Exam Two

Calculus III
Professor D. Olles
Summer II 2009
Monroe Community College

Name Solutions

You must **SHOW ALL WORK** on this exam to receive partial credit for incorrect answers, or ANY credit for correct answers.
Simplify and reduce all answers as much as possible.
Please indicate the location of your final answers and write your solutions clearly to receive all possible credit.
You have until 7:30 pm to complete this portion of the exam, at which time I will come around and collect them. NO extended time!
If you finish before the time limit, you may leave the room and return at 7:45pm for lecture.
This exam is worth 100 points.

Problem	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**Total**
Pts Worth	5	5	10	10	15	10	10	10	10	15	**100**
Pts Earned											

1. Find the limit (if it exists) and discuss the continuity of the function. (5 points)

$$\lim_{(x,y)\to(0,0)} \frac{y+e^x}{1+x^2}$$

$$= \frac{0+e^0}{1+0^2}$$

$$= \frac{0+1}{1+0}$$

$$= 1$$

$1+x^2 \neq 0$

$x^2 \neq -1$ never true in $\mathbb{R}$

so $x, y \in \mathbb{R}$

2. Find all first partial derivatives. (5 points)

$$f(x,y) = xe^y + e^x \cos y$$

$f_x(x,y) = (1)e^y + (e^x)\cos y$

$= e^y + e^x \cos y$

$f_y(x,y) = x(e^y) + e^x(-\sin y)$

$= xe^y - e^x \sin y$

3. Find all second partial derivatives and verify that the mixed partials are equal. (10 points)

$$z = \frac{x}{x+y}$$

$$z_x = \frac{(x+y)(1) - x(1+0)}{(x+y)^2} = \frac{y}{(x+y)^2}$$

$$z_y = \frac{(x+y)(0) - x(0+1)}{(x+y)^2} = \frac{-x}{(x+y)^2}$$

$$z_{xx} = \frac{(x+y)^2(0) - y[2(x+y)(1)]}{(x+y)^4} = \boxed{\frac{-2xy-2y^2}{(x+y)^4}}$$

$$z_{yy} = \frac{(x+y)^2(0) - (-x)[2(x+y)(1)]}{(x+y)^4} = \boxed{\frac{2x^2+2xy}{(x+y)^4}}$$

$$z_{xy} = \frac{(x+y)^2(1) - y[2(x+y)(1)]}{(x+y)^4} = \frac{(x+y)^2 - 2y(x+y)}{(x+y)^4}$$

$$= \frac{x+y-2y}{(x+y)^4}$$

$$= \boxed{\frac{x-y}{(x+y)^4}} \leftarrow z_{xy} = z_{yx}$$

$$z_{yx} = \frac{(x+y)^2(-1) - (-x)(2(x+y))}{(x+y)^4}$$

$$= \frac{-(x+y)^2 + 2x(x+y)}{(x+y)^4}$$

$$= \frac{-x-y+2x}{(x+y)^4} = \boxed{\frac{x-y}{(x+y)^4}}$$

4. Find the total differential of the following function. (10 points)

$$f(x,y) = \frac{xy}{\sqrt{x^2+y^2}} = xy(x^2+y^2)^{-1/2}$$

$$f_x(x,y) = y(x^2+y^2)^{-1/2} + xy\left[-\tfrac{1}{2}(x^2+y^2)^{-3/2}(2x)\right]$$

$$= \frac{y}{\sqrt{x^2+y^2}} - \frac{x^2y}{(x^2+y^2)^{3/2}} = \frac{y(x^2+y^2) - x^2y}{(x^2+y^2)^{3/2}} = \frac{y^3}{(x^2+y^2)^{3/2}}$$

$$f_y(x,y) = x(x^2+y^2)^{-1/2} + xy\left[-\tfrac{1}{2}(x^2+y^2)^{-3/2}(2y)\right]$$

$$= \frac{x}{\sqrt{x^2+y^2}} - \frac{xy^2}{(x^2+y^2)^{3/2}} = \frac{x(x^2+y^2) - xy^2}{(x^2+y^2)^{3/2}} = \frac{x^3}{(x^2+y^2)^{3/2}}$$

$$\boxed{dz = \frac{y^3}{(x^2+y^2)^{3/2}}\,dx + \frac{x^3}{(x^2+y^2)^{3/2}}\,dy}$$

