# Exam Three

Calculus III
Professor D. Olles
Summer II 2009
Monroe Community College

Name______________________________

You must **SHOW ALL WORK** on this exam to receive partial credit for incorrect answers, or ANY credit for correct answers.
Simplify and reduce all answers as much as possible.
Please indicate the location of your final answers and write your solutions clearly to receive all possible credit.
You have until 7:30 pm to complete this portion of the exam, at which time I will come around and collect them. NO extended time!
Please wait until I have graded and returned your exam to you before leaving. You will need it to study for the Final Exam on Thursday.

| **Problem** | **1** | **2** | **3** | **4** | **5** | **6** | **7** | **8** | **9** | **10** | **Total** |
|---|---|---|---|---|---|---|---|---|---|---|---|
| **Pts Worth** | 8 | 12 | 15 | 10 | 15 | 8 | 10 | 8 | 6 | 8 | **100** |
| **Pts Earned** | | | | | | | | | | | |

1. Evaluate the double integral.

$$\int_0^2 \int_{x^2}^{2x} (x^2 + 2y)\,dy\,dx$$

$$= \int_0^2 (x^2 y + y^2)\Big|_{x^2}^{2x}\,dx$$

$$\int_0^2 \left[(x^2 \cdot 2x + (2x)^2) - (x^2 \cdot x^2 + (x^2)^2)\right] dx$$

$$= \int_0^2 \left[2x^3 + 4x^2 - x^4 - x^4\right] dx$$

$$= \int_0^2 \left[2x^3 + 4x^2 - 2x^4\right] dx$$

$$= \left[\frac{2x^4}{\cancel{4}_2} + \frac{4x^3}{3} - \frac{2x^5}{5}\right]_0^2$$

$$= \left(\frac{1}{2}(2)^4 + \frac{4}{3}(2)^3 - \frac{2}{5}(2)^5\right) - 0$$

$$= 8 + \frac{32}{3} - \frac{64}{5}$$

$$= \frac{120 + 150 - 192}{15}$$

$$= \boxed{\frac{88}{15}}$$

$$= \cancel{\boxed{\frac{26}{3}}}$$

2. Set up and evaluate the double integral that would give the area of the region bounded by the graphs of $x = y^2 + 1,\ x = 0,\ y = 0,\ y = 2.$

$0 \le y \le 2$

$A = \int_0^2 \int_0^{y^2+1} dx\,dy$

$= \int_0^2 x \Big|_0^{y^2+1} dy$

$\int_0^2 (y^2+1)\,dy$

$= \left[\frac{y^3}{3} + y\right]_0^2$

$8+6$

3. Set up and evaluate the triple integral that would give the volume of the solid bounded by the graphs of $z = x + y,\ z = 0,\ x = 0,\ x = 3,\ y = x$.

$0 \le z \le x +$

$\le 3$

$$V = \int_0^3 \int_0^x \int_0^{x+y} dz\,dy\,dx$$

$$= \int_0^3 \int_0^x z \Big|_0^{x+y} dy\,dx$$

$$= \int_0^3 \int_0^x (x+y)\,dy\,dx$$

$$= \int_0^3 \Big[xy + \tfrac{1}{2}y^2\Big]_0^x dx$$

$$= \int_0^3 \left(x \cdot x + \tfrac{1}{2}x^2\right) dx$$

$$= \frac{3}{2}\int_0^3 x^2\,dx$$

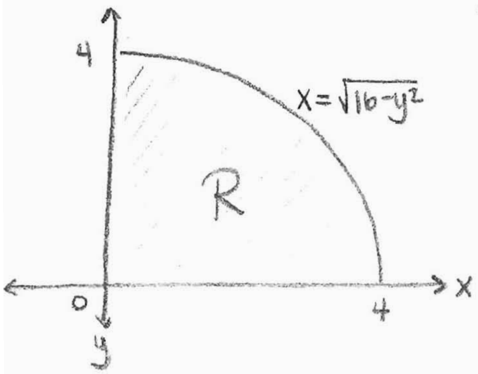
$$= \frac{3}{2} \cdot \frac{1}{3}x^3 \Big|_0^3$$

