|}{[(r')^2 + r^2]^{3/2}}$$

$$= \frac{|2\cos^2\theta - (1 + \sin\theta)(-\sin\theta) + (1 + \sin\theta)^2|}{\sqrt{[\cos^2\theta + (1 + \sin\theta)^2]^3}}$$

$$= \frac{3(1 + \sin\theta)}{\sqrt{8(1 + \sin\theta)^3}} = \frac{3}{2\sqrt{2(1 + \sin\theta)}}$$

(b) $r = \theta$

$r' = 1$

$r'' = 0$

$$K = \frac{|2(r')^2 - rr'' + r^2|}{[(r')^2 + r^2]^{3/2}} = \frac{2 + \theta^2}{(1 + \theta^2)^{3/2}}$$

(c) $r = a\sin\theta$

$r' = a\cos\theta$

$r'' = -a\sin\theta$

$$K = \frac{|2(r')^2 - rr'' + r^2|}{[(r')^2 + r^2]^{3/2}}$$

$$= \frac{|2a^2\cos^2\theta + a^2\sin^2\theta + a^2\sin^2\theta|}{\sqrt{[a^2\cos^2\theta + a^2\sin^2\theta]^3}}$$

$$= \frac{2a^2}{a^3} = \frac{2}{a}, a > 0$$

(d) $r = e^\theta$

$r' = e^\theta$

$r'' = e^\theta$

$$K = \frac{|2(r')^2 - rr'' + r^2|}{[(r')^2 + r^2]^{3/2}} = \frac{2e^{2\theta}}{(2e^{2\theta})^{3/2}} = \frac{1}{\sqrt{2}e^\theta}$$

77. $r = e^{a\theta}, a > 0$

$r' = ae^{a\theta}$

$r'' = a^2e^{a\theta}$

$$K = \frac{|2(r')^2 - rr'' + r^2|}{[(r')^2 + r^2]^{3/2}} = \frac{|2a^2e^{2a\theta} - a^2e^{2a\theta} + e^{2a\theta}|}{[a^2e^{2a\theta} + e^{2a\theta}]^{3/2}}$$

$$= \frac{1}{e^{a\theta}\sqrt{a^2 + 1}}$$

(a) As $\theta \Rightarrow \infty$, $K \Rightarrow 0$.

(b) As $a \Rightarrow \infty$, $K \Rightarrow 0$.

78. At the pole, $r = 0$.

$$K = \frac{|2(r')^2 - rr'' + r^2|}{[(r')^2 + r^2]^{3/2}}$$

$$= \frac{|2(r')^2|}{|r'|^3} = \frac{2}{|r'|}$$

79. $r = 4\sin 2\theta$

$r' = 8\cos 2\theta$

At the pole: $K = \dfrac{2}{|r'(0)|} = \dfrac{2}{8} = \dfrac{1}{4}$

80. $r = 6\cos 3\theta$

$r' = -18\sin 3\theta$

At the pole,

$$\theta = \frac{\pi}{6},\ r'\left(\frac{\pi}{6}\right) = -18,$$

and

$$K = \frac{2}{|r'(\pi/6)|} = \frac{2}{|-18|} = \frac{1}{9}.$$

81. $x = f(t), y = g(t)$

$$y' = \frac{dy}{dx} = \frac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}} = \frac{g'(t)}{f'(t)}$$

$$y'' = \frac{\dfrac{d}{dt}\left[\dfrac{g'(t)}{f'(t)}\right]}{\dfrac{dx}{dt}} = \frac{\dfrac{f'(t)g''(t) - g'(t)f''(t)}{[f'(t)]^2}}{f'(t)} = \frac{f'(t)g''(t) - g'(t)f''(t)}{[f'(t)]^3}$$

$$K = \frac{|y''|}{[1 + (y')^2]^{3/2}} = \frac{\left|\dfrac{f'(t)g''(t) - g'(t)f''(t)}{[f'(t)]^3}\right|}{\left[1 + \left(\dfrac{g'(t)}{f'(t)}\right)^2\right]^{3/2}} = \frac{\left|\dfrac{f'(t)g''(t) - g'(t)f''(t)}{[f'(t)]^3}\right|}{\sqrt{\left\{\dfrac{[f'(t)]^2 + [g'(t)]^2}{[f'(t)]^2}\right\}^3}} = \frac{|f'(t)g''(t) - g'(t)f''(t)|}{([f'(t)]^2 + [g'(t)]^2)^{3/2}}$$

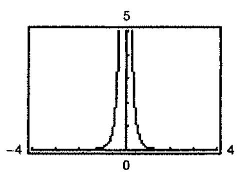
82. $x(t) = t^3,\ x'(t) = 3t^2,\ x''(t) = 6t$

$y(t) = \frac{1}{2}t^2,\ y'(t) = t,\ y''(t) = 1$

$$K = \frac{|(3t^2)(1) - (t)(6t)|}{[(3t^2)^2 + (t)^2]^{3/2}}$$

$$= \frac{3t^2}{|t^3|(9t^2 + 1)^{3/2}} = \frac{3}{|t|(9t^2 + 1)^{3/2}}$$

$K \to 0$ as $t \to \pm\infty$



83. $x(\theta) = a(\theta - \sin\theta) \qquad y(\theta) = a(1 - \cos\theta)$

$x'(\theta) = a(1 - \cos\theta) \qquad y'(\theta) = a\sin\theta$

$x''(\theta) = a\sin\theta \qquad y''(\theta) = a\cos\theta$

$$K = \frac{|x'(\theta)y''(\theta) - y'(\theta)x''(\theta)|}{[x'(\theta)^2 + y'(\theta)^2]^{3/2}}$$

$$= \frac{|a^2(1 - \cos\theta)\cos\theta - a^2\sin^2\theta|}{[a^2(1 - \cos\theta)^2 + a^2\sin^2\theta]^{3/2}}$$

$$= \frac{1}{a}\frac{|\cos\theta - 1|}{[2 - 2\cos\theta]^{3/2}}$$

$$= \frac{1}{a}\frac{1 - \cos\theta}{2\sqrt{2}[1 - \cos\theta]^{3/2}} \qquad (1 - \cos \geq 0)$$

$$= \frac{1}{2a\sqrt{2 - 2\cos\theta}} = \frac{1}{4a}\csc\left(\frac{\theta}{2}\right)$$

Minimum: $\frac{1}{4a}$ $\qquad (\theta = \pi)$

Maximum: none $\qquad (K \to \infty$ as $\theta \to 0)$

84. (a) $\mathbf{r}(t) = 3t^2\mathbf{i} + (3t - t^3)\mathbf{j}$

$\mathbf{v}(t) = 6t\mathbf{i} + (3 - 3t^2)\mathbf{j}$

$\frac{ds}{dt} = \|\mathbf{v}(t)\| = 3(1 + t^2),\ \frac{d^2s}{dt^2} = 6t$

$$K = \frac{2}{3(1 + t^2)^2}$$

$$a_{\mathbf{T}} = \frac{d^2s}{dt^2} = 6t$$

$$a_{\mathbf{N}} = K\left(\frac{ds}{dt}\right)^2 = \frac{2}{3(1 + t^2)^2} \cdot 9(1 + t^2)^2 = 6$$

(b) $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{1}{2}t^2\mathbf{k}$

$\mathbf{v}(t) = \mathbf{i} + 2t\mathbf{j} + t\mathbf{k}$

$$\frac{ds}{dt} = \|\mathbf{v}(t)\| = \sqrt{5t^2 + 1}$$

$$\frac{d^2s}{dt^2} = \frac{5t}{\sqrt{5t^2 + 1}}$$

$\mathbf{a}(t) = 2\mathbf{j} + \mathbf{k}$

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \mathbf{v}(t) \times \mathbf{a}(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2t & t \\ 0 & 2 & 1 \end{vmatrix} = -\mathbf{j} + 2\mathbf{k}$$

$$K = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{\sqrt{5}}{(5t^2 + 1)^{3/2}}$$

$$a_{\mathbf{T}} = \frac{d^2s}{dt^2} = \frac{5t}{\sqrt{5t^2 + 1}}$$

$$a_{\mathbf{N}} = K\left(\frac{ds}{dt}\right)^2 = \frac{\sqrt{5}}{(5t^2 + 1)^{3/2}}(5t^2 + 1) = \frac{\sqrt{5}}{\sqrt{5t^2 + 1}}$$

85. $F = ma_{\mathbf{N}} = mK\left(\frac{ds}{dt}\right)^2 = \left(\frac{5500\text{ lb}}{32\text{ ft/sec}^2}\right)\left(\frac{1}{100\text{ ft}}\right)\left(\frac{30(5280)\text{ ft}}{3600\text{ sec}}\right)^2 = 3327.5\text{ lbs}$

86. $F = ma_{\mathbf{N}} = mK\left(\frac{ds}{dt}\right)^2 = \left(\frac{6400\text{ lb}}{32\text{ ft/sec}^2}\right)\left(\frac{1}{250\text{ ft}}\right)\left(\frac{35(5280)\text{ ft}}{3600\text{ sec}}\right)^2 = \frac{94864}{45} \approx 2108.1\text{ lbs}$

87. $y = \cosh x = \frac{e^x + e^{-x}}{2}$

$$y' = \frac{e^x - e^{-x}}{2} = \sinh x$$

$$y'' = \frac{e^x + e^{-x}}{2} = \cosh x$$

$$K = \frac{|\cosh x|}{[1 + (\sinh x)^2]^{3/2}} = \frac{\cosh x}{(\cosh^2 x)^{3/2}} = \frac{1}{\cosh^2 x} = \frac{1}{y^2}$$

88. (a) $K = \|\mathbf{T}'(s)\| = \left\|\frac{d\mathbf{T}}{ds}\right\| = \left\|\frac{d\mathbf{T}}{dt} \cdot \frac{dt}{ds}\right\|$, by the Chain Rule

$$= \left\|\frac{d\mathbf{T}/dt}{ds/dt}\right\| = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{v}(t)\|} = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|}$$

(b) $$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{\mathbf{r}'(t)}{ds/dt}$$

$$\mathbf{r}'(t) = \frac{ds}{dt}\mathbf{T}(t)$$

$$\mathbf{r}''(t) = \left(\frac{d^2s}{dt^2}\right)\mathbf{T}(t) + \frac{ds}{dt}\mathbf{T}'(t)$$

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \left(\frac{ds}{dt}\right)\left(\frac{d^2s}{dt^2}\right)[\mathbf{T}(t) \times \mathbf{T}(t)] + \left(\frac{ds}{dt}\right)^2[\mathbf{T}(t) \times \mathbf{T}'(t)]$$

Since $\mathbf{T}(t) \times \mathbf{T}(t) = \mathbf{0}$ and $\frac{ds}{dt} = \|\mathbf{r}'(t)\|$, we have:

