Exam One

Calculus III
Professor D. Olles
Summer II 2009
Monroe Community College

Name Solutions

You must **SHOW ALL WORK** on this exam to receive partial credit for incorrect answers, or ANY credit for correct answers.
Simplify and reduce all answers as much as possible.
Please indicate the location of your final answers and write your solutions clearly to receive all possible credit.
You have until 7:30 pm to complete this portion of the exam, at which time I will come around and collect them. NO extended time!
If you finish before the time limit, you may leave the room and return at 7:45pm for lecture.
This exam is worth 100 points.

Problem	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	Total
Pts Worth	5	4	6	8	5	6	8	5	5	5	8	8	6	6	6	9	100
Pts Earned																	

For #'s 1- 6, consider the vectors $\vec{u} = \langle -1, 4, 2\rangle, \vec{v} = \langle 2, -4, -3\rangle, \vec{w} = \langle 2, -2, -1\rangle$.

1. Find the unit vector in the direction of $\vec{u}$. (5 points)

$$\|\vec{u}\| = \sqrt{(-1)^2 + 4^2 + 2^2} = \sqrt{1+} = \sqrt{21}$$

$$\frac{1}{\|\vec{u}\|}\vec{u} = \frac{1}{\sqrt{21}}\langle -1, 4, 2\rangle = \left\langle \frac{-1}{\sqrt{21}}, \frac{4}{\sqrt{21}}, \frac{2}{\sqrt{21}} \right\rangle = \boxed{\left\langle \frac{-\sqrt{21}}{21}, \frac{4\sqrt{21}}{21}, \frac{2\sqrt{21}}{} \right\rangle}$$

2. Find the magnitude of $\vec{v}$. (4 points)

$$\|\vec{v}\| = \sqrt{2^2 + (-4)^2 + (-3)^2} = \sqrt{4+16+9} = \boxed{\sqrt{29}}$$

3. Find the angle between the vectors $\vec{v}$ and $\vec{w}$. (6 points)

$$\cos\theta = \frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\|\|\vec{w}\|} = \frac{(2)(2) + (-4)(-2) + (-3)(-1)}{\sqrt{29} \cdot 3} = \frac{4+8+3}{3\sqrt{29}}$$

$$\|\vec{w}\| = \sqrt{2^2 + (-2)^2 + (-1)^2} = \sqrt{4+4+1} = \sqrt{9} = 3$$

$$\cos\theta = \frac{15}{3\sqrt{29}} = \frac{5}{\sqrt{29}}$$

$$\boxed{\theta = \arccos\left(\frac{5}{\sqrt{29}}\right) = 21.8^\circ = 0.38 \text{ rad}}$$

4. Find the area of the parallelogram generated by $\vec{v}$ and $\vec{w}$. (8 points)

$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -4 & -3 \\ 2 & -2 & -1 \end{vmatrix} = (4-6)\vec{i} - (-2+6)\vec{j} + (-4+8)\vec{k}$$
$$= -2\vec{i} - 4\vec{j} + 4\vec{k}$$

$$A = \|\vec{v} \times \vec{w}\| = \sqrt{(-2)^2 + (-4)^2 + (4)^2} = \sqrt{4+16+16} = \sqrt{36} = \boxed{6}$$

5. Find the volume of parallelepiped generated by $\vec{u}$, $\vec{v}$ and $\vec{w}$. (5 points)

$$V = |\vec{u} \cdot (\vec{v} \times \vec{w})| = |(-1)(-2) + (4)(-4) + (2)(4)|$$
$$= |2 - 16 + 8|$$
$$= |-6|$$
$$= \boxed{6}$$

6. Determine the projection of $\vec{u}$ onto $\vec{w}$. (6 points)

$$\text{proj}_{\vec{w}} \vec{u} = \left(\frac{\vec{u} \cdot \vec{w}}{\|\vec{w}\|^2}\right)\vec{w}$$
$$= \left(\frac{(-1)(2) + (4)(-2) + (2)(-1)}{(2)^2 + (-2)^2 + (-1)^2}\right)\langle 2, -2, -1 \rangle$$
$$= \left(\frac{-2-8-2}{4+4+1}\right)\langle 2, -2, -1 \rangle$$
$$= \frac{-12}{9}\langle 2, -2, -1 \rangle$$
$$= -\frac{4}{3}\langle 2, -2, -1 \rangle = \boxed{\left\langle \frac{-8}{3}, \frac{8}{3}, \frac{4}{3} \right\rangle}$$

