Exam One

Multivariable Calculus
Monroe Community College
Summer 2008
Professor D. Olles

Name: Solutions

You must **SHOW ALL WORK** on this exam in order to receive partial credit for incorrect answers or any credit for correct answers. Please indicate all answers clearly on this exam. This is a two hour long exam worth 100 points.

$\vec{u} = \langle 2, -6, -2 \rangle$
$\vec{v} = \langle -2, -4, 2 \rangle$

1. Consider the points $P(1,2,3)$, $Q(3,-4,1)$ and $R(-1,-2,5)$ and let $\bar{u} = \overline{PQ}$ and $\bar{v} = \overline{PR}$. Find the following:

a. $2\bar{u} \times 3\bar{v} = \langle 4, -12, -4 \rangle \times \langle -6, -12, 6 \rangle = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & -12 & -4 \\ -6 & -12 & +6 \end{vmatrix}$

$= (-72 - 48)\vec{i} - (24 - 24)\vec{j} + (-48 - 72)\vec{k}$

$= -120\vec{i} + 0\vec{j} - 120\vec{k} = \langle -120, 0, -120 \rangle$

b. $\bar{u} \cdot \bar{v} = (2)(-2) + (-6)(-4) + (-2)(2)$

$= -4 + 24 - 4$

$= 16$

c. $\|\bar{v}\| = \sqrt{(-2)^2 + (-4)^2 + (2)^2} = \sqrt{4 + 16 + 4} = \sqrt{24} = 2\sqrt{6}$

d. Unit vector in the direction of $\bar{v}$.

$\vec{u} = \frac{1}{\|\vec{v}\|}\vec{v} = \frac{1}{2\sqrt{6}}\langle -2, -4, 2 \rangle = \left\langle \frac{-1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right\rangle$

$= \left\langle -\frac{\sqrt{6}}{6}, \frac{-2\sqrt{6}}{6}, +\frac{\sqrt{6}}{6} \right\rangle$

$= \left\langle -\frac{\sqrt{6}}{6}, -\frac{\sqrt{6}}{3}, \frac{\sqrt{6}}{6} \right\rangle$

2. Find the set of parametric equation of the line through the following points.

$$P(1,2,-1),\ Q(3,-4,1)$$

$$\vec{v} = \vec{PQ} = \langle 2,-6,-2\rangle = \langle a,b,c\rangle$$

$$\begin{Bmatrix} x = x_1 + at \\ y = y_1 + bt \\ z = z_1 + ct \end{Bmatrix} \Rightarrow \begin{Bmatrix} x = 1 + 2t \\ y = 2 - 6t \\ z = -1 - 2t \end{Bmatrix}$$

3. Find the angle between the given vectors.

$$\vec{u} = \langle 4,-1,5\rangle$$
$$\vec{v} = \langle 3,2,-2\rangle$$

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}$$

$$\cos\theta = \frac{(4)(3)+(-1)(2)+(5)(-2)}{\sqrt{16+1+25}\sqrt{9+4+4}}$$

$$\cos\theta = \frac{12-2-10}{\sqrt{42}\sqrt{17}}$$

$$\cos\theta = 0$$

$$\theta = \arccos(0) = 0 \text{ rad}$$
$$= 0^\circ$$

4. Find the equation of the sphere by completing the square. Then, state the center and radius of the sphere.

$$x^2+y^2+z^2-2x+3y-8z-20=0$$

$(x^2-2x)+(y^2+3y)+(z^2-8z)=20$

$\left[\frac{1}{2}(-2)\right]^2=(-1)^2=1$ *

$\left[\frac{1}{2}(3)\right]^2=\left(\frac{3}{2}\right)^2=\frac{9}{4}$ *

$\left[\frac{1}{2}(-8)\right]^2=(-4)^2=16$ *

$(x^2-2x+1)+(y^2+3y+9/4)+(z^2-8z+16)=20+1+\frac{9}{4}+16$

$(x-1)(x-1)+(y+\frac{3}{2})(y+\frac{3}{2})+(z-4)(z-4)=37+\frac{9}{4}=\frac{148+9}{4}=\frac{157}{4}$