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \|\mathbf{r}'(t)\|^2[\mathbf{T}(t) \times \mathbf{T}'(t)]$$

$$\|\mathbf{r}'(t) \times \mathbf{r}''(t)\| = \|\mathbf{r}'(t)\|^2\|\mathbf{T}(t) \times \mathbf{T}'(t)\| = \|\mathbf{r}'(t)\|^2\|\mathbf{T}(t)\|\,\|\mathbf{T}'(t)\| = \|\mathbf{r}'(t)\|^2(1)K\|\mathbf{r}'(t)\| \quad \text{from (a)}$$

Therefore, $\frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = K.$

(c) $$K = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)^3\|} = \frac{\frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|}}{\|\mathbf{r}'(t)\|^2} = \frac{\frac{\|\mathbf{v}(t) \times \mathbf{a}(t)\|}{\|\mathbf{v}(t)\|}}{\|\mathbf{r}'(t)\|^2} = \frac{\mathbf{a}(t) \cdot \mathbf{N}(t)}{\|\mathbf{r}'(t)\|^2}$$

89. False

90. False

Curvature $= \frac{1}{\text{radius}}$

91. True

92. True

$a_{\mathrm{N}} = K\left(\frac{ds}{dt}\right)^2$

93. Let $\mathbf{r} = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$. Then $r = \|\mathbf{r}\| = \sqrt{[x(t)]^2 + [y(t)]^2 + [z(t)]^2}$ and $r' = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}$. Then,

$$r\left(\frac{dr}{dt}\right) = \sqrt{[x(t)]^2 + [y(t)]^2 + [z(t)]^2}\left[\frac{1}{2}\{[x(t)]^2 + [y(t)]^2 + [z(t)]^2\}^{-1/2} \cdot (2x(t)x'(t) + 2y(t)y'(t) + 2z(t)z'(t))\right]$$

$$= x(t)x'(t) + y(t)y'(t) + z(t)z'(t) = \mathbf{r} \cdot \mathbf{r}'.$$

94. $\mathbf{F} = m\mathbf{a} \implies m\mathbf{a} = \frac{-GmM}{r^3}\mathbf{r}$

$$\mathbf{a} = -\frac{GM}{r^3}\mathbf{r}$$

Since $\mathbf{r}$ is a constant multiple of $\mathbf{a}$, they are parallel. Since $\mathbf{a} = \mathbf{r}''$ is parallel to $\mathbf{r}$, $\mathbf{r} \times \mathbf{r}'' = \mathbf{0}$. Also,

$$\left(\frac{d}{dt}\right)(\mathbf{r} \times \mathbf{r}') = \mathbf{r}' \times \mathbf{r}' + \mathbf{r} \times \mathbf{r}'' = \mathbf{0} + \mathbf{0} = \mathbf{0}.$$

Thus, $\mathbf{r} \times \mathbf{r}'$ is a constant vector which we will denote by $\mathbf{L}$.

95. Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ where x, y, and z are functions of t, and $r = \|\mathbf{r}\|$.

$$\frac{d}{dt}\left[\frac{\mathbf{r}}{r}\right] = \frac{r\mathbf{r}' - \mathbf{r}(dr/dt)}{r^2} = \frac{r\mathbf{r}' - \mathbf{r}[(\mathbf{r}\cdot\mathbf{r}')/r]}{r^2} = \frac{r^2\mathbf{r}' - (\mathbf{r}\cdot\mathbf{r}')\mathbf{r}}{r^3} \quad \text{(using Exercise 93)}$$

$$= \frac{(x^2+y^2+z^2)(x'\mathbf{i}+y'\mathbf{j}+z'\mathbf{k}) - (xx'+yy'+zz')(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})}{r^3}$$

$$= \frac{1}{r^3}[(x'y^2 + x'z^2 - xyy' - xzz')\mathbf{i} + (x^2y' + z^2y' - xx'y - zz'y)\mathbf{j} + (x^2z' + y^2z' - xx'z - yy'z)\mathbf{k}]$$

$$= \frac{1}{r^3}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ yz' - y'z & -(xz' - x'z) & xy' - x'y \\ x & y & z \end{vmatrix} = \frac{1}{r^3}\{[\mathbf{r}\times\mathbf{r}']\times\mathbf{r}\}$$

96. $$\frac{d}{dt}\left[\frac{\mathbf{r}'}{GM}\times\mathbf{L} - \frac{\mathbf{r}}{r}\right] = \frac{1}{GM}[\mathbf{r}'\times\mathbf{0} + \mathbf{r}''\times\mathbf{L}] - \frac{1}{r^3}\{[\mathbf{r}\times\mathbf{r}']\times\mathbf{r}\}$$

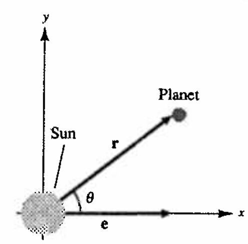
$$= \frac{1}{GM}\left[\mathbf{0} + \left(\frac{-GM\mathbf{r}}{r^3}\right)\times[\mathbf{r}\times\mathbf{r}']\right] - \frac{1}{r^3}\{[\mathbf{r}\times\mathbf{r}']\times\mathbf{r}\}$$

$$= -\frac{\mathbf{r}}{r^3}\times[\mathbf{r}\times\mathbf{r}'] - \frac{1}{r^3}\{[\mathbf{r}\times\mathbf{r}']\times\mathbf{r}\}$$

$$= \frac{1}{r^3}\{[\mathbf{r}\times\mathbf{r}']\times\mathbf{r} - [\mathbf{r}\times\mathbf{r}']\times\mathbf{r}\} = \mathbf{0}$$

Thus, $\left(\frac{\mathbf{r}'}{GM}\right)\times\mathbf{L} - \left(\frac{\mathbf{r}}{r}\right)$ is a constant vector which we will denote by $\mathbf{e}$.

97. From Exercise 94, we have concluded that planetary motion is planar. Assume that the planet moves in the xy-plane with the sun at the origin. From Exercise 96, we have



$$\mathbf{r}'\times\mathbf{L} = GM\left(\frac{\mathbf{r}}{r} + \mathbf{e}\right).$$

Since $\mathbf{r}'\times\mathbf{L}$ and $\mathbf{r}$ are both perpendicular to $\mathbf{L}$, so is $\mathbf{e}$. Thus, $\mathbf{e}$ lies in the xy-plane. Situate the coordinate system so that $\mathbf{e}$ lies along the positive x-axis and θ is the angle between $\mathbf{e}$ and $\mathbf{r}$. Let $e = \|\mathbf{e}\|$. Then $\mathbf{r}\cdot\mathbf{e} = \|\mathbf{r}\|\,\|\mathbf{e}\|\cos\theta = re\cos\theta$. Also,

$$\|\mathbf{L}\|^2 = \mathbf{L}\cdot\mathbf{L} = (\mathbf{r}\times\mathbf{r}')\cdot\mathbf{L}$$

$$= \mathbf{r}\cdot(\mathbf{r}'\times\mathbf{L}) = \mathbf{r}\cdot\left[GM\left(\mathbf{e} + \frac{\mathbf{r}}{r}\right)\right] = GM\left[\mathbf{r}\cdot\mathbf{e} + \frac{\mathbf{r}\cdot\mathbf{r}}{r}\right] = GM[re\cos\theta + r].$$

Thus,

$$\frac{\|\mathbf{L}\|^2/GM}{1 + e\cos\theta} = r$$

and the planetary motion is a conic section. Since the planet returns to its initial position periodically, the conic is an ellipse.

98. $\|\mathbf{L}\| = \|\mathbf{r}\times\mathbf{r}'\|$

Let: $\mathbf{r} = r(\cos\theta\mathbf{i} + \sin\theta\mathbf{j})$

$$\mathbf{r}' = r(-\sin\theta\mathbf{i} + \cos\theta\mathbf{j})\frac{d\theta}{dt} \quad \left(\frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{d\theta}\cdot\frac{d\theta}{dt}\right)$$

Then: $$\mathbf{r}\times\mathbf{r}' = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ r\cos\theta & r\sin\theta & 0 \\ -r\sin\theta\,\frac{d\theta}{dt} & r\cos\theta\,\frac{d\theta}{dt} & 0 \end{vmatrix}$$

$$= r^2\frac{d\theta}{dt}\mathbf{k} \text{ and } \|\mathbf{L}\| = \|\mathbf{r}\times\mathbf{r}'\| = r^2\frac{d\theta}{dt}.$$

99. $A = \frac{1}{2}\int_\alpha^\beta r^2\,d\theta$

Thus,

$$\frac{dA}{dt} = \frac{dA}{d\theta}\,\frac{d\theta}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt} = \frac{1}{2}\|\mathbf{L}\|$$

and $\mathbf{r}$ sweeps out area at a constant rate.

100. Let P denote the period. Then

$$A = \int_0^P \frac{dA}{dt}\,dt = \frac{1}{2}\|\mathbf{L}\|P.$$

Also, the area of an ellipse is πab where $2a$ and $2b$ are the lengths of the major and minor axes.

$$\pi ab = \frac{1}{2}\|\mathbf{L}\|P$$

$$P = \frac{2\pi ab}{\|\mathbf{L}\|}$$

$$P^2 = \frac{4\pi^2 a^2}{\|\mathbf{L}\|^2}(a^2 - c^2) = \frac{4\pi^2 a^2}{\|\mathbf{L}\|^2}a^2(1 - e^2)$$

$$= \frac{4\pi^2 a^4}{\|\mathbf{L}\|^2}\left(\frac{ed}{a}\right) = \frac{4\pi^2 ed}{\|\mathbf{L}\|^2}a^3$$

$$= \frac{4\pi^2(\|\mathbf{L}\|^2/GM)}{\|\mathbf{L}\|^2}a^3 = \frac{4\pi^2}{GM}a^3 = Ka^3$$

Review Exercises for Chapter 12

1. $\mathbf{r}(t) = t\mathbf{i} + \csc t\mathbf{k}$

(a) Domain: $t \neq n\pi$, n an integer

(b) Continuous except at $t = n\pi$, n an integer

2. $\mathbf{r}(t) = \sqrt{t}\mathbf{i} + \dfrac{1}{t-4}\mathbf{j} + \mathbf{k}$

(a) Domain: $[0, 4)$ and $(4, \infty)$

(b) Continuous except at $t = 4$

3. $\mathbf{r}(t) = \ln t\mathbf{i} + t\mathbf{j} + t\mathbf{k}$

(a) Domain: $(0, \infty)$

(b) Continuous for all $t > 0$

4. $\mathbf{r}(t) = (2t + 1)\mathbf{i} + t^2\mathbf{j} + t\mathbf{k}$

(a) Domain: $(-\infty, \infty)$

(b) Continuous for all t

5. (a) $\mathbf{r}(0) = \mathbf{i}$

(b) $\mathbf{r}(-2) = -3\mathbf{i} + 4\mathbf{j} + \frac{8}{3}\mathbf{k}$

(c) $\mathbf{r}(c-1) = (2(c-1)+1)\mathbf{i} + (c-1)^2\mathbf{j} - \frac{1}{3}(c-1)^3\mathbf{k}$
$= (2c-1)\mathbf{i} + (c-1)^2\mathbf{j} - \frac{1}{3}(c-1)^3\mathbf{k}$