7. Complete the square to write the equation of the sphere in standard form, then state the center and radius of the sphere. (8 points)

$$x^2 + y^2 + z^2 + 2x - 4y + 6z + 10 = 0$$

$$(x^2+2x)+(y^2-4y)+(z^2+6z) = -10$$

$$(x^2+2x+1)+(y^2-4y+4)+(z^2+6z+9) = -10+1+4+9$$

$$\boxed{(x+1)^2+(y-2)^2+(z+3)^2 = 4}$$

Center: $(-1, 2, -3)$

Radius: 2

8. Find the component form of the vector $\vec{v}$ given that it's magnitude is given by $\|\vec{v}\| = 16$ and the angle it makes with the positive x- axis is given by $\theta = \frac{\pi}{3}$. Sketch the vector in the plane including the angle. (5 points)

$$\vec{v} = \|\vec{v}\|\cos\theta\,\vec{i} + \|\vec{v}\|\sin\theta\,\vec{j}$$

$$\vec{v} = 16\cos\left(\frac{\pi}{3}\right)\vec{i} + 16\sin\left(\frac{\pi}{3}\right)\vec{j}$$

$$\vec{v} = 16\left(\frac{1}{2}\right)\vec{i} + 16\left(\frac{\sqrt{3}}{2}\right)\vec{j}$$

$$\boxed{\vec{v} = 8\vec{i} + 8\sqrt{3}\,\vec{j}}$$

9. Find the parametric equations of the line passing through the points $(-1,3,4)$ and $(2,0,6)$. (5 points)

$x_1 \; y_1 \; z_1$

$\vec{v} = \langle 2+1, 0-3, 6-4 \rangle = \langle 3,-3,2 \rangle = \langle a,b,c \rangle$

$$\boxed{\begin{aligned} x &= -1+3t \\ y &= 3-3t \\ z &= 4+2t \end{aligned}}$$

10. Find the equation of the plane passing through the point $(-3,-6,2)$ and with normal vector $\vec{n} = \langle 4,2,-1 \rangle$. (5 points)

$a(x-x_1)+b(y-y_1)+c(z-z_1)=0$

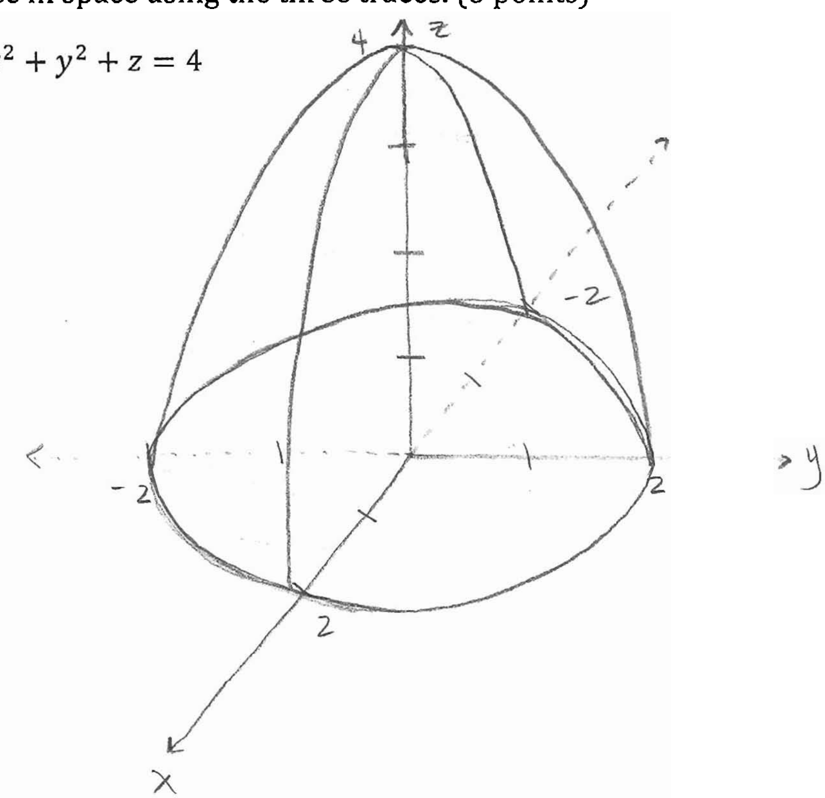
$4(x+3)+2(y+6)-(z-2)=0$

$4x+12+2y+12-z+2=0$

$\boxed{4x+2y-z+26=0}$

11. Sketch the following surface in space using the three traces. (8 points)

$x^2+y^2+z=4$



xy trace

$x^2+y^2=4$

circle w/ radius 2

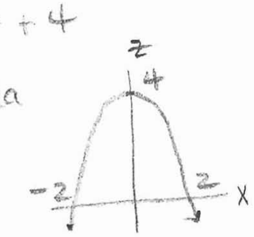
center $(0,0)$

xz trace

$x^2+z=4$

$z=-x^2+4$

Parabola



yz trace

same as xz

12. Convert the following point from rectangular to cylindrical and spherical coordinates (being sure to label you answers appropriately). (8 points)