$(x-1)^2+(y+\frac{3}{2})^2+(z-4)^2=\left(\frac{\sqrt{157}}{2}\right)^2$

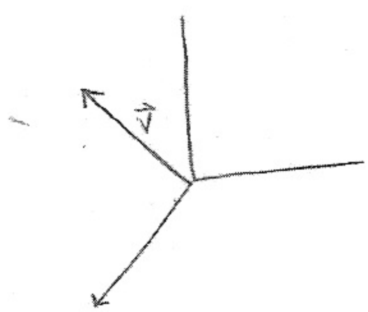
Center: $(1,-\frac{3}{2},4)$

Radius: $\frac{\sqrt{157}}{2}$

5. Find the component form of the vector $\vec{u}$ such that it is perpendicular to the plane $x-3y+4z=0$ and $\|\vec{u}\|=3$.

from the plane: $a=1$ $b=-3$ $c=4 \Rightarrow \vec{v}=\langle 1,-3,4\rangle$ is $\perp$ to the plane (direction vector)

$\|\vec{v}\|=\sqrt{1+9+16}=\sqrt{26}$



$k\vec{v}=\langle k,-3k,4k\rangle$

$\|k\vec{v}\|=3=\sqrt{k^2+9k^2+4k^2}=k\sqrt{26}$

$k=\frac{3}{\sqrt{26}}=\frac{3\sqrt{26}}{26}$

$\vec{u}=\left\langle \frac{3\sqrt{26}}{26},\frac{-9\sqrt{26}}{26},\frac{12\sqrt{26}}{26}\right\rangle$

6. Prove that $\vec{u}\cdot(\vec{v}+\vec{w})=\vec{u}\cdot\vec{v}+\vec{u}\cdot\vec{w}$.

$$\vec{u}\cdot(\vec{v}+\vec{w}) = \vec{u}\cdot\langle v_1+w_1, v_2+w_2\rangle = \langle u_1, u_2\rangle\cdot\langle v_1+w_1, v_2+w_2\rangle$$
$$= u_1(v_1+w_1) + u_2(v_2+w_2)$$
$$= u_1v_1 + u_1w_1 + u_2v_2 + u_2w_2$$
$$= \underbrace{u_1v_1 + u_2v_2} + \underbrace{u_1w_1 + u_2w_2}$$
$$= \vec{u}\cdot\vec{v} + \vec{u}\cdot\vec{w} \quad ///$$

7. Prove that $\|\vec{u}-\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\vec{u}\cdot\vec{v}$.

$$\vec{u}-\vec{v} = \langle u_1-v_1, u_2-v_2\rangle$$
$$\|\vec{u}-\vec{v}\|^2 = (u_1-v_1)^2 + (u_2-v_2)^2$$
$$= (u_1-v_1)(u_1-v_1) + (u_2-v_2)(u_2-v_2)$$
$$= u_1^2 + 2u_1v_1 + v_1^2 + u_2^2 - 2u_2v_2 + v_2^2$$
$$= (u_1^2+u_2^2) + (v_1^2+v_2^2) - 2(u_1v_1 + u_2v_2)$$
$$= \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\vec{u}\cdot\vec{v} \quad ///$$

8. Find the projection of $\bar{u}$ onto $\bar{v}$ where $\bar{u} = \langle 4,-1,5\rangle$, $\bar{v} = \langle 3,2,+2\rangle$.

$$\vec{u}\cdot\vec{v} = (4)(3) + (-1)(2) + (5)(+2) = 12 - 2 + 10 = 20$$

$$\|\vec{v}\|^2 = (3)^2 + (2)^2 + (2)^2 = 9 + 4 + 4 = 17$$

$$\text{proj}_{\vec{v}}\vec{u} = \left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\right)\vec{v} = \left(\frac{20}{17}\right)\langle 3,2,2\rangle$$

$$= \left\langle \frac{60}{17}, \frac{40}{17}, \frac{40}{17}\right\rangle$$

9. Find the equation of the plane that passes through the points $P(-3,-4,2)$, $Q(-3,4,1)$, $R(1,1,-2)$.

$$\vec{u} = \overrightarrow{PQ} = \langle 0, 8, -1\rangle$$

$$\vec{v} = \overrightarrow{PR} = \langle 4, 5, -4\rangle$$

$$\vec{n} = \vec{u}\times\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 8 & -1 \\ 4 & 5 & -4 \end{vmatrix} = (-32+5)\vec{i} - (0+4)\vec{j} + (0-32)\vec{k}$$

