Exam Two

Multivariable Calculus
Monroe Community College
Summer 2008
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Name: Solutions

You must ***SHOW ALL WORK*** on this exam in order to receive partial credit for incorrect answers or any credit for correct answers. Please indicate all answers clearly on this exam. This is a two hour long exam worth 100 points.

1. State the domain and discuss the continuity of the following functions. (1 pt each)

   a. $f(x,y) = \frac{xy}{x^2+y^2}$

   $x^2+y^2 \neq 0 \Rightarrow x,y \neq 0$ simultaneously

   Continuous for all $(x,y) \neq (0,0)$

   b. $g(x,y) = \frac{y+xe^{-y^2}}{1+x^2}$

   $1+x^2 \neq 0 \Rightarrow x^2 \neq -1$

   Continuous for all reals $(-\infty,\infty)$

2. Find the following limits (if they exist). (2 pts each)

   a. $\lim_{(x,y)\to(1,1)} \frac{xy}{x^2+y^2} = \frac{(1)(1)}{1^2+1^2} = \frac{1}{2}$

   b. $\lim_{(x,y)\to(0,0)} \frac{y+xe^{-y^2}}{1+x^2} = \frac{0+0e^0}{1+0} = \frac{0}{1} = 0$

3. Find the partial derivatives with respect to each variable.

a. $f(x, y) = 2xy + x^2 - e^{xy}$ (6 pts)

$f_x = 2y + 2x - ye^{xy}$

$f_y = 2x - xe^{xy}$

b. $g(x, y, z) = \arctan(xy + z) - z^3$ (9 pts)

$g_x = \frac{1}{1+(xy+z)^2}(y)$

$g_y = \frac{1}{1+(xy+z)^2}(x)$

$g_z = \frac{1}{1+(xy+z)^2}(1) - 3z^2$

4. Use implicit differentiation to find the first partials of z. (8 pts each)

a. $x^2y - yz + 2xz = 0 = F(x,y,z)$

$F_x = 2xy + 2z$

$F_y = x^2 - z$

$F_z = -y + 2x$

$$\frac{\partial z}{\partial x} = \frac{-F_x}{F_z} = -\left(\frac{2xy+2z}{-y+2x}\right)$$

$$\frac{\partial z}{\partial y} = \frac{-F_y}{F_z} = -\left(\frac{x^2-z}{-y+2x}\right)$$

b. $\sin(xy + z) + 4xz = 0 = F(x,y,z)$

$F_x = \cos(xy+z)(y) + 4z$

$F_y = \cos(xy+z)(x)$

$F_z = \cos(xy+z)(1) + 4x$

$$\frac{\partial z}{\partial x} = -\left(\frac{y\cos(xy+z)+4z}{\cos(xy+z)+4x}\right)$$

$$\frac{\partial z}{\partial y} = -\left(\frac{x\cos(xy+z)}{\cos(xy+z)+4x}\right)$$

5. Use the chain rule of multiple variables to find the derivative of w.

a. $w = \ln(x^2 + y^2), x = 2t + 3, y = -t + 4$ (8 pts)

$$\frac{\partial w}{\partial x} = \frac{1}{x^2+y^2}(2x) = \frac{2x}{x^2+y^2} = \frac{2(2t+3)}{(2t+3)^2+(-t+4)^2} = \frac{4t+6}{4t^2+12t+9+t^2-8t+16}$$

$$= \frac{4t+6}{5t^2+4t+25}$$

$$\frac{\partial w}{\partial y} = \frac{2y}{x^2+y^2} = \frac{2(-t+4)}{5t^2+4t+25} = \frac{-2t+8}{5t^2+4t+25}$$

$$\frac{dx}{dt} = 2 \qquad \frac{dy}{dt} = -1$$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} = \left(\frac{4t+6}{5t^2+4t+25}\right)(2) + \left(\frac{-2t+8}{5t^2+4t+25}\right)(-1)$$

$$= \frac{8t+12+2t-8}{5t^2+4t+25} = \boxed{\frac{10t+4}{5t^2+4t+25}}$$

b. $w = \frac{xy}{z}, x = 2r + t, y = rt, z = 2r - t$ (10 pts)

