Exam Three

Multivariable Calculus
Monroe Community College
Summer 2008
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Name Solutions

> You must **SHOW ALL WORK** on this exam in order to receive partial credit for incorrect answers or any credit for correct answers. Please indicate answers clearly. If I cannot read or follow your work, I reserve the right to award no points for that problem. This exam is worth 100 points.

1. Evaluate the iterated integral.

$$\int_0^2 \int_0^{\sqrt{4-y^2}} \frac{2}{\sqrt{4-y^2}}\,dx\,dy$$

$$= \int_0^2 \frac{2x}{\sqrt{4-y^2}}\Big|_0^{\sqrt{4-y^2}} dy = \int_0^2 \frac{2\sqrt{4-y^2}}{\sqrt{4-y^2}}\,dy = \int_0^2 2\,dy = 2y\Big|_0^2 = 2(2-0) = 4$$

2. Use an iterated integral to find the area of the region bounded by the following.

$$2x-3y=0,\ x+y=5,\ y=0$$

$2x-3y=0$
$3y=2x$
$y=\frac{2}{3}x$ or
$x=\frac{3}{2}y$

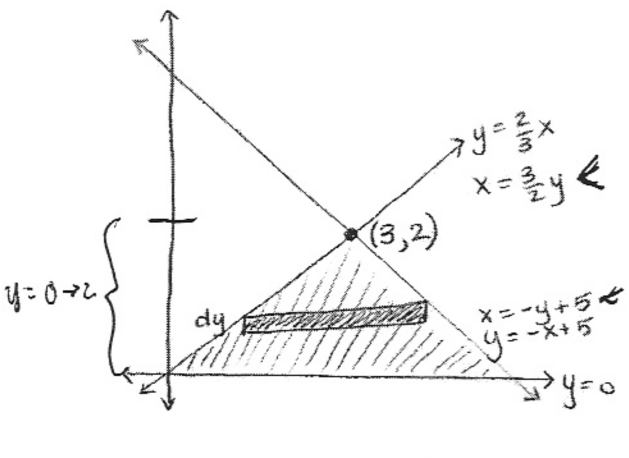
$x+y=5$
$y=-x+5$
or
$x=-y+5$

$$A = \int_0^2 \int_{\frac{3}{2}y}^{-y+5} dx\,dy = \int_0^2 x\Big|_{\frac{3}{2}y}^{-y+5} dy = \int_0^2 \left(-y+5-\frac{3}{2}y\right) dy$$

$$= \int_0^2 \left(-\frac{5}{2}y+5\right) dy = \left[\frac{-5y^2}{4}+5y\right]_0^2$$

$$= -5+10-0$$

$$= 5$$



3. Set up and evaluate the integral over the region R.

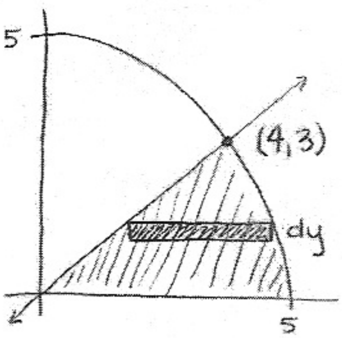
$$\iint_R x\,dA$$

R: sector of a circle in the first quadrant bounded by

$$y=\sqrt{25-x^2},\ 3x-4y=0,\ y=0$$

$3x-4y=0$
$4y=3x$
$y=\frac{3}{4}x$
or
$x=\frac{4}{3}y$

$y=\sqrt{25-x^2}$
or
$x=\sqrt{25-y^2}$



$$\int_0^3\int_{\frac{4}{3}y}^{\sqrt{25-y^2}} x\,dx\,dy = \int_0^3 \frac{x^2}{2}\Bigg|_{\frac{4}{3}y}^{\sqrt{25-y^2}} dy$$

$$= \int_0^3 \left(\frac{25-y^2}{2} - \frac{16y^2}{18}\right) dy = \int_0^3 \frac{225-9y^2-16y^2}{18}\,dy$$

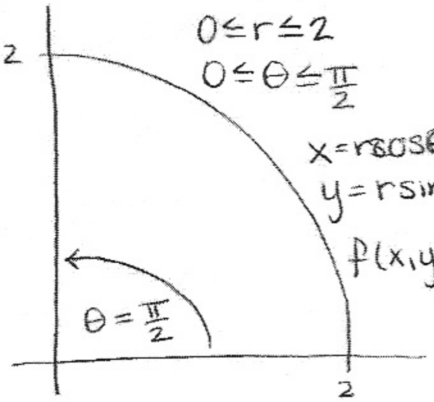
$$= \frac{1}{18}\int_0^3 (225-25y^2)\,dy = \frac{25}{18}\int_0^3 (9-y^2)\,dy = \frac{25}{18}\left(9y-\frac{y^3}{3}\right)\Bigg|_0^3$$

$$= \frac{25}{18}\left[\left(27-\frac{27}{3}\right)-0\right] = \frac{25}{\cancel{18}}\left(\frac{\cancel{2}}{\cancel{3}}\right)\left(\cancel{27}^{\,9}\right) = 25$$