(d) $\mathbf{r}(1+\Delta t) - \mathbf{r}(1) = ([2(1+\Delta t)+1]\mathbf{i} + [1+\Delta t]^2\mathbf{j} - \frac{1}{3}[1+\Delta t]^3\mathbf{k}) - (3\mathbf{i} + \mathbf{j} - \frac{1}{3}\mathbf{k})$
$= 2\Delta t\mathbf{i} + \Delta t(\Delta t + 2)\mathbf{j} - \frac{1}{3}(\Delta t^3 + 3\Delta t^2 + 3\Delta t)\mathbf{k}$

6. (a) $\mathbf{r}(0) = 3\mathbf{i} + \mathbf{j}$

(b) $\mathbf{r}\left(\frac{\pi}{2}\right) = -\frac{\pi}{2}\mathbf{k}$

(c) $\mathbf{r}(s-\pi) = 3\cos(s-\pi)\mathbf{i} + (1 - \sin(s-\pi))\mathbf{j} - (s-\pi)\mathbf{k}$

(d) $\mathbf{r}(\pi+\Delta t) - \mathbf{r}(\pi) = (3\cos(\pi+\Delta t)\mathbf{i} + (1 - \sin(\pi+\Delta t))\mathbf{j} - (\pi+\Delta t)\mathbf{k}) - (-3\mathbf{i} + \mathbf{j} - \pi\mathbf{k})$
$= (-3\cos\Delta t + 3)\mathbf{i} + \sin\Delta t - \Delta t\mathbf{k}$

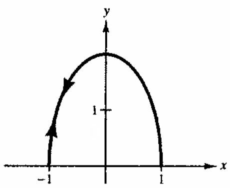
7. $\mathbf{r}(t) = \cos t\mathbf{i} + 2\sin^2 t\mathbf{j}$

$x(t) = \cos t,\ y(t) = 2\sin^2 t$

$x^2 + \frac{y}{2} = 1$

$y = 2(1 - x^2)$

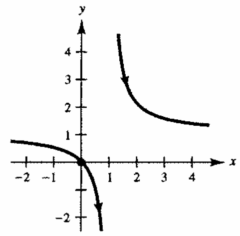
$-1 \le x \le 1$



8. $\mathbf{r}(t) = t\mathbf{i} + \frac{t}{t-1}\mathbf{j}$

$x(t) = t,\ y(t) = \frac{t}{t-1}$

$y = \frac{x}{x-1}$

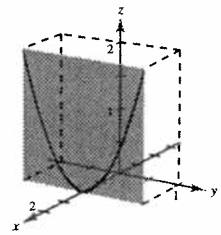


9. $\mathbf{r}(t) = \mathbf{i} + t\mathbf{j} + t^2\mathbf{k}$

$x = 1$

$y = t$

$z = t^2 \implies z = y^2$

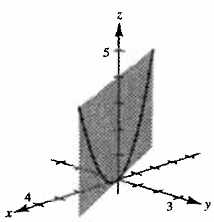


10. $\mathbf{r}(t) = 2t\mathbf{i} + t\mathbf{j} + t^2\mathbf{k}$

$x = 2t,\ y = t,\ z = t^2,$

$y = \frac{1}{2}x,\ z = y^2$

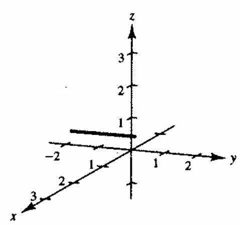
t	0	1	−1	2
x	0	2	−2	4
y	0	1	−1	2
z	0	1	1	4



11. $\mathbf{r}(t) = \mathbf{i} + \sin t\mathbf{j} + \mathbf{k}$

$x = 1,\ y = \sin t,\ z = 1$

t	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
x	1	1	1	1
y	0	1	0	−1
z	1	1	1	1

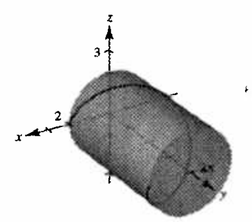


12. $\mathbf{r}(t) = 2\cos t\mathbf{i} + t\mathbf{j} + 2\sin t\mathbf{k}$

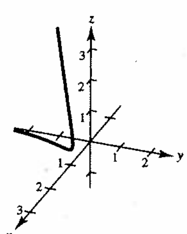
$x = 2\cos t,\ y = t,\ z = 2\sin t$

$x^2 + z^2 = 4$

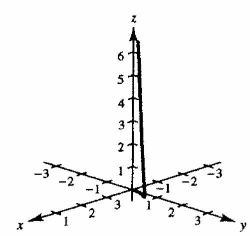
t	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
x	2	0	−2	0
y	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
z	0	2	0	−2



13. $\mathbf{r}(t) = t\mathbf{i} + \ln t\mathbf{j} + \frac{1}{2}t^2\mathbf{k}$



14. $\mathbf{r}(t) = \frac{1}{2}t\mathbf{i} + \sqrt{t}\mathbf{j} + \frac{1}{4}t^3\mathbf{k}$



15. One possible answer is:

$\mathbf{r}_1(t) = 4t\mathbf{i} + 3t\mathbf{j}, \quad 0 \le t \le 1$

$\mathbf{r}_2(t) = 4\mathbf{i} + (3 - t)\mathbf{j}, \quad 0 \le t \le 3$

$\mathbf{r}_3(t) = (4 - t)\mathbf{i}, \quad 0 \le t \le 4$

16. One possible answer is:

$\mathbf{r}_1(t) = 4t\mathbf{i}, \quad 0 \le t \le 1$

$\mathbf{r}_2(t) = 4\cos t\mathbf{i} + 4\sin t\mathbf{j}, \quad 0 \le t \le \frac{\pi}{2}$

$\mathbf{r}_3(t) = (4 - t)\mathbf{j}, \quad 0 \le t \le 4$

17. The vector joining the points is $\langle 7, 4, -10\rangle$. One path is

$\mathbf{r}(t) = \langle -2 + 7t, -3 + 4t, 8 - 10t\rangle.$

18. The x- and y-components are $2 \cos t$ and $2 \sin t$. At

$$t = \frac{3\pi}{2},$$

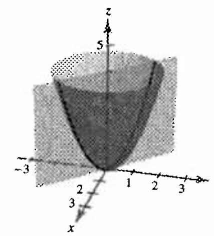
the staircase has made $\frac{3}{4}$ of a revolution and is 2 meters high. Thus, one answer is

$$\mathbf{r}(t) = 2 \cos t\mathbf{i} + 2 \sin t\mathbf{j} + \frac{4}{3\pi}t\mathbf{k}.$$

19. $z = x^2 + y^2,\ x + y = 0,\ t = x$

$x = t,\ y = -t,\ z = 2t^2$

$\mathbf{r}(t) = t\mathbf{i} - t\mathbf{j} + 2t^2\mathbf{k}$

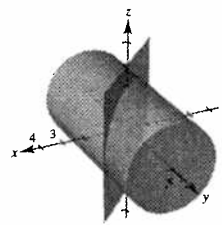


20. $x^2 + z^2 = 4,\ x - y = 0,\ t = x$

$x = t,\ y = t,\ z = \pm\sqrt{4 - t^2}$

$\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + \sqrt{4 - t^2}\mathbf{k}$

$\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} - \sqrt{4 - t^2}\mathbf{k}$



21. $\lim_{t\to 2^-} \left(t^2\mathbf{i} + \sqrt{4 - t^2}\mathbf{j} + \mathbf{k}\right) = 4\mathbf{i} + \mathbf{k}$

22. $\lim_{t\to 0} \left(\frac{\sin 2t}{t}\mathbf{i} + e^{-t}\mathbf{j} + e^t\mathbf{k}\right) = \left(\lim_{t\to 0} \frac{2 \cos 2t}{1}\right)\mathbf{i} + \mathbf{j} + \mathbf{k} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$

23. $\mathbf{r}(t) = 3t\mathbf{i} + (t - 1)\mathbf{j},\ \mathbf{u}(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{2}{3}t^3\mathbf{k}$

(a) $\mathbf{r}'(t) = 3\mathbf{i} + \mathbf{j}$

(b) $\mathbf{r}''(t) = \mathbf{0}$

(c) $\mathbf{r}(t) \cdot \mathbf{u}(t) = 3t^2 + t^2(t - 1) = t^3 + 2t^2$

$D_t[\mathbf{r}(t) \cdot \mathbf{u}(t)] = 3t^2 + 4t$

(d) $\mathbf{u}(t) - 2\mathbf{r}(t) = -5t\mathbf{i} + (t^2 - 2t + 2)\mathbf{j} + \frac{2}{3}t^3\mathbf{k}$

$D_t[\mathbf{u}(t) - 2\mathbf{r}(t)] = -5\mathbf{i} + (2t - 2)\mathbf{j} + 2t^2\mathbf{k}$

(e) $\|\mathbf{r}(t)\| = \sqrt{10t^2 - 2t + 1}$

$D_t[\|\mathbf{r}(t)\|] = \dfrac{10t - 1}{\sqrt{10t^2 - 2t + 1}}$

(f) $\mathbf{r}(t) \times \mathbf{u}(t) = \frac{2}{3}(t^4 - t^3)\mathbf{i} - 2t^4\mathbf{j} + (3t^3 - t^2 + t)\mathbf{k}$

$D_t[\mathbf{r}(t) \times \mathbf{u}(t)] = \left(\frac{8}{3}t^3 - 2t^2\right)\mathbf{i} - 8t^3\mathbf{j} + (9t^2 - 2t + 1)\mathbf{k}$

24. $\mathbf{r}(t) = \sin t\mathbf{i} + \cos t\mathbf{j} + t\mathbf{k},\ \mathbf{u}(t) = \sin t\mathbf{i} + \cos t\mathbf{j} + \frac{1}{t}\mathbf{k}$

(a) $\mathbf{r}'(t) = \cos t\mathbf{i} - \sin t\mathbf{j} + \mathbf{k}$

(b) $\mathbf{r}''(t) = -\sin t\mathbf{i} - \cos t\mathbf{j}$

(c) $\mathbf{r}(t) \cdot \mathbf{u}(t) = 2$

$D_t[\mathbf{r}(t) \cdot \mathbf{u}(t)] = 0$

(d) $\mathbf{u}(t) - 2\mathbf{r}(t) = -\sin t\mathbf{i} - \cos t\mathbf{j} + \left(\frac{1}{t} - 2t\right)\mathbf{k}$

$D_t[\mathbf{u}(t) - 2\mathbf{r}(t)] = -\cos t\mathbf{i} + \sin t\mathbf{j} + \left(-\frac{1}{t^2} - 2\right)\mathbf{k}$

(e) $\|\mathbf{r}(t)\| = \sqrt{1 + t^2}$

$D_t[\|\mathbf{r}(t)\|] = \dfrac{t}{\sqrt{1 + t^2}}$

(f) $\mathbf{r}(t) \times \mathbf{u}(t) = \left(\frac{1}{t}\cos t - t \cos t\right)\mathbf{i} - \left(\frac{1}{t}\sin t - t \sin t\right)\mathbf{j}$

$$D_t[\mathbf{r}(t) \times \mathbf{u}(t)] = \left(-\frac{1}{t}\sin t - \frac{1}{t^2}\cos t + t \sin t - \cos t\right)\mathbf{i} - \left(\frac{1}{t}\cos t - \frac{1}{t^2}\sin t - t \cos t - \sin t\right)\mathbf{j}$$

25. $x(t)$ and $y(t)$ are increasing functions at $t = t_0$, and $z(t)$ is a decreasing function at $t = t_0$.