$$= \frac{1}{2}(27-0)$$

$$= \boxed{\frac{27}{2}}$$

4. Evaluate the iterated integral by converting to polar coordinates.

$$\int_0^4 \int_0^{\sqrt{16-y^2}} (x^2 + y^2)\,dx\,dy$$




$0 \le r \le 4$

$0 \le \theta \le \frac{\pi}{2}$

$$= \int_0^{\pi/2} \int_0^4 r^2 \cdot r\,dr\,d\theta$$

$$= \int_0^{\pi/2} \int_0^4 r^3\,dr\,d\theta$$

$$= \int_0^{\pi/2} \frac{1}{4} r^4 \Big|_0^4 \,d\theta$$

$$= \int_0^{\pi/2} 64\,d\theta$$

$$= 64\theta \Big|_0^{\pi/2}$$

$$= 64\left(\frac{\pi}{2} - 0\right)$$

$$= \boxed{32\pi}$$

5. Use the indicated change in variables and the Jacobian to evaluate the double integral where $R$ is the square with vertices $(1,2), (2,1), (3,2), (2,3)$.

$$\iint_R (x+y)\,dA$$

| $(x,y)$ | $\Rightarrow (u,v)$ |
|---|---|
| $(1,2)$ | $(3,-1)$ |
| $(2,1)$ | $(3,1)$ |
| $(3,2)$ | $(5,1)$ |
| $(2,3)$ | $(5,-1)$ |

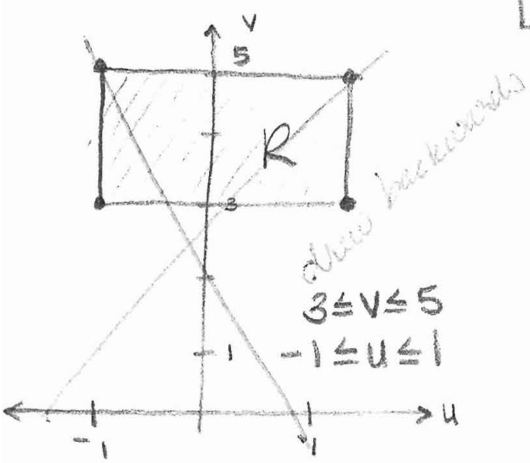
$$x = \frac{1}{2}(u+v) \qquad y = \frac{1}{2}(u-v)$$

$2X = u + V \qquad 2y = u - V$

$2X = 2y + 2V \leftarrow u = 2y + V$

$X = y + V \qquad u = 2y + X - y$

$\boxed{V = X - y} \qquad \boxed{u = X + y}$




$3 \le V \le 5$

$-1 \le u \le 1$

$3 \le u \le 5$

$-1 \le V \le 1$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{-1}{2} \end{vmatrix} = \frac{-1}{4} - \frac{1}{4} = \frac{-1}{2}$$

$$\Rightarrow \int_3^5 \int_{-1}^{1} \left(\tfrac{1}{2}(u+v) + \tfrac{1}{2}(u-v)\right)\left(\tfrac{-1}{2}\right) du\,dv$$

$$= \frac{-1}{2} \int_3^5 \int_{-1}^{1} u\,du\,dv = \frac{-1}{2} \int_3^5 \frac{1}{2}u^2 \Big|_{-1}^{1} dv$$

$$= \frac{-1}{2} \int_3^5 \frac{1}{2}(1-1)\,dv = \frac{-1}{4} \int_3^5 0\,dv.$$

$$= \boxed{0}$$

$= \frac{-1}{4} \int (25-9)\,d$

$= \frac{-1}{4} \int 16\,dv = \frac{-1}{4} 16v \Big|$

6. Determine if the vector field is conservative and if so, find its potential function.

$$\vec{F}(x,y) = -\frac{y}{x^2}\vec{\imath} + \frac{1}{x}\vec{\jmath}$$

$M = -yx^{-2}$ $\qquad$ $N = x^{-1}$

$\frac{\partial M}{\partial y} = -x^{-2}$ $\qquad$ $\frac{\partial N}{\partial x} = -x^{-2}$

equal so conservative!