$$(-\sqrt{2}, 2\sqrt{2}, 4)$$

$X = -\sqrt{2}$ $\quad y = 2\sqrt{2}$ $\quad z = 4$

$r = \sqrt{(-\sqrt{2})^2 + (2\sqrt{2})^2} = \sqrt{2+8} = \sqrt{10}$

$\rho = \sqrt{(-\sqrt{2})^2 + (2\sqrt{2})^2 + 4^2} = \sqrt{2+8+16} = \sqrt{26}$

$\theta = \arctan\left(\frac{2\sqrt{2}}{-\sqrt{2}}\right) = \arctan(-2) =$

$\phi = \arccos\left(\frac{4}{\sqrt{26}}\right) =$

Cylindrical: $(\sqrt{10}, \quad , 4)$

Spherical: $(\sqrt{26}, \quad , \quad)$

For #'s 13-18, consider the following vector valued function.

$$\vec{r}(t) = \tan t\,\vec{\imath} + \frac{1}{1+t^2}\vec{\jmath} - 3t\vec{k}$$

13. State the domain of the vector valued function, and therefore the intervals on which it is continuous on $-\pi \le t \le \pi$. (6 points)

$\tan t = \frac{\sin t}{\cos t}$ $\qquad 1 + t^2 \neq 0$ $\qquad -3t$

$t^2 \neq -1$ $\qquad t \in (-\infty, \infty)$

$\cos t \neq 0$ $\qquad$ no problem

$t \neq \left\{-\frac{\pi}{2}, \frac{\pi}{2}\right\}$

$$\boxed{\left[-\pi, -\frac{\pi}{2}\right), \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \left(\frac{\pi}{2}, \pi\right]}$$

14. Evaluate $\lim_{t\to\frac{\pi}{2}^-} \vec{r}(t)$. (6 points)

$$= \lim_{t\to\frac{\pi}{2}^-} \left[\tan t\,\vec{i} + \frac{1}{1+t^2}\vec{j} - 3t\vec{k}\right]$$

$$= \left[\lim_{t\to\frac{\pi}{2}^-} \tan t\right]\vec{i} + \left[\lim_{t\to\frac{\pi}{2}^-} \frac{1}{1+t^2}\right]\vec{j} - \left[\lim_{t\to\frac{\pi}{2}^-} 3t\right]\vec{k}$$

$$= \boxed{\infty\vec{i} + \frac{1}{1+\frac{\pi^2}{4}}\vec{j} - \frac{3\pi}{2}\vec{k}}$$

15. Find the derivative of the vector valued function. (6 points)

$$\frac{d}{dt}\left[\tan t\,\vec{i} + \frac{1}{1+t^2}\vec{j} - 3t\vec{k}\right]$$

$$= \boxed{\sec^2 t\,\vec{i} - \frac{2t}{(1+t^2)^2}\vec{j} - 3\vec{k}}$$

16. Integrate the vector valued function. (9 points)

$$\left[\int \tan t\,dt\right]\vec{i} + \left[\int \frac{1}{1+t^2}\,dt\right]\vec{j} - \left[\int 3t\,dt\right]\vec{k}$$

$$\boxed{= \left[\ln|\sec t| + C_1\right]\vec{i} + \left[\arctan t + C_2\right]\vec{j} - \left[\frac{3t^2}{2} + C_3\right]\vec{k}}$$

Bonus (5 points):

Prove that the magnitude of a scalar multiple of a vector is equal to the absolute value of the scalar multiplied by the magnitude of the vector. That is:

$$\|c\vec{v}\| = |c|\|\vec{v}\|$$

Let $\vec{v} = \langle v_1, v_2, v_3 \rangle$

Then $c\vec{v} = \langle cv_1, cv_2, cv_3 \rangle$

and $\|c\vec{v}\| = \sqrt{(cv_1)^2 + (cv_2)^2 + (cv_3)^2}$

$= \sqrt{c^2 v_1^2 + c^2 \cdot v_2^2 + c^2 v_3^2}$

$= \sqrt{c^2 (v_1^2 + v_2^2 + v_3^2)}$

$= \sqrt{c^2}\sqrt{v_1^2 + v_2^2 + v_3^2}$

$= |c| \cdot \|\vec{v}\|$ ///