4. Set up and evaluate the double integral by converting to polar coordinates.

$$f(x,y)=x+y$$

$$R: x^2+y^2\le 4,\ x\ge 0,\ y\ge 0$$



$0\le r\le 2$
$0\le\theta\le\frac{\pi}{2}$

$x=r\cos\theta$
$y=r\sin\theta$
$f(x,y)=r\cos\theta+r\sin\theta$

$$\int_0^{\pi/2}\int_0^2 (r\cos\theta+r\sin\theta)\,r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\int_0^2 r^2(\cos\theta+\sin\theta)\,dr\,d\theta$$

$$= \int_0^{\pi/2} \frac{r^3}{3}(\cos\theta+\sin\theta)\Bigg|_0^2 d\theta$$

$$= \int_0^{\pi/2} \frac{8}{3}(\cos\theta+\sin\theta)\,d\theta$$

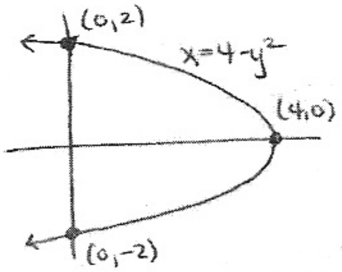
$$= \frac{8}{3}\left(-\sin\theta+\cos\theta\right)\Bigg|_0^{\pi/2} = \frac{8}{3}\left[\left(-\sin(\pi/2)+\cos(\pi/2)\right)-\left(-\sin(0)+\cos(0)\right)\right]$$

$$= \frac{8}{3}\left[(-1+0)-(0+1)\right] = \frac{8}{3}\left[-1-1\right] = \frac{8}{3}(-2) = \frac{-16}{3}$$

5. Use a triple integral to find the volume of the solid bounded by the following.

$$z = x,\ z = 0,\ x = 4 - y^2$$

$0 \le z \le x$
$0 \le x \le 4 - y^2$
$-2 \le y \le 2$



$$\int_{-2}^{2}\int_{0}^{x}\int_{0}^{4-y^2} dx\,dz\,dy = \cancel{\int_0^2\int_0^x x\Big|_0^{4-y^2} dz\,dy} = \cancel{\int_0^2\int_0^x (4-y^2)\,dz\,dy}$$

$$\cancel{= \int_0^2 (4-y^2)z}$$

$$= \int_0^2\int_0^{4-y^2} z\Big|_0^x dx\,dy = \int_0^2\int_0^{4-y^2} x\,dx\,dy = \int_0^2 \frac{x^2}{2}\Big|_0^{4-y^2} dy$$

$$= \int_0^2 \frac{(4-y^2)^2}{2}\,dy = \int_0^2 \frac{16-8y^2+y^4}{2}\,dy = \left[\frac{16y}{2} - \frac{8y^3}{6} + \frac{y^5}{10}\right]_0^2$$

$$= \left[8y - \frac{4y^3}{3} + \frac{y^5}{40}\right]_0^2 = 16 - \frac{32}{3} + \frac{32}{10} = 16 - \frac{32}{3} + \frac{16}{5} = \frac{240-160+48}{15}$$

$$= \frac{128}{15}$$

6. Verify that the following vector field is conservative. Then, find its potential function.

$$\vec{F}(x,y) = \frac{x\vec{i} + y\vec{j}}{x^2+y^2}$$

$M = \dfrac{x}{x^2+y^2} = x(x^2+y^2)^{-1}$

$\dfrac{\partial M}{\partial y} = -x(x^2+y^2)^{-2}(2y)$

$= \dfrac{-2xy}{(x^2+y^2)^2}$ ✓

$N = \dfrac{y}{x^2+y^2} = y(x^2+y^2)^{-1}$

$\dfrac{\partial N}{\partial x} = -y(x^2+y^2)^{-2}(2x)$

$= \dfrac{-2xy}{(x^2+y^2)^2}$ ✓

$\int f_x\,dx = \int \dfrac{x}{x^2+y^2}\,dx = \frac{1}{2}\ln(x^2+y^2) + g(y)$

$\int f_y\,dy = \int \dfrac{y}{x^2+y^2}\,dy = \frac{1}{2}\ln(x^2+y^2) + h(x)$

$f(x,y) = \frac{1}{2}\ln(x^2+y^2) + K$

7. Find the curl for the vector field at the given point.

$$\vec{F}(x,y,z) = e^x \sin y\vec{i} - e^x \cos y\vec{j} + z\vec{k} \qquad (0,0,3)$$