26. The graph of $\mathbf{u}$ is parallel to the yz-plane.

27. $\int(\cos t\mathbf{i} + t\cos t\mathbf{j})\,dt = \sin t\mathbf{i} + (t\sin t + \cos t)\mathbf{j} + \mathbf{C}$

28. $\int(\ln t\mathbf{i} + t\ln t\mathbf{j} + \mathbf{k})\,dt = (t\ln t - t)\mathbf{i} + \frac{t^2}{4}(-1 + 2\ln t)\mathbf{j} + t\mathbf{k} + \mathbf{C}$

29. $\int\|\cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}\|\,dt = \int\sqrt{1 + t^2}\,dt = \frac{1}{2}\left[t\sqrt{1 + t^2} + \ln\left|t + \sqrt{1 + t^2}\right|\right] + \mathbf{C}$

30. $\int(t\mathbf{j} + t^2\mathbf{k}) \times (\mathbf{i} + t\mathbf{j} + t\mathbf{k})\,dt = \int[(t^2 - t^3)\mathbf{i} + t^2\mathbf{j} - t\mathbf{k}]\,dt = \left(\frac{t^3}{3} - \frac{t^4}{4}\right)\mathbf{i} + \frac{t^3}{3}\mathbf{j} - \frac{t^2}{2}\mathbf{k} + \mathbf{C}$

31. $\mathbf{r}(t) = \int(2t\mathbf{i} + e^t\mathbf{j} + e^{-t}\mathbf{k})\,dt = t^2\mathbf{i} + e^t\mathbf{j} - e^{-t}\mathbf{k} + \mathbf{C}$

$\mathbf{r}(0) = \mathbf{j} - \mathbf{k} + \mathbf{C} = \mathbf{i} + 3\mathbf{j} - 5\mathbf{k} \Longrightarrow \mathbf{C} = \mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$

$\mathbf{r}(t) = (t^2 + 1)\mathbf{i} + (e^t + 2)\mathbf{j} - (e^{-t} + 4)\mathbf{k}$

32. $\mathbf{r}(t) = \int(\sec t\mathbf{i} + \tan t\mathbf{j} + t^2\mathbf{k})\,dt$

$= \ln|\sec t + \tan t|\mathbf{i} - \ln|\cos t|\mathbf{j} + \frac{t^3}{3}\mathbf{k} + \mathbf{C}$

$\mathbf{r}(0) = \mathbf{C} = 3\mathbf{k}$

$\mathbf{r}(t) = \ln|\sec t + \tan t|\mathbf{i} - \ln|\cos t|\mathbf{j} + \left(\frac{t^3}{3} + 3\right)\mathbf{k}$

33. $\int_{-2}^{2}(3t\mathbf{i} + 2t^2\mathbf{j} - t^3\mathbf{k})\,dt = \left[\frac{3t^2}{2}\mathbf{i} + \frac{2t^3}{3}\mathbf{j} - \frac{t^4}{4}\mathbf{k}\right]_{-2}^{2} = \frac{32}{3}\mathbf{j}$

34. $\int_0^1\left(\sqrt{t}\mathbf{j} + t\sin t\,\mathbf{k}\right)dt = \left[\frac{2}{3}t^{3/2}\mathbf{j} + (\sin t - t\cos t)\mathbf{k}\right]_0^1$

$= \frac{2}{3}\mathbf{j} + (\sin 1 - \cos 1)\mathbf{k}$

35. $\int_0^2(e^{t/2}\mathbf{i} - 3t^2\mathbf{j} - \mathbf{k})\,dt = \left[2e^{t/2}\mathbf{i} - t^3\mathbf{j} - t\mathbf{k}\right]_0^2 = (2e - 2)\mathbf{i} - 8\mathbf{j} - 2\mathbf{k}$

36. $\int_{-1}^{1}(t^3\mathbf{i} - \arcsin t\mathbf{j} - t^2\mathbf{k})\,dt = \left[\frac{t^4}{4}\mathbf{i} - \left(t\arcsin t + \sqrt{1 - t^2}\right)\mathbf{j} - \frac{t^3}{3}\mathbf{k}\right]_{-1}^{1} = -\frac{2}{3}\mathbf{k}$

37. $\mathbf{r}(t) = \langle\cos^3 t, \sin^3 t, 3t\rangle$

$\mathbf{v}(t) = \mathbf{r}'(t) = \langle -3\cos^2 t\sin t, 3\sin^2 t\cos t, 3\rangle$

$\|\mathbf{v}(t)\| = \sqrt{9\cos^4 t\sin^2 t + 9\sin^4 t\cos^2 t + 9}$

$= 3\sqrt{\cos^2 t\sin^2 t(\cos^2 t + \sin^2 t) + 1}$

$= 3\sqrt{\cos^2 t\sin^2 t + 1}$

$\mathbf{a}(t) = \mathbf{v}'(t) = \langle -6\cos t(-\sin^2 t) + (-3\cos^2 t)\cos t, 6\sin t\cos^2 t + 3\sin^2 t(-\sin t), 0\rangle$

$= \langle 3\cos t(2\sin^2 t - \cos^2 t), 3\sin t(2\cos^2 t - \sin^2 t), 0\rangle$

38. $\mathbf{r}(t) = \langle t, -\tan t, e^t\rangle$

$\mathbf{r}'(t) = \mathbf{v}(t) = \langle 1, -\sec^2 t, e^t\rangle$

$\|\mathbf{v}(t)\| = \sqrt{1 + \sec^4 t + e^{2t}}$

$\mathbf{r}''(t) = \mathbf{a}(t) = \langle 0, -2\sec^2 t \cdot \tan t, e^t\rangle$

39. $\mathbf{r}(t) = \left\langle \ln(t - 3), t^2, \frac{1}{2}t\right\rangle,\ t_0 = 4$

$\mathbf{r}'(t) = \left\langle \frac{1}{t - 3}, 2t, \frac{1}{2}\right\rangle$

$\mathbf{r}'(4) = \left\langle 1, 8, \frac{1}{2}\right\rangle$ direction numbers

Since $\mathbf{r}(4) = \langle 0, 16, 2\rangle$, the parametric equations are $x = t,\ y = 16 + 8t,\ z = 2 + \frac{1}{2}t$.

$\mathbf{r}(t_0 + 0.1) = \mathbf{r}(4.1) \approx \langle 0.1, 16.8, 2.05\rangle$

40. $\mathbf{r}(t) = \langle 3\cosh t, \sinh t, -2t\rangle,\ t_0 = 0$

$\mathbf{r}'(t) = \langle 3\sinh t, \cosh t, -2\rangle$

$\mathbf{r}'(0) = \langle 0, 1, -2\rangle$ direction numbers

Since $\mathbf{r}(0) = \langle 3, 0, 0\rangle$, the parametric equations are $x = 3$, $y = t$, $z = -2t$.

$\mathbf{r}(t_0 + 0.1) = \mathbf{r}(0.1) \approx \langle 3, 0.1, -0.2\rangle$

41. $\mathbf{r}(t) = \left\langle v_0 t\cos\theta, v_0 t\sin\theta - \frac{1}{2}gt^2\right\rangle$

$= \left\langle \frac{75\sqrt{3}}{2}t, \frac{75}{2}t - 16t^2\right\rangle$

$\frac{75}{2}t - 16t^2 = 0 \Rightarrow t = \frac{75}{32}$

$\text{Range} = \frac{75\sqrt{3}}{2}\left(\frac{75}{32}\right) = \frac{5625}{64}\sqrt{3} \approx 152.2 \text{ feet}$

or, $\text{Range} = v_0\cos\theta\left[\frac{v_0\sin\theta}{(1/2)g}\right] = \frac{v_0^2\sin 2\theta}{g}$

$= \frac{75^2\sin(60°)}{32} \approx 152.2 \text{ feet}$

42. $y = -16t^2 + 6 = 0 \Rightarrow t = \frac{\sqrt{6}}{4}$

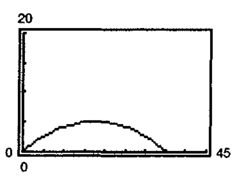
$x = v_0 t \Rightarrow 4 = v_0\left(\frac{\sqrt{6}}{4}\right) \Rightarrow v_0 = \frac{16}{\sqrt{6}}$

$v_0 = \frac{8\sqrt{6}}{3} \approx 6.532 \text{ ft/sec}$

43. $\text{Range} = x = \frac{v_0^2}{9.8}\sin 2\theta = 80 \Rightarrow v_0 = \sqrt{\frac{(80)(9.8)}{\sin 40°}} \approx 34.9 \text{ m/sec}$ (see Exercise 41)

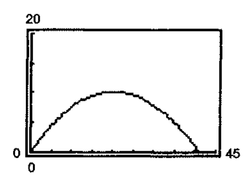
44. $\mathbf{r}(t) = [(v_0\cos\theta)t]\mathbf{i} + \left[(v_0\sin\theta)t - \frac{1}{2}(9.8)t^2\right]\mathbf{j}$

(a) $\mathbf{r}(t) = [(20\cos 30°)t]\mathbf{i} + [(20\sin 30°)t - 4.9t^2]\mathbf{j}$



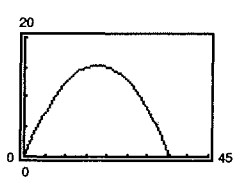
Maximum height ≈ 5.1 m; Range ≈ 35.3 m

(b) $\mathbf{r}(t) = [(20\cos 45°)t]\mathbf{i} + [(20\sin 45°)t - 4.9t^2]\mathbf{j}$



Maximum height ≈ 10.2 m; Range ≈ 40.8 m

(c) $\mathbf{r}(t) = [(20\cos 60°)t]\mathbf{i} + [(20\sin 60°)t - 4.9t^2]\mathbf{j}$



Maximum height ≈ 15.3 m; Range ≈ 35.3 m

(Note that 45° gives the longest range)

45. $\mathbf{r}(t) = 5t\mathbf{i}$

$\mathbf{v}(t) = 5\mathbf{i}$

$\|\mathbf{v}(t)\| = 5$

$\mathbf{a}(t) = \mathbf{0}$

$\mathbf{T}(t) = \mathbf{i}$

$\mathbf{N}(t)$ does not exist.