5. Find the derivative of w with respect to t using the chain rule (not substitution). (15 points)

$$w = \ln(x^2 + y^2), \quad x = 2t + 3, \quad y = 4 - t$$

$$\frac{\partial w}{\partial x} = \frac{1}{x^2+y^2}(2x) = \frac{2(2t+3)}{(2t+3)^2+(4-t)^2} = \frac{4t+6}{4t^2+12t+9+16-8t+t^2}$$

$$= \frac{4t+6}{5t^2+4t+25}$$

$$\frac{\partial w}{\partial y} = \frac{1}{x^2+y^2}(2y) = \frac{2(4-t)}{5t^2+4t+25} = \frac{8-2t}{5t^2+4t+25}$$

$$\frac{dx}{dt} = 2 \qquad \frac{dy}{dt} = -1$$

$$\frac{dw}{dt} = 2\left(\frac{4t+6}{5t^2+4t+25}\right) + (-1)\left(\frac{8-2t}{5t^2+4t+25}\right) = \frac{8t+12-8+2t}{5t^2+4t+25}$$

$$= \boxed{\frac{10t+4}{5t^2+4t+25}}$$

6. Use implicit differentiation to find the first partial derivatives of z. (10 points)

$$x^2y - 2yz - xz - z^2 = 0 = F(x,y,z)$$

$$\frac{\partial F}{\partial x} = F_x = 2xy - z$$

$$F_y = x^2 - 2z$$

$$F_z = -2y - x - 2z$$

$$\frac{\partial z}{\partial x} = -\left(\frac{2xy-z}{-2y-x-2z}\right) = \boxed{\frac{2xy-z}{2y+x+2z}}$$

$$\frac{\partial z}{\partial y} = -\left(\frac{x^2-2z}{-2y-x-2z}\right) = \boxed{\frac{x^2-2z}{2y+x+2z}}$$

7. Find the directional derivative of the function $f(x,y) = \frac{1}{4}y^2 - x^2$ at the point $(1,4)$ in the direction of $\vec{v} = \langle 2,1 \rangle$. (10 points)

$$\|\vec{v}\| = \sqrt{4+1} = \sqrt{5} \Rightarrow \vec{u} = \frac{1}{\|\vec{v}\|}\vec{v} = \left\langle \frac{2\sqrt{5}}{5}, \frac{\sqrt{5}}{5} \right\rangle$$

$f_x(x,y) = -2x \qquad f_y(x,y) = \frac{1}{2}y$

$f_x(1,4) = -2 \qquad f_y(1,4) = 2$

$$D_{\vec{u}} f(1,4) = (-2)\left(\frac{2\sqrt{5}}{5}\right) + (2)\left(\frac{\sqrt{5}}{5}\right)$$

$$= \frac{-4\sqrt{5}}{5} + \frac{2\sqrt{5}}{5}$$

$$= \boxed{\frac{-2\sqrt{5}}{5}}$$

8. Find the gradient of the following function $f(x,y) = e^{-x}\cos y$ and the maximum value of the directional derivative at the point $\left(0,\frac{\pi}{4}\right)$. (10 points)

$f_x(x,y) = -e^{-x}\cos y \qquad f_y(x,y) = -e^{-x}\sin y$

$f_x\left(0,\frac{\pi}{4}\right) = -e^{0}\cos\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2} \qquad f_y\left(0,\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

$$\boxed{\nabla f\left(0,\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}\vec{i} - \frac{\sqrt{2}}{2}\vec{j}}$$

$$\max = \left\|\nabla f\left(0,\frac{\pi}{4}\right)\right\| = \sqrt{\left(-\frac{\sqrt{2}}{2}\right)^2 + \left(-\frac{\sqrt{2}}{2}\right)^2} = \sqrt{\frac{2}{4} + \frac{2}{4}} = \sqrt{1} = \boxed{1}$$

9. Find the equation of the tangent plane and the parametric equation of the normal line to the function $f(x, y) = x^2y$ at the point (2,1,4). (10 points)

$z = x^2y$

$0 = x^2y - z$

$F(x,y,z) = x^2y - z$

$F_x(x,y,z) = 2xy$ $\qquad$ $F_y(x,y,z) = x^2$ $\qquad$ $F_z(x,y,z) = -1$

$F_x(2,1,4) = 4$ $\qquad$ $F_y(2,1,4) = 4$ $\qquad$ $\forall x,y,z$

Tangent Plane:

$4(x-2) + 4(y-1) + -1(z-4) = 0$

$\cancel{z - 4 = 0}$

$4x - 8 + 4y - 4 - z + 4 = 0$

$\boxed{4x + 4y - z = 8}$

Normal Line:

$\boxed{\begin{aligned} x &= 2 + 4t \\ y &= 1 + 4t \\ z &= 4 - t \end{aligned}}$

$\cancel{x = 2 \quad y = 1 \quad z = 4 - t}$

10. Use the Second Partials Test to find any possible extrema of the function $f(x, y) = x^2y - y^2 - 2xy - 4y - 4$. (15 points)

$f_x(x,y) = 2xy - 2y$ $\qquad$ $f_y(x,y) = x^2 - 2y - 2x - 4$

$0 = 2xy - 2y$ $\qquad$ $0 = x^2 - 2y - 2x - 4$

$0 = (x-1)2y$ $\qquad$ $2y = x^2 - 2x - 4$

$0 = (x-1)(x^2 - 2x - 4)$

$x = 1 \quad x = \frac{2 \pm \sqrt{4+16}}{2} = \frac{2 \pm 2\sqrt{5}}{2} = 1 \pm \sqrt{5}$

$x = 1 \Rightarrow y = \frac{-5}{2}$

$x = 1 + \sqrt{5} \Rightarrow y =$

$f_{xx} = 2y$

$f_{xx}(1, -\frac{5}{2})$

finished on bonus page

Bonus: Show how the gradient can be used to find the directional derivative of f in the direction of $\vec{u}$. (5 points)

$$\nabla f(x,y)\cdot\vec{u} = \langle f_x(x,y), f_y(x,y)\rangle \cdot \langle a,b\rangle$$
$$= af_x(x,y) + bf_y(x,y)$$
$$= D_{\vec{u}} f(x,y)$$

you could be more specific by saying $\langle a,b\rangle = \langle \cos\theta, \sin$

#10 cont

$f(x,y) = x^2y - y^2 - 2xy - 4y - 4$

$f_x(x,y) = 2xy - 2y$

$0 = 2xy - 2y$

$0 = x(2y) - 2y$

$0 = x(x^2-2x-4) - (x^2-2x-4)$

$0 = x^3 - 2x^2 - 4x - x^2 + 2x + 4$

$0 = x^3 - 3x^2 - 2x + 4$

$f_y(x,y) = x^2 - 2y - 2x - 4$

$0 = x^2 - 2y - 2x - 4$

$2y = x^2 - 2x - 4$

1	1	-3	-2	4
		1	-2	-4
	1	-2	-4	0

$(x-1)(x^2-2x-4) = 0$

$x = 1 \qquad x = \frac{2 \pm \sqrt{4+16}}{2}$

$= \frac{2 \pm 2\sqrt{5}}{2}$

$= 1 \pm \sqrt{5}$

$x = \{1-\sqrt{5}, 1, 1+\sqrt{5}\}$

$y = \{0, \frac{-5}{2}, 0\}$

$f_{xx}(x,y) = 2y$

$f_{yy}(x,y) = -2$

$f_{xy}(x,y) = 2x - 2$

① $d = f_{xx}(1-\sqrt{5},0)f_{yy}(1-\sqrt{5},0) - [f_{xy}(1-\sqrt{5},0)]^2$

$= (2\cdot 0)(-2) - (2-2\sqrt{5}-2)^2$

$= 0 - 4(5)$

$= -20 \qquad f_{xx} = 0 \quad d = -20$

$\boxed{\text{Saddle Point @ } (1-\sqrt{5}, 0, -4)}$

② $d = f_{xx}(1,\frac{-5}{2})f_{yy}(1,\frac{-5}{2}) - [f_{xy}(1,\frac{-5}{2})]^2$

$= (-5)(-2) - (0)^2$

$= 10 \qquad f_{xx} = -5 \quad d = 10$

$\boxed{\text{maximum value @ } (1, \frac{-5}{2}) \text{ of } \frac{9}{4}}$

③ $d = f_{xx}(1+\sqrt{5},0)f_{yy}(1+\sqrt{5},0) - [f_{xy}(1+\sqrt{5},0)]^2$

$= (0)(-2) - (2+2\sqrt{5}-2)^2$

$= 0 - 4(5)$

$= -20 \qquad f_{xx} = 0 \quad d = -20$

$\boxed{\text{Saddle Point @ } (1+\sqrt{5}, 0) \text{ of } -4}$