$$\frac{\partial w}{\partial x} = \frac{y}{z} = \frac{rt}{2r-t} \qquad \frac{\partial x}{\partial t} = 1 \qquad \frac{\partial x}{\partial r} = 2$$

$$\frac{\partial w}{\partial y} = \frac{x}{z} = \frac{2r+t}{2r-t} \qquad \frac{\partial y}{\partial t} = r \qquad \frac{\partial y}{\partial r} = t$$

$$\frac{\partial w}{\partial z} = \frac{-xy}{z^2} = \frac{-(rt)(2r+t)}{(2r-t)^2} = \frac{-(2r^2t+rt^2)}{(2r-t)^2} \qquad \frac{\partial z}{\partial t} = -1 \qquad \frac{\partial z}{\partial r} = 2$$

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$

$$= \left(\frac{rt}{2r-t}\right)(2) + \left(\frac{2r+t}{2r-t}\right)(t) + \left(\frac{-2r^2t-rt^2}{(2r-t)^2}\right)(2) = \frac{2rt+2rt+t^2}{2r-t} - \frac{2r^2t+rt^2}{(2r+t)^2}$$

$$\frac{\partial w}{\partial t} = \left(\frac{rt}{2r-t}\right)(1) + \left(\frac{2r+t}{2r-t}\right)(r) + \left(\frac{-(2r^2t+rt^2)}{(2r-t)^2}\right)(-1)$$

$$= \frac{rt+2r^2+rt}{2r-t} + \frac{2r^2t+rt^2}{(2r-t)^2}$$

6. Find the gradient of the following. Then, use the gradient to find the directional derivative and the maximum value of the directional derivative. (6 pts each)

a. $z = e^{-x}\cos y \quad \left(0, \frac{\pi}{4}\right)$

$$z_x = -e^{-x}\cos y \Rightarrow z_x\Big|_{(0,\pi/4)} = -e^{0}\cos(\pi/4) = -\frac{\sqrt{2}}{2}$$

$$z_y = -e^{-x}\sin y \Rightarrow z_y\Big|_{(0,\pi/4)} = -e^{0}\sin(\pi/4) = -\frac{\sqrt{2}}{2}$$

$$\nabla z = -\frac{\sqrt{2}}{2}\vec{i} - \frac{\sqrt{2}}{2}\vec{j}$$

$$\|\nabla z\| = \sqrt{\left(-\frac{\sqrt{2}}{2}\right)^2 + \left(-\frac{\sqrt{2}}{2}\right)^2} = \sqrt{\frac{2}{4}+\frac{2}{4}} = \sqrt{\frac{4}{4}} = 1 = \max(D_{\vec{u}})$$

b. $z = \frac{y}{x^2+y^2} \quad (1,1)$

$$z_x = \frac{(0)(x^2+y^2) + y(2x)}{(x^2+y^2)^2} = \frac{-2xy}{(x^2+y^2)^2} \Rightarrow z_x\Big|_{(1,1)} = \frac{-2}{2^2} = \frac{-1}{2}$$

$$z_y = \frac{(1)(x^2+y^2) - y(2y)}{(x^2+y^2)^2} = \frac{x^2+y^2-2y^2}{(x^2+y^2)^2} \Rightarrow z_y\Big|_{(1,1)} = \frac{0}{2^2} = 0$$

$$\nabla z = \frac{-1}{2}\vec{i} + 0\vec{j}$$

$$\|\nabla z\| = \sqrt{\left(\frac{-1}{2}\right)^2 + 0^2} = \sqrt{\frac{1}{4}} = \frac{1}{2} = \max$$