$$\text{Curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x \sin y & -e^x \cos y & z \end{vmatrix} = (0-0)\vec{i} + (0-0)\vec{j} + (-e^x \cos y - e^x \cos y)\vec{k}$$

$$= 0\vec{i} + 0\vec{j} - 2e^x \cos y \vec{k}$$

$$\text{Curl } \vec{F}\Big|_{(0,0,3)} = 0\vec{i} + 0\vec{j} - 2e^0 \cos(0)\vec{k}$$

$$= \langle 0,0,-2 \rangle$$

8. Find the divergence for the vector field at the given point.

$$\vec{F}(x,y,z) = e^x \sin y\vec{i} - e^x \cos y\vec{j} + z\vec{k} \qquad (0,0,3)$$

$$\text{div } \vec{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}$$

$$= e^x \sin y - e^x(-\sin y) + 1$$

$$= +2e^x \sin y + 1$$

$$\text{div } \vec{F}\Big|_{(0,0,3)} = 2e^0 \sin(0) + 1$$

$$= 0 + 1$$

$$= 1$$

9. Prove that $\operatorname{curl}(\nabla f) = \nabla \times (\nabla f) = 0$.

Let $\nabla f = f_x \vec{i} + f_y \vec{j} + f_z \vec{k}$

Then $\operatorname{curl}(\nabla f) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_x & f_y & f_z \end{vmatrix} = (f_{zy} - f_{yz})\vec{i} - (f_{zx} - f_{xz})\vec{j} + (f_{yx} - f_{xy})\vec{k}$

Since 2^{nd} mixed partials are equal, their differences are 0.

$\Rightarrow \quad = 0\vec{i} + 0\vec{j} + 0\vec{k}$

$= \vec{0}$ ///

10. Find the value of the line integral $\int_C \vec{F} \cdot d\vec{r}$.

$$\vec{F}(x, y) = ye^{xy}\vec{i} + xe^{xy}\vec{j}$$

C: the path of the circle $x^2 + y^2 = 1$ from the point $(0,1)$ to $(1,0)$

$0 \le t \le \frac{\pi}{2}$

$\left.\begin{array}{l} x = \cos t \\ y = \sin t \end{array}\right\} \vec{r}(t) = \cos t\,\vec{i} + \sin t\,\vec{j}$

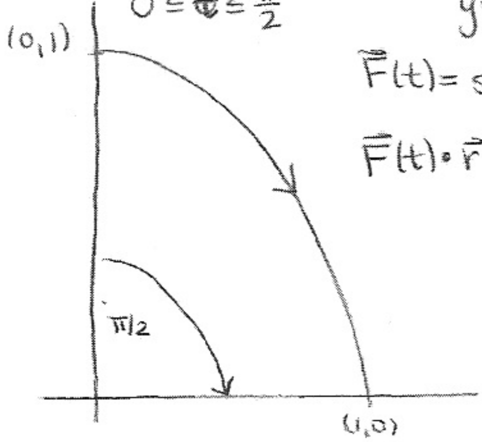
$d\vec{r} = (-\sin t\,\vec{i} + \cos t\,\vec{j})\,dt$

$\vec{F}(t) = \sin t\, e^{\cos t \sin t}\,\vec{i} + \cos t\, e^{\cos t \sin t}\,\vec{j}$

$\vec{F}(t) \cdot \vec{r}\,'(t) = \left(-\sin^2 t\, e^{\cos t \sin t} + \cos^2 t\, e^{\cos t \sin t}\right) dt$

$= e^{\cos t \sin t}(\cos^2 t - \sin^2 t)\,dt$

$\cancel{= e^{\cos t \sin t} \cos(2t)\,dt}$



$\cancel{\int_C \vec{F} \cdot d\vec{r} = \int_0^{\pi/2} e^{\cos t \sin t} \cos(2t)\,dt}$

$\int_C \vec{F} \cdot d\vec{r} = \int_{\frac{\pi}{2}}^{0} e^{\cos t \sin t}(\cos^2 t - \sin^2 t)\,dt$

$u = \cos t \sin t \Rightarrow du = (\cos t \cos t + - \sin t \sin t)\,dt$
$= (\cos^2 t - \sin^2 t)\,dt$

$= \int_{t=\frac{\pi}{2}}^{t=0} e^u\,du = e^u \Big|_{t=\frac{\pi}{2}}^{t=0} = e^{\sin t \cos t} \Big|_{\frac{\pi}{2}}^{0} = e^{\sin(0)\cos(0)} - e^{\sin(\pi/2)\cos(\pi/2)} = e^0 - e^0 = 1 - 1 = 0$