$\mathbf{a} \cdot \mathbf{T} = 0$

$\mathbf{a} \cdot \mathbf{N}$ does not exist.

(The curve is a line.)

46. $\mathbf{r}(t) = (1 + 4t)\mathbf{i} + (2 - 3t)\mathbf{j}$

$\mathbf{v}(t) = 4\mathbf{i} - 3\mathbf{j}$

$\|\mathbf{v}\| = 5$

$\mathbf{a}(t) = \mathbf{0}$

$\mathbf{T}(t) = \frac{1}{5}(4\mathbf{i} - 3\mathbf{j})$

$\mathbf{N}(t)$ does not exist.

$\mathbf{a} \cdot \mathbf{T} = 0$

$\mathbf{a} \cdot \mathbf{N}$ does not exist.

47. $\mathbf{r}(t) = t\mathbf{i} + \sqrt{t}\mathbf{j}$

$\mathbf{v}(t) = \mathbf{i} + \frac{1}{2\sqrt{t}}\mathbf{j}$

$\|\mathbf{v}(t)\| = \frac{\sqrt{4t+1}}{2\sqrt{t}}$

$\mathbf{a}(t) = -\frac{1}{4t\sqrt{t}}\mathbf{j}$

$\mathbf{T}(t) = \frac{\mathbf{i} + \left(1/2\sqrt{t}\right)\mathbf{j}}{\left(\sqrt{4t+1}\right)/2\sqrt{t}} = \frac{2\sqrt{t}\mathbf{i} + \mathbf{j}}{\sqrt{4t+1}}$

$\mathbf{N}(t) = \frac{\mathbf{i} - 2\sqrt{t}\mathbf{j}}{\sqrt{4t+1}}$

$\mathbf{a} \cdot \mathbf{T} = \frac{-1}{4t\sqrt{t}\sqrt{4t+1}}$

$\mathbf{a} \cdot \mathbf{N} = \frac{1}{2t\sqrt{4t+1}}$

48. $\mathbf{r}(t) = 2(t+1)\mathbf{i} + \frac{2}{t+1}\mathbf{j}$

$\mathbf{v}(t) = 2\mathbf{i} - \frac{2}{(t+1)^2}\mathbf{j}$

$\|\mathbf{v}(t)\| = \frac{2\sqrt{(t+1)^4+1}}{(t+1)^2}$

$\mathbf{a}(t) = \frac{4}{(t+1)^3}\mathbf{j}$

$\mathbf{T}(t) = \frac{(t+1)^2\mathbf{i} - \mathbf{j}}{\sqrt{(t+1)^4+1}}$

$\mathbf{N}(t) = \frac{\mathbf{i} + (t+1)^2\mathbf{j}}{\sqrt{(t+1)^4+1}}$

$\mathbf{a} \cdot \mathbf{T} = \frac{-4}{(t+1)^3\sqrt{(t+1)^4+1}}$

$\mathbf{a} \cdot \mathbf{N} = \frac{4(t+1)^2}{(t+1)^3\sqrt{(t+1)^4+1}}$

$= \frac{4}{(t+1)\sqrt{(t+1)^4+1}}$

49. $\mathbf{r}(t) = e^t\mathbf{i} + e^{-t}\mathbf{j}$

$\mathbf{v}(t) = e^t\mathbf{i} - e^{-t}\mathbf{j}$

$\|\mathbf{v}(t)\| = \sqrt{e^{2t} + e^{-2t}}$

$\mathbf{a}(t) = e^t\mathbf{i} + e^{-t}\mathbf{j}$

$\mathbf{T}(t) = \frac{e^t\mathbf{i} - e^{-t}\mathbf{j}}{\sqrt{e^{2t} + e^{-2t}}}$

$\mathbf{N}(t) = \frac{e^{-t}\mathbf{i} + e^t\mathbf{j}}{\sqrt{e^{2t} + e^{-2t}}}$

$\mathbf{a} \cdot \mathbf{T} = \frac{e^{2t} - e^{-2t}}{\sqrt{e^{2t} + e^{-2t}}}$

$\mathbf{a} \cdot \mathbf{N} = \frac{2}{\sqrt{e^{2t} + e^{-2t}}}$

50.

$$\mathbf{r}(t) = t\cos t\mathbf{i} + t\sin t\mathbf{j}$$

$$\mathbf{v}(t) = \mathbf{r}'(t) = (-t\sin t + \cos t)\mathbf{i} + (t\cos t + \sin t)\mathbf{j}$$

$$\|\mathbf{v}(t)\| = \text{speed} = \sqrt{(-t\sin t + \cos t)^2 + (t\cos t + \sin t)^2} = \sqrt{t^2+1}$$

$$\mathbf{a}(t) = \mathbf{r}''(t) = (-t\cos t - 2\sin t)\mathbf{i} + (-t\sin t + 2\cos t)\mathbf{j}$$

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \frac{(-t\sin t + \cos t)\mathbf{i} + (t\cos t + \sin t)\mathbf{j}}{\sqrt{t^2+1}}$$

$$\mathbf{N}(t) = \frac{-(t\cos t + \sin t)\mathbf{i} + (-t\sin t + \cos t)\mathbf{j}}{\sqrt{t^2+1}}$$

$$\mathbf{a}(t)\cdot\mathbf{T}(t) = \frac{t}{\sqrt{t^2+1}}$$

$$\mathbf{a}(t)\cdot\mathbf{N}(t) = \frac{t^2+2}{\sqrt{t^2+1}}$$

51. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{1}{2}t^2\mathbf{k}$

$$\mathbf{v}(t) = \mathbf{i} + 2t\mathbf{j} + t\mathbf{k}$$

$$\|\mathbf{v}\| = \sqrt{1+5t^2}$$

$$\mathbf{a}(t) = 2\mathbf{j} + \mathbf{k}$$

$$\mathbf{T}(t) = \frac{\mathbf{i} + 2t\mathbf{j} + t\mathbf{k}}{\sqrt{1+5t^2}}$$

$$\mathbf{N}(t) = \frac{-5t\mathbf{i} + 2\mathbf{j} + \mathbf{k}}{\sqrt{5}\sqrt{1+5t^2}}$$

$$\mathbf{a}\cdot\mathbf{T} = \frac{5t}{\sqrt{1+5t^2}}$$

$$\mathbf{a}\cdot\mathbf{N} = \frac{5}{\sqrt{5}\sqrt{1+5t^2}} = \frac{\sqrt{5}}{\sqrt{1+5t^2}}$$

52. $\mathbf{r}(t) = (t-1)\mathbf{i} + t\mathbf{j} + \frac{1}{t}\mathbf{k}$

$$\mathbf{v}(t) = \mathbf{i} + \mathbf{j} - \frac{1}{t^2}\mathbf{k}$$

$$\|\mathbf{v}(t)\| = \frac{\sqrt{2t^4+1}}{t^2}$$

$$\mathbf{a}(t) = \frac{2}{t^3}\mathbf{k}$$

$$\mathbf{T}(t) = \frac{t^2\mathbf{i} + t^2\mathbf{j} - \mathbf{k}}{\sqrt{2t^4+1}}$$

$$\mathbf{N}(t) = \frac{\mathbf{i} + \mathbf{j} + 2t^2\mathbf{k}}{\sqrt{2}\sqrt{2t^4+1}}$$

$$\mathbf{a}\cdot\mathbf{T} = \frac{-2}{t^3\sqrt{2t^4+1}}$$

$$\mathbf{a}\cdot\mathbf{N} = \frac{4}{t\sqrt{2}\sqrt{2t^4+1}}$$

53. $\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j} + t\mathbf{k},\ x = 2\cos t,\ y = 2\sin t,\ z = t$

When $t = \frac{3\pi}{4}$, $x = -\sqrt{2}$, $y = \sqrt{2}$, $z = \frac{3\pi}{4}$.

$$\mathbf{r}'(t) = -2\sin t\mathbf{i} + 2\cos t\mathbf{j} + \mathbf{k}$$

Direction numbers when $t = \frac{3\pi}{4}$, $a = -\sqrt{2}$, $b = -\sqrt{2}$, $c = 1$

$$x = -\sqrt{2}t - \sqrt{2},\ y = -\sqrt{2}t + \sqrt{2},\ z = t + \frac{3\pi}{4}$$

54. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{2}{3}t^3\mathbf{k},\ x = t,\ y = t^2,\ z = \frac{2}{3}t^3$

When $t = 2$, $x = 2$, $y = 4$, $z = \frac{16}{3}$.

$$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j} + 2t^2\mathbf{k}$$

Direction numbers when $t = 2$, $a = 1$, $b = 4$, $c = 8$

$$x = t + 2,\ y = 4t + 4,\ z = 8t + \frac{16}{3}$$

55. $v = \sqrt{\dfrac{9.56\times 10^4}{4600}} \approx 4.56$ mi/sec

(see Exercise 56, Section 11.4)

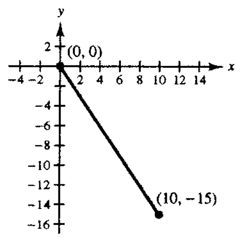
56. Factor of 4

57. $\mathbf{r}(t) = 2t\mathbf{i} - 3t\mathbf{j}, 0 \le t \le 5$

$\mathbf{r}'(t) = 2\mathbf{i} - 3\mathbf{j}$

$$s = \int_a^b \|\mathbf{r}'(t)\|\, dt = \int_0^5 \sqrt{4 + 9}\, dt$$

$$= \sqrt{13}t\Big]_0^5 = 5\sqrt{13}$$



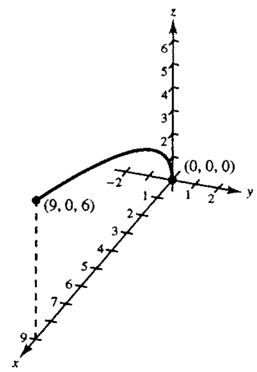
58. $\mathbf{r}(t) = t^2\mathbf{i} + 2t\mathbf{k}, 0 \le t \le 3$