$f_x(x,y) = -yx^{-2} \Rightarrow f(x,y) = \int -yx^{-2}\,dx = -y\frac{x^{-1}}{-1} + K_1(y) = \frac{y}{x} + K_1(y)$

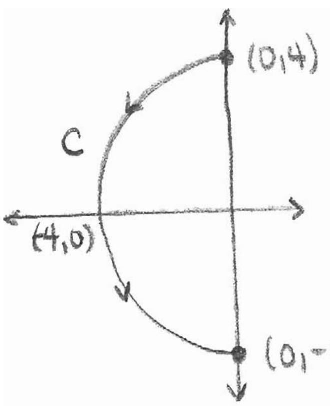
$f_y(x,y) = x^{-1} \Rightarrow f(x,y) = \int x^{-1}\,dy = yx^{-1} + K_2(x) = \frac{y}{x} + K_2(x)$

$$\boxed{f(x,y) = \frac{y}{x} + K}$$

7. Evaluate the line integral along the given path, using the definition of the line integral.

$$\int_C xyds$$

$C$: counterclockwise around the cirle $x^2 + y^2 = 16$ from $(0,4)$ *to* $(0,-4)$




$$\left.\begin{aligned} x &= 4\cos t \\ y &= 4\sin t \end{aligned}\right\} \quad \left.\begin{aligned} x'(t) &= -4\sin t \\ y'(t) &= 4\cos t \end{aligned}\right\} \quad \begin{aligned} [x'(t)]^2 &= 16\sin^2 t \\ [y'(t)]^2 &= 16\cos^2 t \end{aligned}$$

$$\frac{\pi}{2} \le t \le \frac{3\pi}{2}$$

$$\int_C xyds = \int_{\pi/2}^{3\pi/2} 4\cos t \cdot 4\sin t \sqrt{16\sin^2 t + 16\cos^2 t}\, dt = 16\int_{\pi/2}^{3\pi/2} \sin t\cos t \sqrt{16}\, dt$$

$$= 64\int_{\frac{\pi}{2}}^{3\pi/2} \sin t \cos t\, dt$$

$u = \sin t$
$du = \cos t\, dt$
$t = \frac{\pi}{2} \Rightarrow u = 1$
$t = \frac{3\pi}{2} \Rightarrow u = -1$

$$= 64\int_1^{-1} u\,du = 64\left[\frac{u^2}{2}\right]_1^{-1} = \frac{64}{2}\left[(-1)^2 - 1^2\right] = 32(0) = \boxed{0}$$

8. Evaluate $\int_C \vec{F} \cdot d\vec{r}$ for the vector field $\vec{F}(x, y) = xy\vec{\imath} + x^2\vec{\jmath}$ along the path defined by $\vec{r}(t) = t^2\vec{\imath} + t^3\vec{\jmath},\ 0 \leq t \leq 1$.

$$\left.\begin{aligned} x &= t^2 \\ y &= t^3 \end{aligned}\right\} \quad \vec{F} = t^2 \cdot t^3 \vec{\imath} + (t^2)^2 \vec{\jmath} = t^5 \vec{\imath} + t^4 \vec{\jmath}$$

$$d\vec{r} = (2t\vec{\imath} + 3t^2\vec{\jmath})\,dt$$

$$\int_0^1 \langle t^5, t^4 \rangle \cdot \langle 2t, 3t^2 \rangle \, dt = \int_0^1 (2t^6 + 3t^6)\,dt$$

$$= \int_0^1 5t^6\,dt = \left.\frac{5t^7}{7}\right|_0^1 = \boxed{\frac{5}{7}}$$

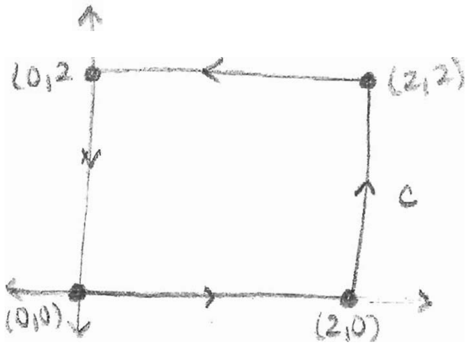
9. Use Green's Theorem to evaluate the Line Integral.