$$= -27\vec{i} - 4\vec{j} - 32\vec{k}$$

$$= \langle -27, -4, -32\rangle = \langle a, b, c\rangle$$

$$a(x-x_1) + b(y-y_1) + c(z-z_1) = 0$$

$$-27(x+3) - 4(y+4) - 32(z-2) = 0$$

$$-27x - 81 - 4y - 16 - 32z + 64 = 0$$

$$-27x - 4y - 32z - 33 = 0$$

$$27x + 4y + 32z = -33$$

10. Convert the following point from rectangular to spherical coordinates.

$$\left(2\sqrt{2},-2\sqrt{2},4\right)$$

$x = 2\sqrt{2} \quad y = -2\sqrt{2} \quad z = 4$

$\rho = \sqrt{x^2+y^2+z^2} = \sqrt{(2\sqrt{2})^2+(-2\sqrt{2})^2+(4)^2} = \sqrt{8+8+16} = 4\sqrt{2}$

$\theta = \arctan\left(\frac{y}{x}\right) = \arctan\left(\frac{-2\sqrt{2}}{2\sqrt{2}}\right) = \arctan(-1) = \frac{3\pi}{4}$

$\phi = \arccos\left(\frac{z}{\rho}\right) = \arccos\left(\frac{4}{4\sqrt{2}}\right) = \arccos\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$

$\left(4\sqrt{2}, \frac{3\pi}{4}, \frac{\pi}{4}\right)$

11. Convert the following equation from rectangular to cylindrical coordinates.

$$x^2 - y^2 = z^2$$

$x = r\cos\theta$
$y = r\sin\theta \Rightarrow$
$z = z$

$(r\cos\theta)^2 - (r\sin\theta)^2 = z^2$

$r^2\cos^2\theta - r^2\sin^2\theta = z^2$

$r^2(\cos^2\theta - \sin^2\theta) = z^2$

$r^2\cos(2\theta) = z^2$

$r^2 = z^2\sec(2\theta)$

12. Consider the following vector valued function.

$$\vec{r}(t)=\sqrt{4-t^2}\,\bar{i}+t^2\bar{j}-6t\bar{k} = (4-t^2)^{1/2}\vec{i}+t^2\vec{j}-6t\vec{k}$$

a. Find the domain and state the intervals on which the vector valued function is continuous.

$$4-t^2 \geq 0$$
$$4 \geq t^2$$
$$t^2 \leq 4$$
$$|t| \leq \sqrt{4}$$
$$|t| \leq 2$$
$$-2 \leq t \leq 2$$

$$[-2,2]$$

b. Find the magnitude of the vector valued function.

$$\|\vec{r}(t)\| = \sqrt{(\sqrt{4-t^2})^2+(t^2)^2+(-6t)^2}$$
$$= \sqrt{4-t^2+t^4+36t^2}$$
$$= \sqrt{t^4+35t^2+4}$$

c. Find the limit of the function as $t \to 2^+$.

$$\lim_{t\to 2^+}\left[\sqrt{4-t^2}\,\vec{i}+t^2\vec{j}-6t\vec{k}\right]$$
$$= 0\vec{i}+4\vec{j}-12\vec{k} = \langle 0,4,-12\rangle$$

d. Find the derivative of the vector valued function.

$$\vec{r}\,'(t) = \tfrac{1}{\cancel{2}}(4-t^2)^{-1/2}(-\cancel{2}t)\vec{i}+2t\vec{j}-6\vec{k}$$
$$= \frac{-t}{\sqrt{4-t^2}}\vec{i}+2t\vec{j}-6\vec{k}$$

e. Find the indefinite integral of the vector valued function.

$$\int \vec{r}(t)\,dt = \left[\int\sqrt{4-t^2}\,dt\right]\vec{i} + \left[\int t^2\,dt\right]\vec{j} - 6\left[\int t\,dt\right]\vec{k}$$