$\mathbf{r}'(t) = 2t\mathbf{i} + 2\mathbf{k}$

$$s = \int_a^b \|\mathbf{r}'(t)\|\, dt = \int_0^3 \sqrt{4t^2 + 4}\, dt$$

$$= \left[\ln\left|\sqrt{t^2+1} + t\right| + t\sqrt{t^2+1}\right]_0^3$$

$$= \ln(\sqrt{10} + 3) + 3\sqrt{10} \approx 11.3053$$



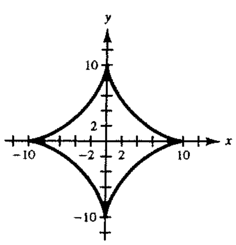
59. $\mathbf{r}(t) = 10\cos^3 t\mathbf{i} + 10\sin^3 t\mathbf{j}$

$\mathbf{r}'(t) = -30\cos^2 t \sin t\mathbf{i} + 30\sin^2 t\cos t\mathbf{j}$

$\|\mathbf{r}'(t)\| = 30\sqrt{\cos^4 t\sin^2 t + \sin^4 t\cos^2 t}$

$= 30|\cos t \sin t|$

$$s = 4\int_0^{\pi/2} 30\cos t \cdot \sin t\, dt = \left[120\frac{\sin^2 t}{2}\right]_0^{\pi/2} = 60$$

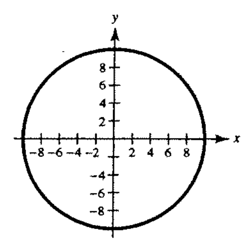


60. $\mathbf{r}(t) = 10\cos t\mathbf{i} + 10\sin t\mathbf{j}$

$\mathbf{r}'(t) = -10\sin t\mathbf{i} + 10\cos t\mathbf{j}$

$\|\mathbf{r}'(t)\| = 10$

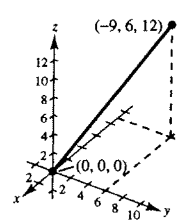
$$s = \int_0^{2\pi} 10\, dt = 20\pi$$



61. $\mathbf{r}(t) = -3t\mathbf{i} + 2t\mathbf{j} + 4t\mathbf{k}, 0 \le t \le 3$

$\mathbf{r}'(t) = -3\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$

$$s = \int_a^b \|\mathbf{r}'(t)\|\, dt = \int_0^3 \sqrt{9 + 4 + 16}\, dt = \int_0^3 \sqrt{29}\, dt = 3\sqrt{29}$$

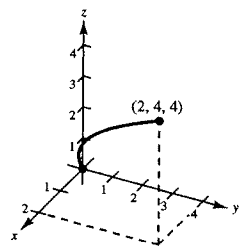


62. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + 2t\mathbf{k}, 0 \le t \le 2$

$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j} + 2\mathbf{k}, \|\mathbf{r}'(t)\| = \sqrt{5 + 4t^2}$

$$s = \int_a^b \|\mathbf{r}'(t)\|\,dt = \int_0^2 \sqrt{5 + 4t^2}\,dt$$

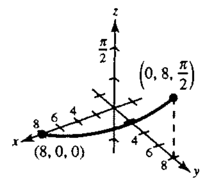
$$= \sqrt{21} + \frac{5}{4}\ln 5 - \frac{5}{4}\ln\left(\sqrt{105} - 4\sqrt{5}\right) \approx 6.2638$$



63. $\mathbf{r}(t) = \langle 8\cos t, 8\sin t, t\rangle, 0 \le t \le \frac{\pi}{2}$

$\mathbf{r}'(t) = \langle -8\sin t, 8\cos t, 1\rangle, \|\mathbf{r}'(t)\| = \sqrt{65}$

$$s = \int_a^b \|\mathbf{r}'(t)\|\,dt = \int_0^{\pi/2} \sqrt{65}\,dt = \frac{\pi\sqrt{65}}{2}$$



64. $\mathbf{r}(t) = \langle 2(\sin t - t\cos t), 2(\cos t + t\sin t), t\rangle, 0 \le t \le \frac{\pi}{2}$

$\mathbf{r}'(t) = \langle 2t\sin t, 2t\cos t, 1\rangle, \|\mathbf{r}'(t)\| = \sqrt{4t^2 + 1}$

$$s = \int_a^b \|\mathbf{r}'(t)\|\,dt = \int_0^{\pi/2} \sqrt{4t^2 + 1}\,dt$$

$$= \frac{1}{4}\ln\left(\sqrt{\pi^2 + 1} + \pi\right) + \frac{\pi}{4}\sqrt{\pi^2 + 1}$$

≈ 3.055

65. $\mathbf{r}(t) = \frac{1}{2}t\mathbf{i} + \sin t\mathbf{j} + \cos t\mathbf{k},\ 0 \le t \le \pi$

$\mathbf{r}'(t) = \frac{1}{2}\mathbf{i} + \cos t\mathbf{j} - \sin t\mathbf{k}$

$$s = \int_0^{\pi} \|\mathbf{r}'(t)\|\,dt$$

$$= \int_0^{\pi} \sqrt{\frac{1}{4} + \cos^2 t + \sin^2 t}\,dt$$

$$= \frac{\sqrt{5}}{2}\int_0^{\pi} dt = \left[\frac{\sqrt{5}}{2}t\right]_0^{\pi} = \frac{\sqrt{5}}{2}\pi$$

66. $\mathbf{r}(t) = e^t\sin t\mathbf{i} + e^t\cos t\mathbf{k},\ 0 \le t \le \pi$

$\mathbf{r}'(t) = (e^t\cos t + e^t\sin t)\mathbf{i} + (-e^t\sin t + e^t\cos t)\mathbf{k}$

$\|\mathbf{r}'(t)\| = \sqrt{(e^t\cos t + e^t\sin t)^2 + (-e^t\sin t + e^t\cos t)^2}$

$= \sqrt{2}e^t$

$$s = \int_0^{\pi} \|\mathbf{r}'(t)\|\,dt$$

$$= \sqrt{2}\int_0^{\pi} e^t\,dt = \left[\sqrt{2}e^t\right]_0^{\pi} = \sqrt{2}(e^{\pi} - 1)$$

67. $\mathbf{r}(t) = 3t\mathbf{i} + 2t\mathbf{j}$

Line

$K = 0$

68. $\mathbf{r}(t) = 2\sqrt{t}\mathbf{i} + 3t\mathbf{j}$

$\mathbf{r}'(t) = \frac{1}{\sqrt{t}}\mathbf{i} + 3\mathbf{j}, \|\mathbf{r}'(t)\| = \sqrt{\frac{1}{t} + 9} = \sqrt{\frac{1 + 9t}{t}}$

$\mathbf{r}''(t) = -\frac{1}{2}t^{-3/2}\mathbf{i}$

$$\mathbf{r}' \times \mathbf{r}'' = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{1}{\sqrt{t}} & 3 & 0 \\ -\frac{1}{2}t^{-3/2} & 0 & 0 \end{vmatrix} = \frac{3}{2}t^{-3/2}\mathbf{k};\ \|\mathbf{r}' \times \mathbf{r}''\| = \frac{3}{2t^{3/2}}$$

$$K = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{3/2t^{3/2}}{(1 + 9t)^{3/2}/t^{3/2}} = \frac{3}{2(1 + 9t)^{3/2}}$$

69. $\mathbf{r}(t) = 2t\mathbf{i} + \frac{1}{2}t^2\mathbf{j} + t^2\mathbf{k}$

$\mathbf{r}'(t) = 2\mathbf{i} + t\mathbf{j} + 2t\mathbf{k}, \|\mathbf{r}'\| = \sqrt{5t^2 + 4}$

$\mathbf{r}''(t) = \mathbf{j} + 2\mathbf{k}$

$$\mathbf{r}' \times \mathbf{r}'' = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & t & 2t \\ 0 & 1 & 2 \end{vmatrix} = -4\mathbf{j} + 2\mathbf{k}, \|\mathbf{r}' \times \mathbf{r}''\| = \sqrt{20}$$

$$K = \frac{\|\mathbf{r}' \times \mathbf{r}''\|}{\|\mathbf{r}'\|^3} = \frac{\sqrt{20}}{(5t^2 + 4)^{3/2}} = \frac{2\sqrt{5}}{(4 + 5t^2)^{3/2}}$$

70. $\mathbf{r}(t) = 2t\mathbf{i} + 5\cos t\mathbf{j} + 5\sin t\mathbf{k}$

$\mathbf{r}'(t) = 2\mathbf{i} - 5\sin t\mathbf{j} + 5\cos t\mathbf{k}, \|\mathbf{r}'(t)\| = \sqrt{29}$

$\mathbf{r}''(t) = 5\cos t\mathbf{j} - 5\sin t\mathbf{k}$

$$\mathbf{r}' \times \mathbf{r}'' = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -5\sin t & 5\cos t \\ 0 & -5\cos t & -5\sin t \end{vmatrix} = 25\mathbf{i} + 10\sin t\mathbf{j} - 10\cos t\mathbf{k}$$

$\|\mathbf{r}' \times \mathbf{r}''\| = \sqrt{725}$

$$K = \frac{\|\mathbf{r}' \times \mathbf{r}''\|}{\|\mathbf{r}'\|^3} = \frac{\sqrt{725}}{(29)^{3/2}} = \frac{\sqrt{25 \cdot 29}}{29\sqrt{29}} = \frac{5}{29}$$

71. $y = \frac{1}{2}x^2 + 2$

$y' = x$

$y'' = 1$

$$K = \frac{|y''|}{[1 + (y')^2]^{3/2}} = \frac{1}{(1 + x^2)^{3/2}}$$

At $x = 4$, $K = \frac{1}{17^{3/2}}$ and $r = 17^{3/2} = 17\sqrt{17}$.

72. $y = e^{-x/2}$

$y' = -\frac{1}{2}e^{-x/2}, y'' = \frac{1}{4}e^{-x/2}$

$$K = \frac{|y''|}{[1 + (y')^2]^{3/2}} = \frac{\frac{1}{4}e^{-x/2}}{\left[1 + \frac{1}{4}e^{-x}\right]^{3/2}}$$

At $x = 0$, $K = \frac{1/4}{(5/4)^{3/2}} = \frac{2}{5^{3/2}} = \frac{2}{5\sqrt{5}} = \frac{2\sqrt{5}}{25}, r = \frac{5\sqrt{5}}{2}$.

73. $y = \ln x$

$y' = \frac{1}{x}, y'' = -\frac{1}{x^2}$

$$K = \frac{|y''|}{[1 + (y')^2]^{3/2}} = \frac{1/x^2}{[1 + (1/x)^2]^{3/2}}$$

At $x = 1$, $K = \frac{1}{2^{3/2}} = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}$ and $r = 2\sqrt{2}$.

74. $y = \tan x$

$y' = \sec^2 x$

$y'' = 2\sec^2 x \tan x$

$$K = \frac{|y''|}{[1 + (y')^2]^{3/2}} = \frac{|2\sec^2 x \tan x|}{[1 + \sec^4 x]^{3/2}}$$

At $x = \frac{\pi}{4}$, $K = \frac{4}{5^{3/2}} = \frac{4}{5\sqrt{5}} = \frac{4\sqrt{5}}{25}$ and $r = \frac{5\sqrt{5}}{4}$.