7. Find the directional derivatives of the following functions at the given points in the direction of the given vector. (6 pts each)

a. $f(x, y) = xy + 2x$, $(2,1)$, $v = \langle 1, -1 \rangle$

$$\vec{u} = \frac{1}{\|\vec{v}\|}\vec{v} = \frac{1}{\sqrt{1+1}}\langle 1, -1\rangle = \frac{1}{\sqrt{2}}\langle 1,-1\rangle = \left\langle \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right\rangle$$

$$f_x = y + 2 \Rightarrow f_x\big|_{(2,1)} = 1 + 2 = 3$$

$$f_y = x \Rightarrow f_y\big|_{(2,1)} = 2$$

$$D_{\vec{u}} = 3\left(\frac{\sqrt{2}}{2}\right) + 2\left(-\frac{\sqrt{2}}{2}\right) = (3-2)\frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2}$$

b. $f(x, y, z) = 6x + 3xy - 4yz$, $(1,0,1)$, $v = \langle 1,1,-1 \rangle$

$$\vec{u} = \frac{1}{\|\vec{v}\|}\vec{v} = \frac{1}{\sqrt{1+1+1}}\langle 1,1,-1\rangle = \frac{1}{\sqrt{3}}\langle 1,1,-1\rangle = \left\langle \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3} \right\rangle$$

$$f_x = 6 + 3y \Rightarrow f_x\big|_{(1,0,1)} = 6 + 0 = 6$$

$$f_y = 3x - 4z \Rightarrow f_y\big|_{(1,0,1)} = 3 - 4 = -1$$

$$f_z = -4y \Rightarrow f_z\big|_{(1,0,1)} = 0$$

$$D_{\vec{u}} = 6\left(\frac{\sqrt{3}}{3}\right) + (-1)\left(\frac{\sqrt{3}}{3}\right) + 0\left(-\frac{\sqrt{3}}{3}\right) = \frac{6\sqrt{3}}{3} - \frac{\sqrt{3}}{3} = \frac{5\sqrt{3}}{3}$$

8. Find the equations of the tangent plane AND normal line to the following surface. (12 pts)

$$f(x,y) = \sqrt{25-y^2} \qquad (2,3,4)$$

$z = \sqrt{25-y^2}$

$0 = \sqrt{25-y^2} - z = F(x,y,z)$

$F_x = 0 = a$

$F_y = \frac{1}{2}(25-y^2)^{-1/2}(-2y) = \frac{-y}{\sqrt{25-y^2}} \Rightarrow F_y\Big|_{(2,3,4)} = \frac{-3}{4} = b$

$F_z = -1 = c$

T. Plane

$a(x-x_1)+b(y-y_1)+c(z-z_1)=0$

$0(x-2)+\frac{-3}{4}(y-3)+-1(z-4)=0$

$0-\frac{3}{4}y+\frac{9}{4}-z+4=0$

$-3y+9-4z+16=0 \quad \boxed{3y+4z=25}$

N. Line

$\cancel{\frac{x-2}{0}}$ not needed!! $= \frac{y-3}{\left(\frac{-3}{4}\right)} = \frac{z-4}{-1}$

9. Show that the function satisfies Laplace's Equation $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$. (4 pts)

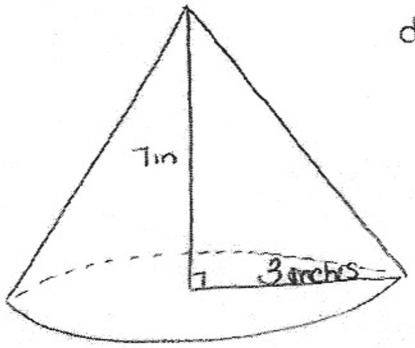
$$z = x^3 - 3xy^2$$

$z_x = 3x^2 - 3y^2 \qquad z_y = -6xy$

$z_{xx} = 6x \qquad z_{yy} = -6x$

$z_{xx} + z_{yy} = 6x - 6x = 0$ ✓

10. A right circular cone is measured and the radius and height are found to be 3 inches and 7 inches respectively. The possible error in measurement for each is 0.10 inches. Approximate the maximum possible error in the computation of the volume. (5 pts)



$dr = dh = \frac{1}{10}$

$$V = \frac{\pi}{3} r^2 h$$

$$dV = \frac{\pi}{3}\left[2rh\,dr + r^2\,dh\right]$$

$$dV = \frac{\pi}{3}\left[2(3)(7)\left(\frac{1}{10}\right) + (3)^2\left(\frac{1}{10}\right)\right]$$

$$= \frac{\pi}{3}\left[\frac{42}{10} + \frac{9}{10}\right]$$

$$= \frac{\pi}{3}\left[\frac{51}{10}\right]$$

$$= \boxed{\frac{51\pi}{30}}$$