$$\int_C y dx + 2x dy$$

$C$: boundary of the square with vertices $(0,0), (0,2), (2,0), (2,2)$

$$M = y \qquad N = 2x$$

$$\left.\begin{array}{ll} \dfrac{\partial M}{\partial y} = 1 & \dfrac{\partial N}{\partial x} = 2 \end{array}\right\} \quad \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 2 - 1 = 1$$




$$0 \leq x \leq 2$$
$$0 \leq y \leq 2$$

$$\Rightarrow \int_0^2 \int_0^2 1 \, dx \, dy$$

$$= \int_0^2 x \Big|_0^2 dy$$

$$= \int_0^2 2 \, dy$$

$$= 2y \Big|_0^2$$

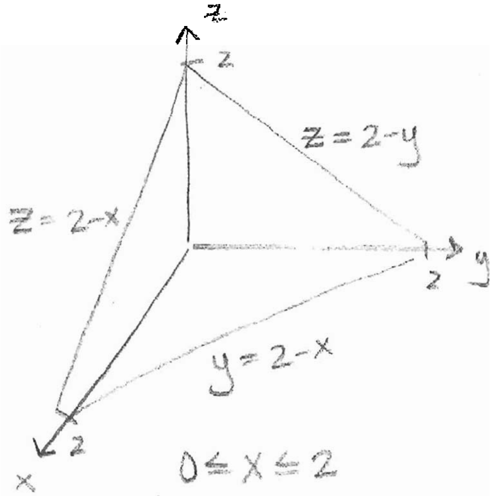
$$= \boxed{4}$$

10. Use the Divergence Theorem to evaluate $\iint_S \vec{F} \cdot \vec{N} dS$ for $\vec{F}(x, y, z) = x\vec{i} + y\vec{j} + z\vec{k}$ and $Q$: solid region bounded by the coordinate planes and $x + y + z = 2$.

$M = x \qquad N = y \qquad P = z$

$\frac{\partial M}{\partial x} = 1 \qquad \frac{\partial N}{\partial y} = 1 \qquad \frac{\partial P}{\partial z} = 1$

$\text{div}\vec{F} = 1 + 1 + 1 = 3$




$0 \le x \le 2$

$0 \le y \le 2-x$

$0 \le z \le 2-x-y$

$$\Rightarrow \int_0^2 \int_0^{2-x} \int_0^{2-x-y} 3\,dz\,dy\,dx$$

$$= \int_0^2 \int_0^{2-x} 3z \Big|_0^{2-x-y} dy\,dx$$

$$= \int_0^2 \int_0^{2-x} 3(2-x-y)\,dy\,dx$$

$$= 3\int_0^2 \left(2y - xy - \tfrac{1}{2}y^2\right)\Big|_0^{2-x} dx$$

$$= 3\int_0^2 \left(2(2-x) - x(2-x) - \tfrac{1}{2}(2-x)^2\right) dx$$

$$= 3\int_0^2 \left(4 - 2x - 2x + x^2 - 2 + 2x - \tfrac{1}{2}x^2\right) dx$$

$$= 3\int_0^2 \left(2 - 2x + \tfrac{1}{2}x^2\right) dx$$

$\boxed{4}$

$$= 3\left[2x - x^2 + \tfrac{1}{6}x^3\right]_0^2 = 3\left[4 - 4 + \tfrac{64}{6} - 0\right] = \cancel{\tfrac{64}{2}} = \cancel{\boxed{32}}$$

**Bonus:**

Explain why the curl$\vec{F}$ cannot be found for the vector in problem 6.

curl must be found on a function of 3 variables

$\vec{F}(x, y, z)$ and [illegible] s.t. domain $(-\infty, \infty)$

Crossproduct needs $\vec{i}, \vec{j}, \vec{k}$