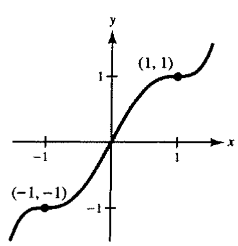
75. The curvature changes abruptly from zero to a nonzero constant at the points B and C.

76. $y = ax^5 + bx^3 + cx$

$y' = 5ax^4 + 3bx^2 + c$

$y'' = 20ax^3 + 6bx$

$$K = \frac{|20ax^4 + 6bx|}{[1 + (5ax^4 + 3bx^2 + c)^2]^{3/2}}$$



At $x = 1$: $k = 0 \Longrightarrow 20a + 6b = 0$

$y' = 0 \Longrightarrow 5a + 3b + c = 0$

$y(1) = 1 \Longrightarrow a + b + c = 1$

Solving these 3 equations for a, b, c, you obtain $a = \frac{3}{8}, b = -\frac{5}{4}, c = \frac{15}{8}$. By symmetry, the same holds at $x = -1$.

$$y = \frac{3}{8}x^5 - \frac{5}{4}x^3 + \frac{15}{8}x$$

Problem Solving for Chapter 12

1. $x(t) = \int_0^t \cos\left(\frac{\pi u^2}{2}\right) du, y(t) = \int_0^t \sin\left(\frac{\pi u^2}{2}\right) du$

$x'(t) = \cos\left(\frac{\pi t^2}{2}\right), y'(t) = \sin\left(\frac{\pi t^2}{2}\right)$

(a) $s = \int_0^a \sqrt{x'(t)^2 + y'(t)^2}\, dt = \int_0^a dt = a$

(b) $x''(t) = -\pi t \sin\left(\frac{\pi t^2}{2}\right), y''(t) = \pi t \cos\left(\frac{\pi t^2}{2}\right)$

$$K = \frac{\left|\pi t \cos^2\left(\frac{\pi t^2}{2}\right) + \pi t \sin^2\left(\frac{\pi t^2}{2}\right)\right|}{1} = \pi t$$

At $t = a, K = \pi a$.

(c) $K = \pi a = \pi(\text{length})$

2. $x^{2/3} + y^{2/3} = a^{2/3}$

$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}y' = 0$

$y' = \frac{-y^{1/3}}{x^{1/3}}$ Slope at $P(x, y)$.

$\mathbf{r}(t) = \cos^3 t\mathbf{i} + \sin^3 t\mathbf{j}$

$\mathbf{r}'(t) = -3\cos^2 t \sin t\mathbf{i} + 3\sin^2 t \cos t\mathbf{j}$

$\|\mathbf{r}'(t)\text{i} = |3\cos t \sin t|$

$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = -\cos t\mathbf{i} + \sin t\mathbf{j}$

$\mathbf{T}'(t) = \sin t\mathbf{i} + \cos t\mathbf{j}$

$Q(0, 0, 0)$ origin

$P = (\cos^3 t, \sin^3 t, 0)$ on curve.

$$\overrightarrow{PQ} \times \mathbf{T} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos^3 t & \sin^3 t & 0 \\ -\cos t & \sin t & 0 \end{vmatrix}$$

$= (\cos^3 t \sin t - \sin^3 t \cos t)\mathbf{k}$

$D = \frac{\|\overrightarrow{PQ} \times \mathbf{T}\|}{\|\mathbf{T}\|} = |\cos t \sin t|$

$K = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{1}{|3\cos t \sin t|}$

Thus, the radius of curvature, $\frac{1}{K}$, is three times the distance from the origin to the tangent line.

3. Bomb: $\mathbf{r}_1(t) = \langle 5000 - 400t, 3200 - 16t^2\rangle$

Projectile: $\mathbf{r}_2(t) = \langle (v_0 \cos\theta)t, (v_0 \sin\theta)t - 16t^2\rangle$

At 1600 feet: Bomb:

$3200 - 16t^2 = 1600 \Rightarrow t = 10$ seconds.

Projectile will travel 5 seconds:

$5(v_0 \sin\theta) - 16(25) = 1600$

$v_0 \sin\theta = 400.$

Horizontal position:

At $t = 10$, bomb is at $5000 - 400(10) = 1000$.

At $t = 5$, projectile is at $5v_0 \cos\theta$.

Thus, $v_0 \cos\theta = 200$.

Combining, $\frac{v_0 \sin\theta}{v_0 \cos\theta} = \frac{400}{200} \Rightarrow \tan\theta = 2 \Rightarrow \theta \approx 63.4°$.

$v_0 = \frac{200}{\cos\theta} \approx 447.2$ ft/sec

4. Bomb: $\mathbf{r}_1(t) = \langle 5000 + 400t, 3200 - 16t^2\rangle$

Projectile: $\mathbf{r}_2(t) = \langle (v_0 \cos\theta)t, (v_0 \sin\theta)t - 16t^2\rangle$

At 1600 feet: Bomb:

$3200 - 16t^2 = 1600 \Rightarrow t = 10$

Projectile will travel 5 seconds:

$5(v_0 \sin\theta) - 16(25) = 1600$

$v_0 \sin\theta = 400.$

Horizontal position:

At $t = 10$, bomb is at $5000 + 400(10) = 9000$.

At $t = 5$, projectile is at $(v_0 \cos\theta)5$.

Thus,

$5v_0 \cos\theta = 9000$

$v_0 \cos\theta = 1800.$

Combining, $\frac{v_0 \sin\theta}{v_0 \cos\theta} = \frac{400}{1800} \Rightarrow \tan\theta = \frac{2}{9} \Rightarrow \theta \approx 12.5°$.

$v_0 = \frac{1800}{\cos\theta} \approx 1843.9$ ft/sec

5. $x'(\theta) = 1 - \cos\theta, y'(\theta) = \sin\theta, 0 \le \theta \le 2\pi$

$$\sqrt{x'(\theta)^2 + y'(\theta)^2} = \sqrt{(1-\cos\theta)^2 + \sin^2\theta}$$

$$= \sqrt{2 - 2\cos\theta} = \sqrt{4\sin^2\frac{\theta}{2}}$$

$$s(t) = \int_\pi^t 2\sin\frac{\theta}{2}\,d\theta = \left[-4\cos\frac{\theta}{2}\right]_\pi^t = -4\cos\frac{t}{2}$$

$$x''(\theta) = \sin\theta, y''(\theta) = \cos\theta$$

$$K = \frac{|(1-\cos\theta)\cos\theta - \sin\theta\sin\theta|}{\left(2\sin\frac{\theta}{2}\right)^3} = \frac{|\cos\theta - 1|}{8\sin^3\frac{\theta}{2}}$$

$$= \frac{1}{4\sin\frac{\theta}{2}}$$

Thus, $\rho = \frac{1}{K} = 4\sin\frac{t}{2}$ and

$$s^2 + \rho^2 = 16\cos^2\left(\frac{t}{2}\right) + 16\sin^2\left(\frac{t}{2}\right) = 16.$$

6. $r = 1 - \cos\theta$

$r' = \sin\theta$

$$s(t) = \int_\pi^t \sqrt{(1-\cos\theta)^2 + \sin^2\theta}\,d\theta = \int_\pi^t \sqrt{2 - 2\cos\theta}\,d\theta$$

$$= \int_\pi^t 2\sin\frac{\theta}{2}\,d\theta = \left[-4\cos\frac{\theta}{2}\right]_\pi^t = -4\cos\frac{t}{2}$$

$$K = \frac{|2(r')^2 - rr'' + r^2|}{[(r')^2 + r^2]^{3/2}}$$

$$= \frac{|2\sin^2\theta - (1-\cos\theta)(\cos\theta) + (1-\cos\theta)^2|}{8\sin^3\frac{\theta}{2}}$$

$$= \frac{|3 - 3\cos\theta|}{8\sin^3\frac{\theta}{2}}$$

$$= \frac{3}{4}\frac{\sin^2\frac{\theta}{2}}{\sin^3\frac{\theta}{2}} = \frac{3}{4\sin\frac{\theta}{2}}$$

$$\rho = \frac{1}{K} = \frac{4\sin\frac{\theta}{2}}{3}$$

$$s^2 + 9\rho^2 = 16\cos^2\frac{\theta}{2} + 16\sin^2\frac{\theta}{2} = 16$$

7. $\|\mathbf{r}(t)\|^2 = \mathbf{r}(t)\cdot\mathbf{r}(t)$

$$\frac{d}{dt}(\|\mathbf{r}(t)\|)^2 = 2\|\mathbf{r}(t)\|\frac{d}{dt}\|\mathbf{r}(t)\| = \mathbf{r}(t)\cdot\mathbf{r}'(t) + \mathbf{r}'(t)\cdot\mathbf{r}(t) \implies \frac{d}{dt}\|\mathbf{r}(t)\| = \frac{\mathbf{r}(t)\cdot\mathbf{r}'(t)}{\|\mathbf{r}(t)\|}$$

8. (a) $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$ position vector

$\mathbf{r} = r\cos\theta\mathbf{i} + r\sin\theta\mathbf{j}$

$$\frac{d\mathbf{r}}{dt} = \left[\frac{dr}{dt}\cos\theta - r\sin\theta\frac{d\theta}{dt}\right]\mathbf{i} + \left[\frac{dr}{dt}\sin\theta + r\cos\theta\frac{d\theta}{dt}\right]\mathbf{j}$$

$$\mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = \left[\frac{d^2r}{dt^2}\cos\theta - \frac{dr}{dt}\sin\theta\frac{d\theta}{dt} - \frac{dr}{dt}\sin\theta\frac{d\theta}{dt} - r\cos\theta\left(\frac{d\theta}{dt}\right)^2 - r\sin\theta\frac{d^2\theta}{dt^2}\right]\mathbf{i}$$

$$+ \left[\frac{d^2r}{dt^2}\sin\theta + \frac{dr}{dt}\cos\theta\frac{d\theta}{dt} + \frac{dr}{dt}\cos\theta\frac{d\theta}{dt} - r\sin\theta\left(\frac{d\theta}{dt}\right)^2 + r\cos\theta\frac{d^2\theta}{dt^2}\right]$$

$$a_r = \mathbf{a}\cdot\mathbf{u}_r = \mathbf{a}\cdot(\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j})$$

$$= \left[\frac{d^2r}{dt^2}\cos^2\theta - 2\frac{dr}{dt}\sin\theta\cos\theta\frac{d\theta}{dt} - r\cos^2\theta\left(\frac{d\theta}{dt}\right)^2 - r\cos\theta\sin\theta\frac{d^2\theta}{dt^2}\right]$$

$$+ \left[\frac{d^2r}{dt^2}\sin^2\theta + 2\frac{dr}{dt}\sin\theta\cos\theta\frac{d\theta}{dt} - r\sin^2\theta\left(\frac{d\theta}{dt}\right)^2 + r\cos\theta\sin\theta\frac{d^2\theta}{dt^2}\right]$$

$$= \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2$$

$$a_\theta = \mathbf{a}\cdot u_\theta = \mathbf{a}\cdot(-\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}) = 2\frac{dr}{dr}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}$$

$$\mathbf{a} = (\mathbf{a}\cdot\mathbf{u}_r)\mathbf{u}_r + (\mathbf{a}\cdot\mathbf{u}_\theta)u_\theta$$

$$= \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\mathbf{u}_r + \left[2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\right]\mathbf{u}_\theta$$

—CONTINUED—

8. —CONTINUED—

(b) $\mathbf{r} = 42{,}000\cos\left(\frac{\pi t}{12}\right)\mathbf{i} + 42{,}000\sin\left(\frac{\pi t}{12}\right)\mathbf{j}$

$$\mathbf{r} = 42{,}000, \frac{dr}{dt} = 0, \frac{d^2r}{dt^2} = 0$$

$$\frac{d\theta}{dt} = \frac{\pi}{12}, \frac{d^2\theta}{dt^2} = 0$$

Therefore, $\mathbf{a} = -42000\left(\frac{\pi}{12}\right)^2\mathbf{u}_r = -\frac{875}{3}\pi^2\mathbf{u}_r$.