$\int\sqrt{4-t^2}\,dt$ let $t=2\sin\theta \Rightarrow \frac{{}^{O}t}{{}_{H}2} = \sin\theta \Rightarrow \theta = \arcsin\left(\frac{t}{2}\right)$

$dt = 2\cos\theta\,d\theta$

$$= \int\sqrt{4-(2\sin\theta)^2}\cdot 2\cos\theta\,d\theta = 2\int\sqrt{4-4\sin^2\theta}\cdot\cos\theta\,d\theta = 2\int\sqrt{4(1-\sin^2\theta)}\cdot\cos\theta\,dt$$

$$= 2\int 2\sqrt{\cos^2\theta}\cdot\cos\theta\,d\theta = 4\int\cos^2\theta\,d\theta = 4\int\left(\frac{1+\cos(2\theta)}{2}\right)d\theta = 2\int(1+\cos(2\theta))\,d\theta$$

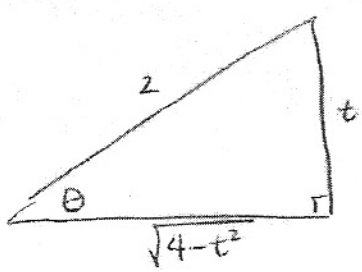
$u = 2\theta$
$du = 2\,d\theta$
$\frac{1}{2}du = d\theta$

$$= \cancel{2}\int(1+\cos(u))\cancel{\tfrac{1}{2}}\,du = \int(1+\cos u)\,du = u + \sin u + C$$

$$= 2\theta + \sin(2\theta) + C = (2\theta + 2\sin\theta\cos\theta) + C$$

$$= 2\arcsin\left(\frac{t}{2}\right) + \boxed{2\left(\frac{t}{2}\right)\left(\frac{\sqrt{4-t^2}}{2}\right)} + C$$

$$= 2\arcsin\left(\frac{t}{2}\right) + \frac{t\sqrt{4-t^2}}{2} + C$$



$\cos\theta = \frac{A}{H} = \frac{\sqrt{4-t^2}}{2}$

$\sin\theta = \arcsin\frac{t}{2}$

$$\int\vec{r}(t)\,dt = \left(2\arcsin\left(\tfrac{t}{2}\right) + \frac{t\sqrt{4-t^2}}{2} + c_1\right)\vec{i} + \left(\frac{t^3}{3} + c_2\right)\vec{j}$$

$$-6\left(\frac{t^2}{2}\right) + c_3\,\vec{k}$$

$$= \left(2\arcsin\left(\tfrac{t}{2}\right) + \frac{t\sqrt{4-t^2}}{2} + c_1\right)\vec{i} + \left(\frac{t^3}{3} + c_2\right)\vec{j} - \left(3t^2 + c_3\right)\vec{k}$$

13. Find $\vec{r}(t)$ given that $\vec{r}\,'(t) = e^{t}\,\vec{i} - e^{-t}\,\vec{j} + \vec{k}$ and $\vec{r}(0) = \frac{1}{2}\vec{i} - \vec{j} + \vec{k}$.

$\cancel{\vec{r}(t) = (}$ $\cancel{u = -t^2 \Rightarrow -u = t^2 \Rightarrow}$

$\cancel{du = -2t\,dt}$

$\cancel{\vec{r}(t) = e^{t}\,\vec{i} + e^{-t}}$

$$\vec{r}(t) = (e^{t} + C_1)\vec{i} + (e^{-t} + C_2)\vec{j} + (t + C_3)\vec{k}$$

$$\vec{r}(0) = (e^{0} + C_1)\vec{i} + (e^{0} + C_2)\vec{j} + (0 + C_3)\vec{k}$$

$$= (1 + C_1)\vec{i} + (1 + C_2)\vec{j} + C_3\vec{k}$$

$$= \frac{1}{2}\vec{i} - \vec{j} + \vec{k}$$

$1 + C_2 = -1$ $\quad C_3 = 1$

$1 + C_1 = \frac{1}{2}$ $\quad C_2 = -2$

$C_1 = -\frac{1}{2}$

$$\boxed{\vec{r}(t) = \left(e^{t} - \frac{1}{2}\right)\vec{i} + \left(e^{-t} - 2\right)\vec{j} + \vec{k}}$$