Radial component: $-\frac{875}{3}\pi^2$

Angular component: 0

9. $\mathbf{r}(t) = 4\cos t\mathbf{i} + 4\sin t\mathbf{j} + 3t\mathbf{k}, t = \frac{\pi}{2}$

$\mathbf{r}'(t) = -4\sin t\mathbf{i} + 4\cos t\mathbf{j} + 3\mathbf{k}, \|\mathbf{r}'(t)\| = 5$

$\mathbf{r}''(t) = -4\cos t\mathbf{i} - 4\sin t\mathbf{j}$

$\mathbf{T} = -\frac{4}{5}\sin t\mathbf{i} + \frac{4}{5}\cos t\mathbf{j} + \frac{3}{5}\mathbf{k}$

$\mathbf{T}' = -\frac{4}{5}\cos t\mathbf{i} - \frac{4}{5}\sin t\mathbf{j}$

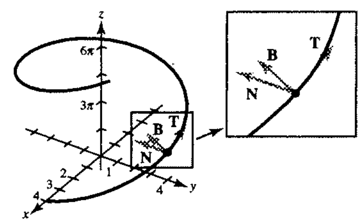
$\mathbf{N} = -\cos t\mathbf{i} - \sin t\mathbf{j}$

$\mathbf{B} = \mathbf{T}\times\mathbf{N} = \frac{3}{5}\sin t\mathbf{i} - \frac{3}{5}\cos t\mathbf{j} + \frac{4}{5}\mathbf{k}$

At $t = \frac{\pi}{2}$, $\mathbf{T}\left(\frac{\pi}{2}\right) = -\frac{4}{5}\mathbf{i} + \frac{3}{5}\mathbf{k}$

$\mathbf{N}\left(\frac{\pi}{2}\right) = -\mathbf{j}$

$\mathbf{B}\left(\frac{\pi}{2}\right) = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{k}$



10. $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} - \mathbf{k}, t = \frac{\pi}{4}$

$\mathbf{r}'(t) = -\sin t\mathbf{i} + \cos t\mathbf{j}, \|\mathbf{r}'(t)\| = 1$

$\mathbf{T} = -\sin t\mathbf{i} + \cos t\mathbf{j}$

$\mathbf{T}' = -\cos t\mathbf{i} - \sin t\mathbf{j}$

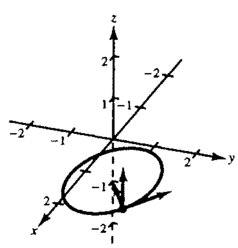
$\mathbf{N} = -\cos t\mathbf{i} - \sin t\mathbf{j}$

$\mathbf{B} = \mathbf{T}\times\mathbf{N} = \mathbf{k}$

At $t = \frac{\pi}{4}$, $\mathbf{T}\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}$

$\mathbf{N}\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}$

$\mathbf{B}\left(\frac{\pi}{4}\right) = \mathbf{k}$



11. (a) $\|\mathbf{B}\| = \|\mathbf{T} \times \mathbf{N}\| = 1$ constant length $\Rightarrow \dfrac{d\mathbf{B}}{ds} \perp \mathbf{B}$

$$\frac{d\mathbf{B}}{ds} = \frac{d}{ds}(\mathbf{T} \times \mathbf{N}) = (\mathbf{T} \times \mathbf{N}') + (\mathbf{T}' \times \mathbf{N})$$

$$\mathbf{T} \cdot \frac{d\mathbf{B}}{ds} = \mathbf{T} \cdot (\mathbf{T} \times \mathbf{N}') + \mathbf{T} \cdot (\mathbf{T}' \times \mathbf{N})$$

$$= (\mathbf{T} \times \mathbf{T}) \cdot \mathbf{N}' + \mathbf{T} \cdot \left(\mathbf{T}' \times \frac{\mathbf{T}'}{\|\mathbf{T}'\|}\right) = 0$$

Hence, $\dfrac{d\mathbf{B}}{ds} \perp \mathbf{B}$ and $\dfrac{d\mathbf{B}}{ds} \perp \mathbf{T} \Rightarrow \dfrac{d\mathbf{B}}{ds} = -\tau\mathbf{N}$ for some scalar τ.

(b) $\mathbf{B} = \mathbf{T} \times \mathbf{N}$. Using Exercise 11.4, number 64,

$$\mathbf{B} \times \mathbf{N} = (\mathbf{T} \times \mathbf{N}) \times \mathbf{N} = -\mathbf{N} \times (\mathbf{T} \times \mathbf{N})$$
$$= -[(\mathbf{N} \cdot \mathbf{N})\mathbf{T} - (\mathbf{N} \cdot \mathbf{T})\mathbf{N}]$$
$$= -\mathbf{T}$$

$$\mathbf{B} \times \mathbf{T} = (\mathbf{T} \times \mathbf{N}) \times \mathbf{T} = -\mathbf{T} \times (\mathbf{T} \times \mathbf{N})$$
$$= -[(\mathbf{T} \cdot \mathbf{N})\mathbf{T} - (\mathbf{T} \cdot \mathbf{T})\mathbf{N}]$$
$$= \mathbf{N}.$$

Now, $K\mathbf{N} = \left\|\dfrac{d\mathbf{T}}{ds}\right\| \dfrac{\mathbf{T}'(s)}{\|\mathbf{T}'(s)\|} = \mathbf{T}'(s) = \dfrac{d\mathbf{T}}{ds}.$

Finally,

$$\mathbf{N}'(s) = \frac{d}{ds}(\mathbf{B} \times \mathbf{T}) = (\mathbf{B} \times \mathbf{T}') + (\mathbf{B}' \times \mathbf{T})$$
$$= (\mathbf{B} \times K\mathbf{N}) + (-\tau\mathbf{N} \times \mathbf{T})$$
$$= -K\mathbf{T} + \tau\mathbf{B}.$$

12. $y = \dfrac{1}{32}x^{5/2}$

$$y' = \frac{5}{64}x^{3/2}$$

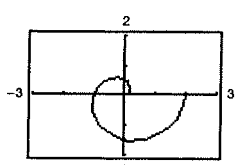
$$y'' = \frac{15}{128}x^{1/2}$$

$$K = \left|\frac{\frac{15}{128}x^{1/2}}{\left(1 + \frac{25}{4096}x^3\right)^{3/2}}\right|$$

At the point (4, 1), $K = \dfrac{120}{(89)^{3/2}} \Rightarrow r = \dfrac{1}{K} = \dfrac{(89)^{3/2}}{120} \approx 7.$

13. $\mathbf{r}(t) = \langle t\cos \pi t, t\sin \pi t\rangle,\ 0 \le t \le 2$

(a)



(b) Length $= \displaystyle\int_0^2 \|\mathbf{r}'(t)\|\,dt$

$$= \int_0^2 \sqrt{\pi^2 t^2 + 1}\,dt \approx 6.766 \quad \text{(graphing utility)}$$

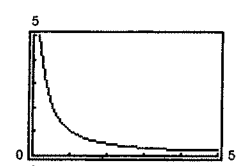
(c) $K = \dfrac{\pi(\pi^2t^2 + 2)}{[\pi^2t^2 + 1]^{3/2}}$

$K(0) = 2\pi$

$K(1) = \dfrac{\pi(\pi^2 + 2)}{(\pi^2 + 1)^{3/2}} \approx 1.04$

$K(2) \approx 0.51$

(d)



(e) $\displaystyle\lim_{t\to\infty} K = 0$

(f) As $t \to \infty$, the graph spirals outward and the curvature decreases.

14. (a) Eliminate the parameter to see that the Ferris wheel has a radius of 15 meters and is centered at $16\mathbf{j}$. At $t = 0$, the friend is located at $\mathbf{r}_1(0) = \mathbf{j}$, which is the low point on the Ferris wheel.

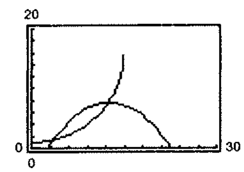
(b) If a revolution takes Δt seconds, then

$$\frac{\pi(t + \Delta t)}{10} = \frac{\pi t}{10} + 2\pi$$

and so $\Delta t = 20$ seconds. The Ferris wheel makes three revolutions per minute.

(c) The initial velocity is $r'_2(t_0) = -8.03\mathbf{i} + 11.47\mathbf{j}$. The speed is $\sqrt{8.03^2 + 11.47^2} \approx 14$ m/sec. The angle of inclination is arctan(11.47/8/03) $\approx$ 0.96 radians or 55°.

(d) Although you may start with other values, $t_0 = 0$ is a fine choice. The graph at the right shows two points of intersection. At $t = 3.15$ sec the friend is near the vertex of the parabola, which the object reaches when



$$t - t_0 = -\frac{11.47}{2(-4.9)} \approx 1.17 \text{ sec.}$$

Thus, after the friend reaches the low point on the Ferris wheel, wait $t_0 = 2$ sec before throwing the object in order to allow it to be within reach.

(e) The approximate time is 3.15 seconds after starting to rise from the low point on the Ferris wheel. The friend has a constant speed of $\|\mathbf{r}'_1(t)\| = 15$ m/sec. The speed of the object at that time is

$$\|\mathbf{r}'_2(3.15)\| = \sqrt{8.03^2 + [11.47 - 9.8(3.15 - 2)]^2} \approx 8.03 \text{ m/sec.}$$